The Logic of Compound Statements

Logical Form and Logical Equivalence

Lecture Notes #2

Peter Blomgren
Department of Mathematics and Statistics
San Diego State University
San Diego, CA 92182-7720
blomgren@terminus.SDSU.EDU
http://terminus.SDSU.EDU

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Two arguments

If the program syntax is faulty or if program execution results in division by zero, then the computer will generate an error message. Therefore, if the computer does not generate an error message, then the program syntax correct and program execution does not result in division by zero.

If x is a real number such that $x < -2$ or $x > 2$, then $x^2 > 4$. Therefore, if $x^2 \leq 4$, then $x \geq -2$ and $x \leq 2$.

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The content of these arguments is very different. Nevertheless, their ***logical form*** is the same:

If p ***or*** **q**, **then r**.
Therefore, if **not r** then **not p** ***and*** **not q**.

Example-1.1.1: Fill in the blanks so that argument (b) has the same form as argument (a). Then represent the common form of the arguments using letters to stand for component structures.

Statement A:

If Jane is a math major or Jane is a CS major,
then Jane will take Math 245.
Jane is a CS major.
Therefore, Jane will take Math 245.

Statement B:

If logic is easy or ________,
then ________.
I will study hard.
Therefore, I will get an A in this course.

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Common Form:

If **p** or **q**, then **r**.

q.

Therefore, **r**.

Statements

In any mathematical theory new terms are defined using previously defined terms. This process has to start somewhere. In ***logic***, the words *sentence*, *true*, and *false* are initial undefined terms.

> ***Definition:* Statement** —
> A **statement** (or proposition) is a sentence that is true or false, but not both.

Examples:

"The square root of 9 is 3."

"The square root of 9 is 81."

are both statements, the first one is true, and the second false.

Non-statements

The sentence:

"She is a college student."

Sure looks like a statement. However, the truth or falsity ***depends*** on the reference for the pronoun *she*.

If the sentence was preceded by additional information that made the pronoun's reference clear, **then** the sentence would be a statement.

On its own, the sentence is ***neither*** true nor false; hence it is not a statement (in the language of mathematics).

Similarly "$x + y > 0$" is not a statement because the truth or falsity depends on the values of x and y.

In order to express complicated statement clearly, we introduce three symbols:

> The symbol $\sim$ denotes **not**.
> The symbol $\wedge$ denotes **and**.
> The symbol $\vee$ denotes **or**.

"$\sim$ **p**" is read "not p" and is called ***the negation of p***.

Side note: In the computer language C, the symbol for **not** is "**!**", hence "**! p**" means "not p" in C.

The conjunction of p and q:

"$\mathbf{p} \wedge \mathbf{q}$" is read "p and q."

The disjunction of p and q:

"$\mathbf{p} \vee \mathbf{q}$" is read "p or q."

The order of evaluation matters — $\sim$ has the highest order of precedence, *e.g.*

$\sim \mathbf{p} \wedge \mathbf{q} = (\sim \mathbf{p}) \wedge \mathbf{q}$.

We use parentheses to override and/or clarify the order of operations, thus "$\sim (\mathbf{p} \wedge \mathbf{q})$" represents the negation of the conjunction of p and q.

The symbols $\wedge$ and $\vee$ are considered coequal in order of operation, and an expression such as

$$\mathbf{p} \wedge \mathbf{q} \vee \mathbf{r}$$

is considered ambiguous.

This expression must be written as either

$$(\mathbf{p} \wedge \mathbf{q}) \vee \mathbf{r} \quad \text{or} \quad \mathbf{p} \wedge (\mathbf{q} \vee \mathbf{r})$$

to have meaning.

Note: The statements $(\mathbf{p} \wedge \mathbf{q}) \vee \mathbf{r}$ and $\mathbf{p} \wedge (\mathbf{q} \vee \mathbf{r})$ are ***not the same.*** We will discuss this in detail soon.

Translating from English to Symbols

Example-1.1.2: Write each of the following sentences symbolically, letting **p**=”it is hot” and **q**=”it is sunny”.

(a) “It is not hot but sunny”

(b) “It is neither hot nor sunny”

Solution:

Translating from English to Symbols

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Solution:

(a) By convention “but” = “and”, so the sentence is equivalent to “It is not hot and it is sunny”, which we write symbolically as $(\sim \mathbf{p}) \wedge \mathbf{q}$.

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(b) The phrase "neither A nor B" means the same as "not A and not B." To say it is neither hot nor sunny means it is not hot and it is not sunny. Therefore, the given sentence can be written symbolically as $(\sim \mathbf{p}) \wedge (\sim \mathbf{q})$.

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In both (a) and (b) the parentheses around the negations are optional.

Translating Mathematical Inequalities to Symbols

Note: the notation for inequalities involves both ***and*** and ***or*** statements. For instance, if x. a, and b are particular real numbers, then

$$x \leq a \quad \text{means} \quad x < a \quad \textbf{\textit{or}} \quad x = a$$

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which expands to

$$(a < x \quad \textbf{\textit{or}} \quad a = x) \quad \textbf{\textit{and}} \quad (x < b \quad \textbf{\textit{or}} \quad x = b)$$

More examples

Example-1.1.3: Suppose x is a particular real number. Let **p=**"$0 < x$", **q=**"$x < 3$", and **r=**"$x = 3$" respectively. Write the following inequalities symbolically:

(a) $x \leq 3$

(b) $0 < x < 3$

(c) $0 < x \leq 3$

Solution:

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Solution:

(a) $\mathbf{q} \vee \mathbf{r}$

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(b) $\mathbf{p} \wedge \mathbf{q}$

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(b) $0 < x < 3$
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Solution:

(a) $\mathbf{q} \vee \mathbf{r}$
(b) $\mathbf{p} \wedge \mathbf{q}$
(c) $\mathbf{p} \wedge (\mathbf{q} \vee \mathbf{r})$

Definition and Truth Tables — Negation (not / $\sim$)

***Definition*: *Negation* —**
If p is a statement variable, the **negation** of p is "not p" or "It is not the case that p". The negation is denoted $(\sim p)$. It has the opposite truth value from p: if p is true, then "not p" is false; if p is false, then "not p" is true.

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The truth values for negation are summarized in a ***truth table:***

p	$\sim p$
T	F
F	T

Truth table for $\sim p$.

Definition and Truth Tables — Conjunction (and / $\wedge$)

***Definition*: Conjunction —**

If p and q are statement variables, the **conjunction** of p and q is "p and q". The conjunction is denoted $p \wedge q$. It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, then $p \wedge q$ is false.

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p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth table for $p \wedge q$.

Definition and Truth Tables — Disjunction (or / $\vee$)

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p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth table for $p \vee q$.

Note that disjunction is an ***inclusive or*** (its truth value is **`true`** when ***both*** p and q are **`true`**).

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We can express ***exclusive or*** as a **compound statement**: For the statement variables p and q, we want an expression which is true ***if exactly one of*** p ***or*** q ***is true*** and false otherwise.

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p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

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T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Sometimes we use the notation $p \oplus q$ for exclusive-or.

General Compound Statements

***Definition*: Statement form —**

A **statement form** (or **propositional form**) is an expression made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variable. The **truth table** for a given statement form displays the truth values that correspond to the different combinations of truth values for the variables.

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To compute the truth values for a statement form: For each combination of truth values for the statement variables, first evaluate the expressions within the innermost parentheses, then evaluate the expressions within the next innermost parentheses, and so forth until you have the truth values for the complete expression.

Example: Truth Table for $(p \wedge q) \vee \sim r$

p	q	r	$(p \wedge q)$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

Logical Equivalence

The statements

$$8 > 3 \quad \text{and} \quad 3 < 8$$

are two different ways of saying the same thing (by the definition of $<$ and $>$).

The statements

"Pigs fly and cats bark" and "Cats bark and pigs fly"

are also two different ways of saying the same thing. The reason is the ***logical form*** of the statement.

Any two statements having the same form as these statements would either be both true or both false. In such a case the statements are said to be ***logically equivalent***.

Logical Equivalence — Truth Table

The expressions $p \wedge q$ and $q \wedge p$ are logically equivalent:

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Since the $p \wedge q$ and $q \wedge p$ ***columns*** in the table ***have the same values***, the statements are **logically equivalent.**

Logical Equivalence — Definition

***Definition*: *Logical equivalence* —**

Two *statement forms* are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of forms P and Q is denoted by writing $P \equiv Q$.

Two *statements* are called **logically equivalent** if, and only if, when the same statement variables are used to represent identical component statements, their forms are logically equivalent.

Checking for Logical Equivalence

To test whether two statement forms P and Q are logically equivalent:

(1) Construct the truth tables for P and Q using the same statement variables for identical component statements.

(2) Check each combination of truth values of the statement variables to see whether the truth value of P is the same as the truth value of Q.

(a) If in each row, the truth value for P is the same as the truth value for Q, then P and Q are logically equivalent.

(b) Otherwise P and Q are not logically equivalent.

De Morgan's Laws: Negations of AND and OR

For the statement "John is tall and Jim is short" to be true, both components must be true. It follows that for the statement to be false, ***one or both*** components must be false.

Thus the negation is "John is not tall or Jim is not short."

In general, the negation of a conjunction is logically equivalent to the disjunction of their negations:

$$\sim(p \wedge q) \quad \equiv \quad \sim p \ \vee \sim q$$

De Morgan's Laws — Truth Table for AND

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

This shows that $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

De Morgan's Laws — Truth Table for OR

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

This shows that $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

De Morgan's Laws — A Warning

According to De Morgan's Laws, the negation of

$$\begin{aligned} p &: \quad \text{Jim is tall and Jim is thin} \\ \sim p &: \quad \text{Jim is not tall or Jim is not thin} \end{aligned}$$

In English we can write the statement p more compactly as "Jim is tall and thin"...

$$\begin{aligned} q &: \quad \text{Jim is tall and thin} \\ \sim q &: \quad \text{Jim is not tall and thin} \end{aligned}$$

The problem here is that we do not have complete statements on both sides of the AND.

Although the laws of logic are extremely useful, they should be used as an aid to thinking, not as a mechanical substitute for it.

Tautologies and Contradictions

Definition: ***Tautology and Contradiction*** —

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is called a **tautological statement.**

A **contradiction** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is called a **contradictory statement.**

Tautologies and Contradictions

Example:

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F

Hence $p \vee \sim p$ is a tautology, and $p \wedge \sim p$ a contradiction.

Logical Equivalences

Given any statement variables p, q and r, a tautology t and a contradiction c, the following equivalences hold:

Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws	$p \wedge t \equiv p$	$p \vee c \equiv p$
Negation laws	$p \vee \sim p \equiv t$	$p \wedge \sim p \equiv c$
Double negative law	$\sim(\sim p) \equiv p$	
Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
De Morgan's laws	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
Universal bound laws	$p \vee t \equiv t$	$p \wedge c \equiv c$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of t and c	$\sim t \equiv c$	$\sim c \equiv t$