Sequences and Mathematical Induction

Sequences

Lecture Notes #7

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Introduction

So far we have talked about the ***fundamentals of logic***; we have looked at compound and quantified statements (Chapters 1–2).

Then we explored some introductory number theory, and tried our hands at a couple of different methods of proof: ***direct proofs, proofs by contradiction,*** and ***proofs by contraposition***; as well as "anti-proofs", *i.e.* ***counterexamples*** (Chapter 3).

Now we'll switch gears a little — we'll look at sequences (the computer scientists among us should think of ***for***- and ***while***-loops). We will look at sequences of numbers — looking for patterns, etc. Also, we will prove things sequentially (using ***mathematical induction***).

Sequences

A powerful tool in mathematics (and other sciences, and life itself) is to discover and make use of **patterns** (*c.f.* Math 596 "Pattern Formation.")

Figure: To the left we see a picture of a chemical reaction (Belouzov-Zhabotinsky) in progress, and to the right a mathematical model of the reaction mimicking the complex time evolution of the pattern.

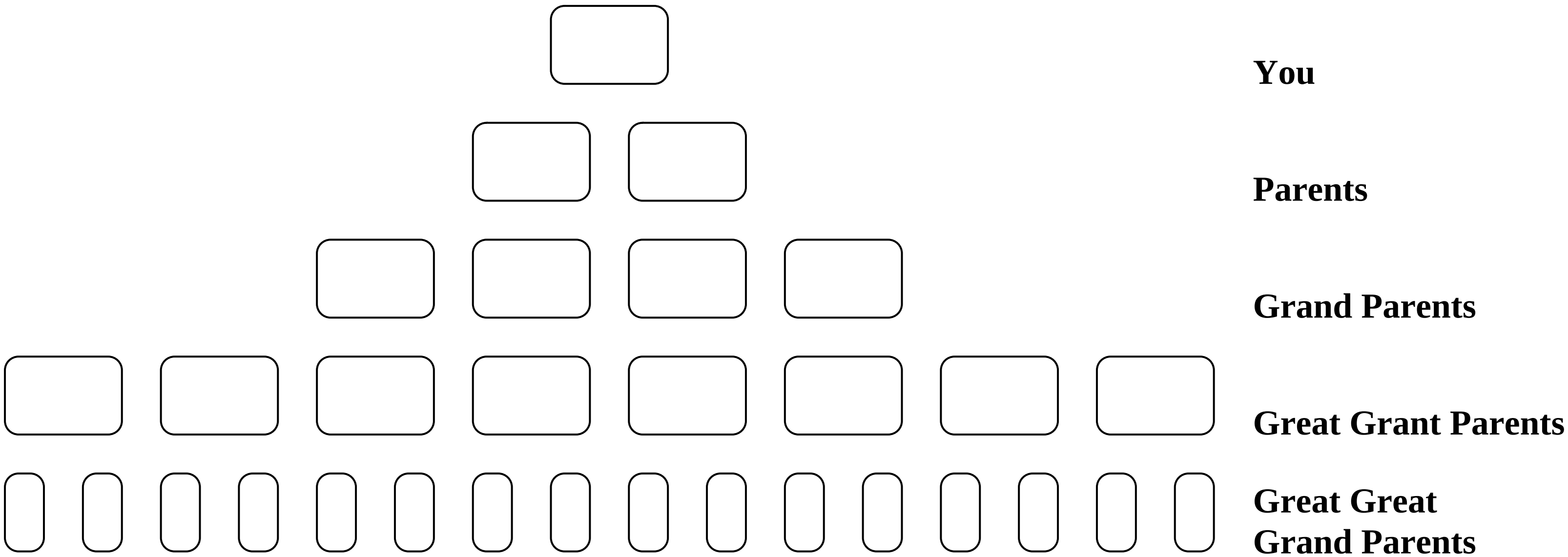
We will study slightly less complicated patterns, starting with ***sequences of numbers***; — and we'll verify conjectures about patterns.

Sequences: Counting your ancestors...

Imagine you want to trace down your family tree, and write down all your ancestors.

First, there is you — the center of the universe as we know it.

You have two parents, 4 grand parents, 8 great-grand-parents, etc...



Sequences: Counting your ancestors...

Generation (g)	1	2	3	4	5	6	...
Number of Ancestors	2	4	8	16	32	64	...
$a_g = 2^g$	2^1	2^2	2^3	2^4	2^5	2^6	...

We can write down the sequence:

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ \ldots$$

The symbol "$\ldots$" is called an ***ellipsis*** and is shorthand for ***"and so forth"*** (showing that the sequence continues in a predictable way).

For a general generation g back, the number of ancestors in that generation is

$$\mathbf{a_g = 2^g}$$

Sequences: Terminology

We write down a **sequence** (a set of elements written in a row)

$$a_1,\ a_2,\ \ldots,\ a_k,\ a_{k+1}, \ldots$$

the individual elements in the sequence are called **terms**, and the element a_k is read "a-sub-k". The k is called a **subscript** or **index**.

The term a_k with the lowest subscript is called the **initial term**. If the sequence is **finite** then the term a_k with the highest subscript is called the **final term**.

For the sequence above a_1 is the initial term, and there is no final terms, since the ellipsis indicates an **infinite** sequence.

An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k. (This is not always available.)

Examples: Finding Terms Given by Explicit Formulas

Let the sequences $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, and $\underline{\mathbf{c}}$ be defined by

$$a_k = 2^k,\ k \geq 1, \quad b_k = \frac{k}{k+1},\ k \geq 1, \quad c_k = \frac{k-1}{k},\ k \geq 2,$$

then we have the following

$\mathbf{k}$	$\mathbf{a_k}$	$\mathbf{b_k}$	$\mathbf{c_k}$
1	2	1/2	—
2	4	2/3	1/2
3	8	3/4	2/3
4	16	4/5	3/4
5	32	5/6	4/5
6	64	6/7	5/6
⋮	⋮	⋮	⋮

Examples: Alternating Sequence

Let

$$c_k = (-1)^k, \ k \geq 0$$

Then,

$$c_0 = 1, \ c_1 = -1, \ c_2 = 1, \ c_3 = -1, \ c_4 = 1, \ c_5 = -1, \ \ldots$$

Even though ***the sequence is infinite***, but it only takes a ***finite number of values: {−1,+1}***.

Given a Sequence, Can We Find an Explicit Formula?

Ponder the sequence

$$1, \quad -\frac{1}{4}, \quad \frac{1}{9}, \quad -\frac{1}{16}, \quad \frac{1}{25}, \quad -\frac{1}{36}, \quad \ldots$$

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Rewriting it a bit helps...

$$\frac{1}{1^2}, \quad \frac{-1}{2^2}, \quad \frac{1}{3^2}, \quad \frac{-1}{4^2}, \quad \frac{1}{5^2}, \quad \frac{-1}{6^2}, \quad \ldots$$

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We can now identify

$$a_k = \frac{(-1)^{k+1}}{k^2}, \; k \geq 1$$

and we can answer the Sunday-Newspaper-Puzzle-Question "what comes next?" — The answer is $\mathbf{\frac{1}{49}}$.

What it the Next Term?

Ponder this sequence

$$4, \quad 14, \quad 23, \quad 34, \quad 42, \quad \ldots$$

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Any New Yorkers in the audience?

These are NYC subway stops (weekdays only) on the F-line — the next stop is on 47th Street.

Moral: Not every sequence (even if it makes sense) can be described with an explicit formula!

Diverging Sequences

Two sequences can start out the same, but **diverge** (have different values) later...

Consider the sequence of odd numbers greater than 1

$$3,\ 5,\ 7,\ 9,\ 11,\ \ldots$$

and the sequence of primes greater than 2

$$3,\ 5,\ 7,\ 11,\ 13,\ \ldots$$

The first three terms are the same, but then they differ...

Sums of Sequences

One thing you're frequently asked to do is to compute the sum of the terms in a sequence...

$$s = a_1 + a_2 + a_3 + \ldots + a_n, \quad \text{(finite sum)}$$

$$t = b_1 + b_2 + b_3 + \ldots, \quad \text{(infinite sum)}$$

We use the following short-hand notation for the sums above:

$$s = \sum_{k=1}^{n} a_k, \qquad t = \sum_{j=1}^{\infty} b_j.$$

We call the sum from the lowest subscript (lower limit) to the highest subscript (upper limit) [possibly ∞].

History: According to Epp, the use of the Greek letter sigma (Σ) to denote summation was introduced by Joseph Louis Lagrange in 1772. However, according to O'Connor and Robertson (see *"Mathematics Personae"* on the class web-page) it was introduced in 1755 by Leonhard Euler.

Know What the Short-Hand Means! HW/Test Warning!

Compute the sum $\displaystyle\sum_{k=1}^{4} k^3$:

Write the following sum in compact form, using the summation notation

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

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$$\sum_{\mathbf{k=1}}^{\mathbf{4}} \mathbf{k^3} = \mathbf{1^3} + \mathbf{2^3} + \mathbf{3^3} + \mathbf{4^3} = \mathbf{1} + \mathbf{8} + \mathbf{27} + \mathbf{64} = \mathbf{100}.$$

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$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

$$\sum_{\mathbf{k=0}}^{\mathbf{n}} \frac{\mathbf{k+1}}{\mathbf{k+n}}$$

"Telescoping Sums"

Sometimes the sum of the terms simplify greatly since terms, or part of terms may cancel each other. — The sum "telescopes" (compresses) down to only a few remaining terms...

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$$\sum_{k=1}^{n} \left[\frac{k}{k+1} - \frac{k+1}{k+2} \right] = \left[\frac{1}{2} - \frac{\mathbf{2}}{\mathbf{3}} \right] + \left[\frac{\mathbf{2}}{\mathbf{3}} - \frac{\mathbf{3}}{\mathbf{4}} \right] + \left[\frac{\mathbf{3}}{\mathbf{4}} - \frac{4}{5} \right] + \cdots$$

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we notice that the second part of the a_k term gets canceled out by the first part of the a_{k+1} term (its successor). The whole sum telescopes down to

$$\sum_{k=1}^{n}\left[\frac{k}{k+1}-\frac{k+1}{k+2}\right]=\frac{1}{2}-\frac{n+1}{n+2}$$

where the only non-canceled parts are the first part of the initial term term, and the second part of the final term. (This is sometimes quite useful in proving complicated theorems... *e.g.* in Math 693b.)

“Dummy Variables” (Place-holder)

Consider the two sums

$$\sum_{k=1}^{3} \sqrt{k} = \sqrt{1} + \sqrt{2} + \sqrt{3}$$

and

$$\sum_{j=1}^{3} \sqrt{j} = \sqrt{1} + \sqrt{2} + \sqrt{3}$$

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The symbol used to represent the index of a summation can be replaced by any other symbol, *as long as* the replacement is made in each location where the symbol occurs. The index-symbol is frequently referred to as a **dummy variable.**

Changes of Variable

If you have a summation

$$\sum_{k=1}^{n} \frac{\sqrt{k}}{(k+1)(k+2)} = \frac{1}{2 \cdot 3} + \cdots + \frac{\sqrt{n}}{(n+1)(n+2)}$$

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$$\sum_{j=2}^{n+1} \frac{\sqrt{j-1}}{j(j+1)} = \frac{1}{2 \cdot 3} + \cdots + \frac{\sqrt{n}}{(n+1)(n+2)}$$

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This may seem like a silly exercise, but sometimes it is really helpful to transform the summation — if you can show that your summation is equivalent to a summation for which you know the value, you're done!

Notation: Products of Sequences

We use the Greek (capital) letter Π to denote products, *e.g.*

$$\prod_{k=1}^{4} k^2 = 1^2 \cdot 2^2 \cdot 3^2 \cdot 4^2 = 1 \cdot 4 \cdot 9 \cdot 16 = 576$$

More generally

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot \ldots \cdot a_{n-1} \cdot a_n$$

Properties of Summations and Products

The following properties hold for summations and products:

Theorem: If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and $c \in \mathbb{R}$, then the following equations hold for any integer $n \geq m$:

1. $$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k).$$

2. $$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k.$$

3. $$\left[\prod_{k=m}^{n} a_k\right] \cdot \left[\prod_{k=m}^{n} b_k\right] = \prod_{k=m}^{n} \left[a_k \cdot b_k\right].$$

We will prove these results later, when we talk about ***recursion***; for now, they enable us to manipulate sums and products.

Definition of n factorial ($n!$)

The product of all consecutive integers up to a given integer occurs in many mathematical formulas — therefore it has been designated its own notation — ***factorial notation***

Definition: For all positive integers n, the quantity ***n factorial***, denoted **n!**, is defined to be the product of all integers from 1 to n:

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

Further ***zero factorial*** is defined to be 1:

$$0! = 1$$

Note: The definition $0! = 1$ is for convenience.

Playing With Factorials

n	n!	Value	n	n!	Value
1	1	1	6	$1\cdot2\cdot3\cdot4\cdot5\cdot6$	720
2	$1\cdot2$	2	7	$1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7$	5,040
3	$1\cdot2\cdot3$	6	8	$1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8$	40,320
4	$1\cdot2\cdot3\cdot4$	24	9	$1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9$	362,880
5	$1\cdot2\cdot3\cdot4\cdot5$	120	10	$1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10$	3,628,800

Note that

$$5\cdot6\cdot7\cdot8\cdot9\cdot10 = \frac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{1\cdot2\cdot3\cdot4} = \frac{10!}{4!} = 151,200.$$

and for integers $n \geq 0$:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n\cdot(n-1)! & \text{if } n \geq 0 \end{cases}$$

We covered a lot of definitions and terms today — the sooner you make them your "friends," the better!

Epp, 3rd Edition:

4.1.3, ***4.1.7***, ***4.1.27***, ***4.1.36***, ***4.1.52***, ***4.1.60***

Epp, 2nd Edition:

4.1.3, ***4.1.7***, ***4.1.24***, —, ***4.1.39***, ***4.1.46***

Next Time: Mathematical Induction — *"Proof by sequence."*

If you do not have the 3rd edition, it is your responsibility to seek out the "missing" questions.
— Phone-a-Friend, or come to office hours!