## Math 245: Discrete Mathematics

Sequences and Mathematical Induction

Sequences

Lecture Notes #7


**Peter Blomgren**
Department of Mathematics and Statistics
San Diego State University
San Diego, CA 92182-7720
**blomgren@terminus.SDSU.EDU**
**http://terminus.SDSU.EDU**



`$Id: lecture.tex,v 1.2 2006/10/16 20:07:05 blomgren Exp $`




## Introduction

So far we have talked about the ***fundamentals of logic***; we have looked at compound and quantified statements (Chapters 1–2).

Then we explored some introductory number theory, and tried our hands at a couple of different methods of proof: ***direct proofs, proofs by contradiction,*** and ***proofs by contraposition***; as well as "anti-proofs", *i.e.* ***counterexamples*** (Chapter 3).

Now we'll switch gears a little — we'll look at sequences (the computer scientists among us should think of ***for***- and ***while***-loops). We will look at sequences of numbers — looking for patterns, etc. Also, we will prove things sequentially (using ***mathematical induction***).



## Sequences

A powerful tool in mathematics (and other sciences, and life itself) is to discover and make use of **patterns** (*c.f.* Math 596 "Pattern Formation.")

**Figure:** To the left we see a picture of a chemical reaction (Belouzov-Zhabotinsky) in progress, and to the right a mathematical model of the reaction mimicking the complex time evolution of the pattern.

We will study slightly less complicated patterns, starting with ***sequences of numbers***; — and we'll verify conjectures about patterns.

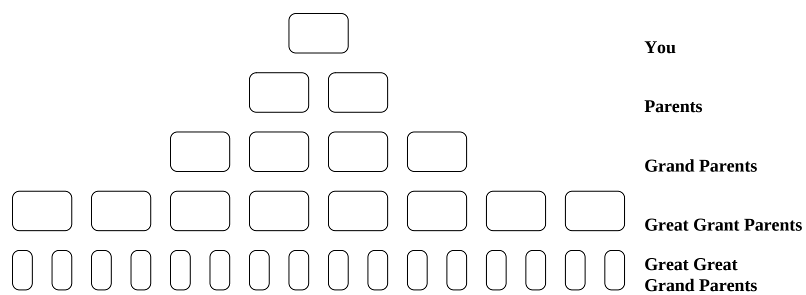


## Sequences: Counting your ancestors...

Imagine you want to trace down your family tree, and write down all your ancestors.

First, there is you — the center of the universe as we know it.

You have two parents, 4 grand parents, 8 great-grand-parents, etc...



## Sequences: Counting your ancestors...

| Generation ($g$) | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|
| Number of Ancestors | 2 | 4 | 8 | 16 | 32 | 64 | ... |
| $a_g = 2^g$ | $2^1$ | $2^2$ | $2^3$ | $2^4$ | $2^5$ | $2^6$ | ... |

We can write down the sequence:

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ \ldots$$

The symbol "..." is called an ***ellipsis*** and is shorthand for ***"and so forth"*** (showing that the sequence continues in a predictable way).

For a general generation $g$ back, the number of ancestors in that generation is

$$\mathbf{a_g = 2^g}$$



## Sequences: Terminology

We write down a **sequence** (a set of elements written in a row)

$$a_1,\ a_2,\ \ldots,\ a_k,\ a_{k+1}, \ldots$$

the individual elements in the sequence are called **terms**, and the element $a_k$ is read "$a$-sub-$k$". The $k$ is called a **subscript** or **index**.

The term $a_k$ with the lowest subscript is called the **initial term**. If the sequence is **finite** then the term $a_k$ with the highest subscript is called the **final term**.

For the sequence above $a_1$ is the initial term, and there is no final terms, since the ellipsis indicates an **infinite** sequence.

An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of $a_k$ depend on $k$. (This is not always available.)



## Examples: Finding Terms Given by Explicit Formulas

Let the sequences $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, and $\underline{\mathbf{c}}$ be defined by

$$a_k = 2^k,\ k \geq 1, \quad b_k = \frac{k}{k+1},\ k \geq 1, \quad c_k = \frac{k-1}{k},\ k \geq 2,$$

then we have the following

| $\mathbf{k}$ | $\mathbf{a_k}$ | $\mathbf{b_k}$ | $\mathbf{c_k}$ |
|---|---|---|---|
| 1 | 2 | 1/2 | — |
| 2 | 4 | 2/3 | 1/2 |
| 3 | 8 | 3/4 | 2/3 |
| 4 | 16 | 4/5 | 3/4 |
| 5 | 32 | 5/6 | 4/5 |
| 6 | 64 | 6/7 | 5/6 |
| ⋮ | ⋮ | ⋮ | ⋮ |



## Examples: Alternating Sequence

Let

$$c_k = (-1)^k,\ k \geq 0$$

Then,

$$c_0 = 1,\ c_1 = -1,\ c_2 = 1,\ c_3 = -1,\ c_4 = 1,\ c_5 = -1,\ \ldots$$

Even though ***the sequence is infinite***, but it only takes a ***finite number of values: {−1,+1}***.

## Given a Sequence, Can We Find an Explicit Formula?

Ponder the sequence

$$1, \quad -\frac{1}{4}, \quad \frac{1}{9}, \quad -\frac{1}{16}, \quad \frac{1}{25}, \quad -\frac{1}{36}, \quad \ldots$$

Rewriting it a bit helps...

$$\frac{1}{1^2}, \quad \frac{-1}{2^2}, \quad \frac{1}{3^2}, \quad \frac{-1}{4^2}, \quad \frac{1}{5^2}, \quad \frac{-1}{6^2}, \quad \ldots$$

We can now identify

$$a_k = \frac{(-1)^{k+1}}{k^2}, \; k \geq 1$$

and we can answer the Sunday-Newspaper-Puzzle-Question "what comes next?" — The answer is $\mathbf{\frac{1}{49}}$.



## What it the Next Term?

Ponder this sequence

$$4, \quad 14, \quad 23, \quad 34, \quad 42, \quad \ldots$$

What is the next term?

Any New Yorkers in the audience?

These are NYC subway stops (weekdays only) on the F-line — the next stop is on 47th Street.

**Moral:** Not every sequence (even if it makes sense) can be described with an explicit formula!



## Diverging Sequences

Two sequences can start out the same, but **diverge** (have different values) later...

Consider the sequence of odd numbers greater than 1

$$3, \; 5, \; 7, \; 9, \; 11, \; \ldots$$

and the sequence of primes greater than 2

$$3, \; 5, \; 7, \; 11, \; 13, \; \ldots$$

The first three terms are the same, but then they differ...



## Sums of Sequences

One thing you're frequently asked to do is to compute the sum of the terms in a sequence...

$$s = a_1 + a_2 + a_3 + \ldots + a_n, \quad \text{(finite sum)}$$

$$t = b_1 + b_2 + b_3 + \ldots, \quad \text{(infinite sum)}$$

We use the following short-hand notation for the sums above:

$$s = \sum_{k=1}^{n} a_k, \qquad t = \sum_{j=1}^{\infty} b_j.$$

We call the sum from the lowest subscript (lower limit) to the highest subscript (upper limit) [possibly $\infty$].

**History:** According to Epp, the use of the Greek letter sigma ($\Sigma$) to denote summation was introduced by Joseph Louis Lagrange in 1772. However, according to O'Connor and Robertson (see *"Mathematics Personae"* on the class web-page) it was introduced in 1755 by Leonhard Euler.

## Know What the Short-Hand Means! **HW/Test Warning!**

**Compute the sum** $\sum_{k=1}^{4} k^3$ :

$$\sum_{\mathbf{k=1}}^{\mathbf{4}} \mathbf{k^3} = \mathbf{1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100}.$$

**Write the following sum in compact form, using the summation notation**

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

$$\sum_{\mathbf{k=0}}^{\mathbf{n}} \frac{\mathbf{k+1}}{\mathbf{k+n}}$$



## "Telescoping Sums"

Sometimes the sum of the terms simplify greatly since terms, or part of terms may cancel each other. — The sum "telescopes" (compresses) down to only a few remaining terms... For example

$$\sum_{k=1}^{n} \left[ \frac{k}{k+1} - \frac{k+1}{k+2} \right] = \left[ \frac{1}{2} - \frac{\mathbf{2}}{\mathbf{3}} \right] + \left[ \frac{\mathbf{2}}{\mathbf{3}} - \frac{\mathbf{3}}{\mathbf{4}} \right] + \left[ \frac{\mathbf{3}}{\mathbf{4}} - \frac{4}{5} \right] + \cdots$$

we notice that the second part of the $a_k$ term gets canceled out by the first part of the $a_{k+1}$ term (its successor). The whole sum telescopes down to

$$\sum_{k=1}^{n} \left[ \frac{k}{k+1} - \frac{k+1}{k+2} \right] = \frac{1}{2} - \frac{n+1}{n+2}$$

where the only non-canceled parts are the first part of the initial term term, and the second part of the final term. (This is sometimes quite useful in proving complicated theorems... *e.g.* in Math 693b.)



## "Dummy Variables" **(Place-holder)**

Consider the two sums

$$\sum_{k=1}^{3} \sqrt{k} = \sqrt{1} + \sqrt{2} + \sqrt{3}$$

and

$$\sum_{j=1}^{3} \sqrt{j} = \sqrt{1} + \sqrt{2} + \sqrt{3}$$

Clearly,

$$\sum_{k=1}^{3} \sqrt{k} = \sum_{j=1}^{3} \sqrt{j}$$

**The symbol used to represent the index of a summation can be replaced by any other symbol, *as long as* the replacement is made in each location where the symbol occurs.** The index-symbol is frequently referred to as a **dummy variable.**



## Changes of Variable

If you have a summation

$$\sum_{k=1}^{n} \frac{\sqrt{k}}{(k+1)(k+2)} = \frac{1}{2 \cdot 3} + \cdots + \frac{\sqrt{n}}{(n+1)(n+2)}$$

you can introduce the **change of variables** $j = k+1$, $(k = j-1)$ and get the (equivalent) summation

$$\sum_{j=2}^{n+1} \frac{\sqrt{j-1}}{j(j+1)} = \frac{1}{2 \cdot 3} + \cdots + \frac{\sqrt{n}}{(n+1)(n+2)}$$

This may seem like a silly exercise, but sometimes it is really helpful to transform the summation — if you can show that your summation is equivalent to a summation for which you know the value, you're done!

## Notation: Products of Sequences

We use the Greek (capital) letter $\Pi$ to denote products, *e.g.*

$$\prod_{k=1}^{4} k^2 = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 = 1 \cdot 4 \cdot 9 \cdot 16 = 576$$

More generally

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot \ldots \cdot a_{n-1} \cdot a_n$$



## Properties of Summations and Products

The following properties hold for summations and products:

**Theorem**: If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and $c \in \mathbb{R}$, then the following equations hold for any integer $n \geq m$:

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k).$
2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k.$
3. $\displaystyle\left[\prod_{k=m}^{n} a_k\right] \cdot \left[\prod_{k=m}^{n} b_k\right] = \prod_{k=m}^{n} [a_k \cdot b_k].$

We will prove these results later, when we talk about ***recursion***; for now, they enable us to manipulate sums and products.



## Definition of $n$ factorial ($n!$)

The product of all consecutive integers up to a given integer occurs in many mathematical formulas — therefore it has been designated its own notation — ***factorial notation***

**Definition**: For all positive integers $n$, the quantity ***n factorial***, denoted **n!**, is defined to be the product of all integers from 1 to $n$:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Further ***zero factorial*** is **defined** to be 1:

$$0! = 1$$

**Note:** The definition $0! = 1$ is for convenience.



## Playing With Factorials

| n | n! | Value | n | n! | Value |
|---|---|---|---|---|---|
| 1 | $1$ | 1 | 6 | $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$ | 720 |
| 2 | $1 \cdot 2$ | 2 | 7 | $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$ | 5,040 |
| 3 | $1 \cdot 2 \cdot 3$ | 6 | 8 | $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$ | 40,320 |
| 4 | $1 \cdot 2 \cdot 3 \cdot 4$ | 24 | 9 | $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$ | 362,880 |
| 5 | $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ | 120 | 10 | $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$ | 3,628,800 |

Note that

$$5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{10!}{4!} = 151,200.$$

and for integers $n \geq 0$:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 0 \end{cases}$$

We covered a lot of definitions and terms today — the sooner you make them your "friends," the better!

***Epp, 3rd Edition:***
***4.1.3**, **4.1.7**, **4.1.27**, **4.1.36**, **4.1.52**, **4.1.60***

***Epp, 2nd Edition:***
***4.1.3**, **4.1.7**, **4.1.24**, —, **4.1.39**, **4.1.46***

**Next Time:** Mathematical Induction — *"Proof by sequence."*

***If you do not have the 3rd edition, it is your responsibility to seek out the "missing" questions.***
***— Phone-a-Friend, or come to office hours!***