

Sequences and Mathematical Induction

Sequences

Lecture Notes #7

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Introduction

So far we have talked about the *fundamentals of logic*; we have looked at compound and quantified statements (Chapters 1–2).

Then we explored some introductory number theory, and tried our hands at a couple of different methods of proof: *direct proofs*, *proofs by contradiction*, and *proofs by contraposition*; as well as “anti-proofs”, i.e. *counterexamples* (Chapter 3).

Now we’ll switch gears a little — we’ll look at sequences (the computer scientists among us should think of *for*- and *while*-loops). We will look at sequences of numbers — looking for patterns, etc. Also, we will prove things sequentially (using *mathematical induction*).

Sequences

A powerful tool in mathematics (and other sciences, and life itself) is to discover and make use of *patterns* (c.f. Math 596 “Pattern Formation.”)



Figure: To the left we see a picture of a chemical reaction (Belousov-Zhabotinsky) in progress, and to the right a mathematical model of the reaction mimicking the complex time evolution of the pattern.

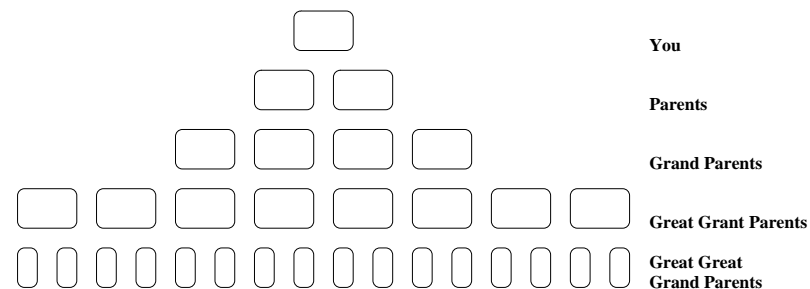
We will study slightly less complicated patterns, starting with *sequences of numbers*; — and we’ll verify conjectures about patterns.

Sequences: Counting your ancestors...

Imagine you want to trace down your family tree, and write down all your ancestors.

First, there is you — the center of the universe as we know it.

You have two parents, 4 grand parents, 8 great-grand-parents, etc...



## Sequences: Counting your ancestors...

Generation ( $g$ )	1	2	3	4	5	6	...
Number of Ancestors	2	4	8	16	32	64	...
$a_g = 2^g$	$2^1$	$2^2$	$2^3$	$2^4$	$2^5$	$2^6$	...

We can write down the sequence:

$$2, 4, 8, 16, 32, 64, 128, \dots$$

The symbol “...” is called an *ellipsis* and is shorthand for “*and so forth*” (showing that the sequence continues in a predictable way).

For a general generation  $g$  back, the number of ancestors in that generation is

$$a_g = 2^g$$

## Sequences: Terminology

We write down a **sequence** (a set of elements written in a row)

$$a_1, a_2, \dots, a_k, a_{k+1}, \dots$$

the individual elements in the sequence are called **terms**, and the element  $a_k$  is read “ $a$ -sub- $k$ ”. The  $k$  is called a **subscript** or **index**.

The term  $a_k$  with the lowest subscript is called the **initial term**. If the sequence is **finite** then the term  $a_k$  with the highest subscript is called the **final term**.

For the sequence above  $a_1$  is the initial term, and there is no final terms, since the ellipsis indicates an **infinite** sequence.

An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of  $a_k$  depend on  $k$ . (This is not always available.)

## Examples: Finding Terms Given by Explicit Formulas

Let the sequences **a**, **b**, and **c** be defined by

$$a_k = 2^k, k \geq 1, \quad b_k = \frac{k}{k+1}, k \geq 1, \quad c_k = \frac{k-1}{k}, k \geq 2,$$

then we have the following

<b>k</b>	<b>a<sub>k</sub></b>	<b>b<sub>k</sub></b>	<b>c<sub>k</sub></b>
1	2	1/2	—
2	4	2/3	1/2
3	8	3/4	2/3
4	16	4/5	3/4
5	32	5/6	4/5
6	64	6/7	5/6
⋮	⋮	⋮	⋮

## Examples: Alternating Sequence

Let

$$c_k = (-1)^k, k \geq 0$$

Then,

$$c_0 = 1, c_1 = -1, c_2 = 1, c_3 = -1, c_4 = 1, c_5 = -1, \dots$$

Even though *the sequence is infinite*, but it only takes a **finite number of values:  $\{-1, +1\}$** .

## Given a Sequence, Can We Find an Explicit Formula?

Ponder the sequence

$$1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, -\frac{1}{36}, \dots$$

Rewriting it a bit helps...

$$\frac{1}{1^2}, \frac{-1}{2^2}, \frac{1}{3^2}, \frac{-1}{4^2}, \frac{1}{5^2}, \frac{-1}{6^2}, \dots$$

We can now identify

$$a_k = \frac{(-1)^{k+1}}{k^2}, \quad k \geq 1$$

and we can answer the Sunday-Newspaper-Puzzle-Question “what comes next?” — The answer is  $\frac{1}{49}$ .

## What is the Next Term?

Ponder this sequence

$$4, 14, 23, 34, 42, \dots$$

What is the next term?

Any New Yorkers in the audience?

These are NYC subway stops (weekdays only) on the F-line — the next stop is on 47th Street.

**Moral:** Not every sequence (even if it makes sense) can be described with an explicit formula!

## Diverging Sequences

Two sequences can start out the same, but **diverge** (have different values) later...

Consider the sequence of odd numbers greater than 1

$$3, 5, 7, 9, 11, \dots$$

and the sequence of primes greater than 2

$$3, 5, 7, 11, 13, \dots$$

The first three terms are the same, but then they differ...

## Sums of Sequences

One thing you're frequently asked to do is to compute the sum of the terms in a sequence...

$$s = a_1 + a_2 + a_3 + \dots + a_n, \quad (\text{finite sum})$$

$$t = b_1 + b_2 + b_3 + \dots, \quad (\text{infinite sum})$$

We use the following short-hand notation for the sums above:

$$s = \sum_{k=1}^n a_k, \quad t = \sum_{j=1}^{\infty} b_j.$$

We call the sum from the lowest subscript (lower limit) to the highest subscript (upper limit) [possibly  $\infty$ ].

**History:** According to Epp, the use of the Greek letter sigma ( $\Sigma$ ) to denote summation was introduced by Joseph Louis Lagrange in 1772. However, according to O'Connor and Robertson (see “*Mathematics Personae*” on the class web-page) it was introduced in 1755 by Leonhard Euler.

## Know What the Short-Hand Means! HW/Test Warning!

Compute the sum  $\sum_{k=1}^4 k^3$ :

$$\sum_{k=1}^4 k^3 = 1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100.$$

Write the following sum in compact form, using the summation notation

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$
$$\sum_{k=0}^n \frac{k+1}{k+n}$$

## “Telescoping Sums”

Sometimes the sum of the terms simplify greatly since terms, or part of terms may cancel each other. — The sum “telescopes” (compresses) down to only a few remaining terms... For example

$$\sum_{k=1}^n \left[ \frac{k}{k+1} - \frac{k+1}{k+2} \right] = \left[ \frac{1}{2} - \frac{2}{3} \right] + \left[ \frac{2}{3} - \frac{3}{4} \right] + \left[ \frac{3}{4} - \frac{4}{5} \right] + \cdots$$

we notice that the second part of the  $a_k$  term gets canceled out by the first part of the  $a_{k+1}$  term (its successor). The whole sum telescopes down to

$$\sum_{k=1}^n \left[ \frac{k}{k+1} - \frac{k+1}{k+2} \right] = \frac{1}{2} - \frac{n+1}{n+2}$$

where the only non-canceled parts are the first part of the initial term term, and the second part of the final term. (This is sometimes quite useful in proving complicated theorems... e.g. in Math 693b.)

## “Dummy Variables”

(Place-holder)

Consider the two sums

$$\sum_{k=1}^3 \sqrt{k} = \sqrt{1} + \sqrt{2} + \sqrt{3}$$

and

$$\sum_{j=1}^3 \sqrt{j} = \sqrt{1} + \sqrt{2} + \sqrt{3}$$

Clearly,

$$\sum_{k=1}^3 \sqrt{k} = \sum_{j=1}^3 \sqrt{j}$$

The symbol used to represent the index of a summation can be replaced by any other symbol, as long as the replacement is made in each location where the symbol occurs. The index-symbol is frequently referred to as a **dummy variable**.

## Changes of Variable

If you have a summation

$$\sum_{k=1}^n \frac{\sqrt{k}}{(k+1)(k+2)} = \frac{1}{2 \cdot 3} + \cdots + \frac{\sqrt{n}}{(n+1)(n+2)}$$

you can introduce the **change of variables**  $j = k + 1$ , ( $k = j - 1$ ) and get the (equivalent) summation

$$\sum_{j=2}^{n+1} \frac{\sqrt{j-1}}{j(j+1)} = \frac{1}{2 \cdot 3} + \cdots + \frac{\sqrt{n}}{(n+1)(n+2)}$$

This may seem like a silly exercise, but sometimes it is really helpful to transform the summation — if you can show that your summation is equivalent to a summation for which you know the value, you're done!

## Notation: Products of Sequences

We use the Greek (capital) letter  $\Pi$  to denote products, e.g.

$$\prod_{k=1}^4 k^2 = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 = 1 \cdot 4 \cdot 9 \cdot 16 = 576$$

More generally

$$\prod_{k=m}^n a_k = a_m \cdot a_{m+1} \cdot \dots \cdot a_{n-1} \cdot a_n$$

## Properties of Summations and Products

The following properties hold for summations and products:

**Theorem:** If  $a_m, a_{m+1}, a_{m+2}, \dots$  and  $b_m, b_{m+1}, b_{m+2}, \dots$  are sequences of real numbers and  $c \in \mathbb{R}$ , then the following equations hold for any integer  $n \geq m$ :

- $$\sum_{k=m}^n a_k + \sum_{k=m}^n b_k = \sum_{k=m}^n (a_k + b_k).$$
- $$c \cdot \sum_{k=m}^n a_k = \sum_{k=m}^n c \cdot a_k.$$
- $$\left[ \prod_{k=m}^n a_k \right] \cdot \left[ \prod_{k=m}^n b_k \right] = \prod_{k=m}^n [a_k \cdot b_k].$$

We will prove these results later, when we talk about *recursion*; for now, they enable us to manipulate sums and products.

## Definition of $n$ factorial ( $n!$ )

The product of all consecutive integers up to a given integer occurs in many mathematical formulas — therefore it has been designated its own notation — *factorial notation*

**Definition:** For all positive integers  $n$ , the quantity  $n$  factorial, denoted  $n!$ , is defined to be the product of all integers from 1 to  $n$ :

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

Further *zero factorial* is **defined** to be 1:

$$0! = 1$$

**Note:** The definition  $0! = 1$  is for convenience.

## Playing With Factorials

n	n!	Value	n	n!	Value
1	1	1	6	1 · 2 · 3 · 4 · 5 · 6	720
2	1 · 2	2	7	1 · 2 · 3 · 4 · 5 · 6 · 7	5,040
3	1 · 2 · 3	6	8	1 · 2 · 3 · 4 · 5 · 6 · 7 · 8	40,320
4	1 · 2 · 3 · 4	24	9	1 · 2 · 3 · 4 · 5 · 6 · 7 · 8 · 9	362,880
5	1 · 2 · 3 · 4 · 5	120	10	1 · 2 · 3 · 4 · 5 · 6 · 7 · 8 · 9 · 10	3,628,800

Note that

$$5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{10!}{4!} = 151,200.$$

and for integers  $n \geq 0$ :

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1 \end{cases}$$

We covered a lot of definitions and terms today — the sooner you make them your “friends,” the better!

*Epp, 3rd Edition:*

4.1.3, 4.1.7, 4.1.27, 4.1.36, 4.1.52, 4.1.60

*Epp, 2nd Edition:*

4.1.3, 4.1.7, 4.1.24, —, 4.1.39, 4.1.46

**Next Time:** Mathematical Induction — “Proof by sequence.”

*If you do not have the 3rd edition, it is your responsibility to seek out the “missing” questions.  
— Phone-a-Friend, or come to office hours!*