Math 245: Discrete Mathematics

Sequences and Mathematical Induction

Sequences

Lecture Notes #7

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Introduction

So far we have talked about the *fundamentals of logic*; we have looked at compound and quantified statements (Chapters 1–2).

Then we explored some introductory number theory, and tried our hands at a couple of different methods of proof: *direct proofs, proofs by contradiction,* and *proofs by contraposition*; as well as "anti-proofs", *i.e. counterexamples* (Chapter 3).

Now we'll switch gears a little — we'll look at sequences (the computer scientists among us should think of *for*- and *while*-loops). We will look at sequences of numbers — looking for patterns, etc. Also, we will prove things sequentially (using *mathematical induction*).

Sequences

A powerful tool in mathematics (and other sciences, and life itself) is to discover and make use of **patterns** (*c.f.* Math 596 "Pattern Formation.")

Figure: To the left we see a picture of a chemical reaction (Belouzov-Zhabotinsky) in progress, and to the right a mathematical model of the reaction mimicking the complex time evolution of the pattern.

We will study slightly less complicated patterns, starting with *sequences of numbers*; — and we'll verify conjectures about patterns.
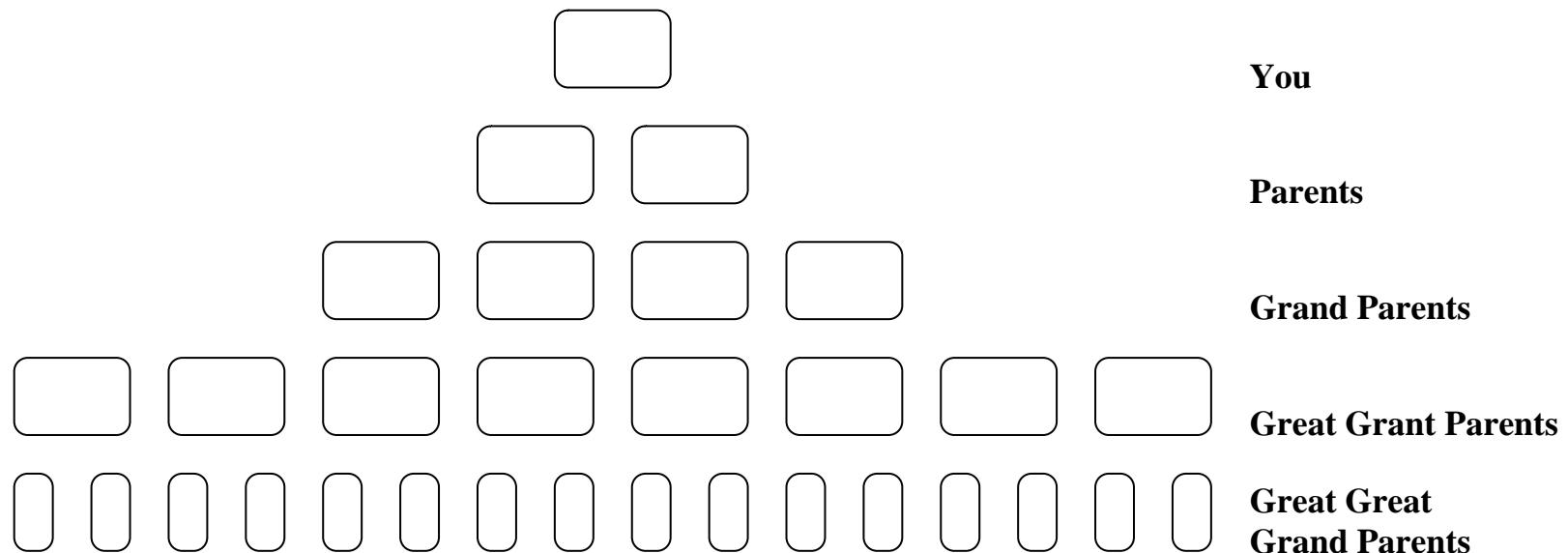
Sequences: Counting your ancestors...

Imagine you want to trace down your family tree, and write down all your ancestors.

First, there is you — the center of the universe as we know it.

You have two parents, 4 grand parents, 8 great-grand-parents, etc...

You

Parents

Grand Parents

Great Grant Parents

Great Great Grand Parents

Sequences: Counting your ancestors...

Generation (g)	1	2	3	4	5	6	...
Number of Ancestors	2	4	8	16	32	64	...
$a_g = 2^g$	2^1	2^2	2^3	2^4	2^5	2^6	...

We can write down the sequence:

$$2, \ 4, \ 8, \ 16, \ 32, \ 64, \ 128, \ \ldots$$

The symbol "..." is called an **ellipsis** and is shorthand for **"and so forth"** (showing that the sequence continues in a predictable way).

For a general generation g back, the number of ancestors in that generation is

$$\mathbf{a_g = 2^g}$$

Sequences: Terminology

We write down a **sequence** (a set of elements written in a row)

$$a_1, \ a_2, \ \ldots, \ a_k, \ a_{k+1}, \ldots$$

the individual elements in the sequence are called **terms**, and the element a_k is read "a-sub-k". The k is called a **subscript** or **index**.

The term a_k with the lowest subscript is called the **initial term**. If the sequence is **finite** then the term a_k with the highest subscript is called the **final term**.

For the sequence above a_1 is the initial term, and there is no final terms, since the ellipsis indicates an **infinite** sequence.

An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k. (This is not always available.)

Examples: Finding Terms Given by Explicit Formulas

Let the sequences $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, and $\underline{\mathbf{c}}$ be defined by

$$a_k = 2^k, \ k \geq 1, \quad b_k = \frac{k}{k+1}, \ k \geq 1, \quad c_k = \frac{k-1}{k}, \ k \geq 2,$$

then we have the following

$\mathbf{k}$	$\mathbf{a_k}$	$\mathbf{b_k}$	$\mathbf{c_k}$
1	2	1/2	—
2	4	2/3	1/2
3	8	3/4	2/3
4	16	4/5	3/4
5	32	5/6	4/5
6	64	6/7	5/6
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Examples: Alternating Sequence

Let

$$c_k = (-1)^k, \ k \geq 0$$

Then,

$$c_0 = 1, \ c_1 = -1, \ c_2 = 1, \ c_3 = -1, \ c_4 = 1, \ c_5 = -1, \ \ldots$$

Even though *the sequence is infinite*, but it only takes a *finite number of values: {−1,+1}*.

Given a Sequence, Can We Find an Explicit Formula?

Ponder the sequence

$$1, \quad -\frac{1}{4}, \quad \frac{1}{9}, \quad -\frac{1}{16}, \quad \frac{1}{25}, \quad -\frac{1}{36}, \quad \ldots$$

Rewriting it a bit helps...

$$\frac{1}{1^2}, \quad \frac{-1}{2^2}, \quad \frac{1}{3^2}, \quad \frac{-1}{4^2}, \quad \frac{1}{5^2}, \quad \frac{-1}{6^2}, \quad \ldots$$

We can now identify

$$a_k = \frac{(-1)^{k+1}}{k^2}, \ k \geq 1$$

and we can answer the Sunday-Newspaper-Puzzle-Question "what comes next?" — The answer is $\dfrac{1}{49}$.

What it the Next Term?

Ponder this sequence

$$4, \quad 14, \quad 23, \quad 34, \quad 42, \quad \ldots$$

What is the next term?

Any New Yorkers in the audience?

These are NYC subway stops (weekdays only) on the F-line — the next stop is on 47th Street.

Moral: Not every sequence (even if it makes sense) can be described with an explicit formula!

Diverging Sequences

Two sequences can start out the same, but **diverge** (have different values) later...

Consider the sequence of odd numbers greater than 1

$$3, \ 5, \ 7, \ 9, \ 11, \ \ldots$$

and the sequence of primes greater than 2

$$3, \ 5, \ 7, \ 11, \ 13, \ \ldots$$

The first three terms are the same, but then they differ...

Sums of Sequences

One thing you're frequently asked to do is to compute the sum of the terms in a sequence...

$$s = a_1 + a_2 + a_3 + \ldots + a_n, \quad \text{(finite sum)}$$

$$t = b_1 + b_2 + b_3 + \ldots, \quad \text{(infinite sum)}$$

We use the following short-hand notation for the sums above:

$$s = \sum_{k=1}^{n} a_k, \qquad t = \sum_{j=1}^{\infty} b_j.$$

We call the sum from the lowest subscript (lower limit) to the highest subscript (upper limit) [possibly ∞].

History: According to Epp, the use of the Greek letter sigma (Σ) to denote summation was introduced by Joseph Louis Lagrange in 1772. However, according to O'Connor and Robertson (see *"Mathematics Personae"* on the class web-page) it was introduced in 1755 by Leonhard Euler.

Know What the Short-Hand Means!   HW/Test Warning!

Compute the sum $\displaystyle\sum_{k=1}^{4} k^3$:

$$\sum_{k=1}^{4} k^3 = 1^3 + 2^3 + 3^3 + 4^3 = 1 + 8 + 27 + 64 = 100.$$

Write the following sum in compact form, using the summation notation

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

$$\sum_{k=0}^{n} \frac{k+1}{k+n}$$

"Telescoping Sums"

Sometimes the sum of the terms simplify greatly since terms, or part of terms may cancel each other. — The sum "telescopes" (compresses) down to only a few remaining terms... For example

$$\sum_{k=1}^{n} \left[\frac{k}{k+1} - \frac{k+1}{k+2} \right] = \left[\frac{1}{2} - \frac{2}{3} \right] + \left[\frac{2}{3} - \frac{3}{4} \right] + \left[\frac{3}{4} - \frac{4}{5} \right] + \cdots$$

we notice that the second part of the a_k term gets canceled out by the first part of the a_{k+1} term (its successor). The whole sum telescopes down to

$$\sum_{k=1}^{n} \left[\frac{k}{k+1} - \frac{k+1}{k+2} \right] = \frac{1}{2} - \frac{n+1}{n+2}$$

where the only non-canceled parts are the first part of the initial term term, and the second part of the final term. (This is sometimes quite useful in proving complicated theorems... *e.g.* in Math 693b.)

Consider the two sums

$$\sum_{k=1}^{3} \sqrt{k} = \sqrt{1} + \sqrt{2} + \sqrt{3}$$

and

$$\sum_{j=1}^{3} \sqrt{j} = \sqrt{1} + \sqrt{2} + \sqrt{3}$$

Clearly,

$$\sum_{k=1}^{3} \sqrt{k} = \sum_{j=1}^{3} \sqrt{j}$$

The symbol used to represent the index of a summation can be replaced by any other symbol, *as long as* the replacement is made in each location where the symbol occurs. The index-symbol is frequently referred to as a **dummy variable**.

Changes of Variable

If you have a summation

$$\sum_{k=1}^{n} \frac{\sqrt{k}}{(k+1)(k+2)} = \frac{1}{2 \cdot 3} + \cdots + \frac{\sqrt{n}}{(n+1)(n+2)}$$

you can introduce the **change of variables** $j = k + 1$, $(k = j - 1)$ and get the (equivalent) summation

$$\sum_{j=2}^{n+1} \frac{\sqrt{j-1}}{j(j+1)} = \frac{1}{2 \cdot 3} + \cdots + \frac{\sqrt{n}}{(n+1)(n+2)}$$

This may seem like a silly exercise, but sometimes it is really helpful to transform the summation — if you can show that your summation is equivalent to a summation for which you know the value, you're done!

Notation: Products of Sequences

We use the Greek (capital) letter Π to denote products, *e.g.*

$$\prod_{k=1}^{4} k^2 = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 = 1 \cdot 4 \cdot 9 \cdot 16 = 576$$

More generally

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot \ldots \cdot a_{n-1} \cdot a_n$$

Properties of Summations and Products

The following properties hold for summations and products:

> **Theorem:** If a_m, a_{m+1}, a_{m+2}, $\ldots$ and b_m, b_{m+1}, b_{m+2}, $\ldots$ are sequences of real numbers and $c \in \mathbb{R}$, then the following equations hold for any integer $n \geq m$:
>
> **1.** $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k).$
>
> **2.** $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k.$
>
> **3.** $\displaystyle\left[\prod_{k=m}^{n} a_k\right] \cdot \left[\prod_{k=m}^{n} b_k\right] = \prod_{k=m}^{n} \left[a_k \cdot b_k\right].$

We will prove these results later, when we talk about **recursion**; for now, they enable us to manipulate sums and products.

Definition of n factorial ($n!$)

The product of all consecutive integers up to a given integer occurs in many mathematical formulas — therefore it has been designated its own notation — *factorial notation*

> **Definition**: For all positive integers n, the quantity **n factorial**, denoted **n!**, is defined to be the product of all integers from 1 to n:
>
> $$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$
>
> Further *zero factorial* is **defined** to be 1:
>
> $$0! = 1$$

Note: The definition $0! = 1$ is for convenience.

Playing With Factorials

n	n!	Value	n	n!	Value
1	1	1	6	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$	720
2	$1 \cdot 2$	2	7	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$	5,040
3	$1 \cdot 2 \cdot 3$	6	8	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$	40,320
4	$1 \cdot 2 \cdot 3 \cdot 4$	24	9	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9$	362,880
5	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$	120	10	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$	3,628,800

Note that

$$5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{10!}{4!} = 151,200.$$

and for integers $n \geq 0$:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 0 \end{cases}$$

We covered a lot of definitions and terms today — the sooner you make them your "friends," the better!

Epp, 3rd Edition:

4.1.3, *4.1.7*, *4.1.27*, *4.1.36*, *4.1.52*, *4.1.60*

Epp, 2nd Edition:

4.1.3, *4.1.7*, *4.1.24*, *—*, *4.1.39*, *4.1.46*

Next Time: Mathematical Induction — *"Proof by sequence."*

If you do not have the 3rd edition, it is your responsibility to seek out the "missing" questions.
— Phone-a-Friend, or come to office hours!