# Math 245: Discrete Mathematics

## Set Theory

## Definition and Basics

## Lecture Notes #9

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$Id: lecture.tex,v 1.3 2006/10/26 17:13:07 blomgren Exp $



## Last Time: Strong Mathematical Induction and Well-Ordering

These principles are useful in mathematics (from time to time they will re-emerge in theorems and proofs).

There are also very tangible applications to computer science (see ***Epp-4.5*** for a very brief discussion); the idea of ***proving*** that an algorithm performs what it is supposed to perform is quite useful.

As our world becomes more an more computerized it is very likely that formal testing and validation of software will become increasingly important.

One current project in "the wild" that I am aware of is the formal checker for the Linux kernel developed at Stanford University. — It is a *"slight"* extension of Epp-4.5. ;–}



## Set Theory: Introduction

In short "Set Theory" is the study of collections of objects (here, usually numbers), abstracting one level away from the objects (elements) that are part of the sets.

Set theory is new kid on the block in the world of mathematics; proposed by Georg Cantor (1845–1918) in the late $19^{\text{th}}$ century.

As with most new ideas, acceptance was slow; but in the present set theory is one of the corner stones of mathematics. In a sense it is the "everyday language" of mathematics. Every mathematical object can be defined in terms of sets.

We have already used some of the set theoretical language and concepts (think of *e.g.* the Well-Ordering Principle). Now we will take a closer, more detailed look...



## Sets and Their Elements

A set is a collection of elements.

For instance $\mathbb{Z}$ — the set of integers.

If $S$ is a set, then $a \in S$ means that $a$ is an element of $S$, and $a \notin S$ means that $a$ is not an element in $S$.

> ***Axiom of Extension***:
> *"A set is completely determined by its elements. — The order in which the elements are listed is irrelevant; it is allowable to list an element more than once."*

## The { } Notation for Sets

- $S_1 = \{1, 2, 3\}$ is the set containing three elements — 1, 2, and 3.
- $S_2 = \{1, 2, 3, \ldots\}$ is the set containing all positive integers.

This is three descriptions of the same set (by the axiom of extension):

- $S_1 = \{1, 2, 3\}$.
- $S_1 = \{2, 1, 3\}$.
- $S_1 = \{3, 1, 2, 1, 3, 2, 2, 2, 2\}$.

The set

- $S_3 = \{1\}$ is the set containing one element — 1. Note that 1 (by itself) is the element, and $\{1\}$ is the set containing 1. Hence

$$\mathbf{1 \in \{1\}, \quad 1 \neq \{1\}} \qquad \textbf{Test/HW Warning!!!}$$



## More Set Notation and Examples

A set can be a member of another set, *e.g,*

$$S_4 = \{1, \{1\}\}$$

the set whose elements are 1, and the set $\{1\}$. Hence

$$1 \in S_4, \quad \{1\} \in S_4$$

Recall Truth Sets?

$$P_{\textit{\textbf{true}}} = \{x \in D \mid P(x)\}$$

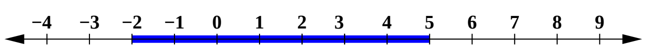
*i.e.* the set containing all the members of the set $D$ which make $P(x)$ true.

This notation is frequently used when specifying sets...

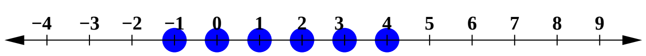


## Sets Given by a Defining Property / Truth Sets

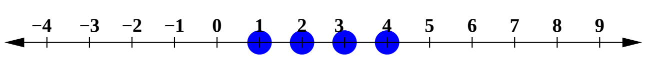
- $S_5 = \{x \in \mathbb{R} \mid -2 < x < 5\}$ — The open interval of real numbers strictly between $-2$ and $5$:




- $S_6 = \{x \in \mathbb{Z} \mid -2 < x < 5\}$ — The set of integers strictly between $-2$ and $5$ — $\{-1, 0, 1, 2, 3, 4\}$:




- $S_7 = \{x \in \mathbb{Z}^+ \mid -2 < x < 5\}$ — The set of positive integers strictly between $-2$ and $5$ — $\{1, 2, 3, 4\}$:






## Subsets

***Definition: Subset —***
If $A$ and $B$ are sets, $A$ is called a **subset** of $B$, written $A \subseteq B$ if and only if, every element of $A$ is also an element in $B$.

Symbolically:

$$A \subseteq B \quad \Leftrightarrow \quad \forall x, \text{ if } x \in A, \text{ then } x \in B.$$

The phrases "$A$ ***is contained in*** $B$," and "$B$ ***contains*** $A$" are alternative ways of saying that $A$ is a subset of $B$.

If $A$ is not a subset of $B$:

$$A \not\subseteq B \quad \Leftrightarrow \quad \exists x, \text{ such that } x \in A, \text{ and } x \notin B.$$

## Proper Subsets

**Definition:** ***Proper Subset*** —
Let $A$ and $B$ be sets. $A$ is a **proper subset** of $B$ if and only if, every element of $A$ is in $B$ but there is at least one element of $B$ that is not in $A$.

Symbolically:

$$A \subset B \quad \Leftrightarrow \quad \forall x, \text{ if } x \in A, \text{ then } x \in B$$
$$\textbf{\textit{and}} \ \exists x \in B \text{ such that } x \notin A.$$

Note that the concept of a proper subset is more restrictive than the concept of a subset. Any proper subset of $B$ is also a subset of $B$ (but the ***converse*** is not true!)



## Venn Diagrams $A \subseteq B$

If we represent the sets $A$ and $B$ as regions in the plane, then we can represent relationships between $A$ and $B$ by pictures (— we did this earlier, recall the homework problem with good food and cafeteria food!)

These pictures are called **Venn Diagrams**, after the British mathematician John Venn (1834–1923),

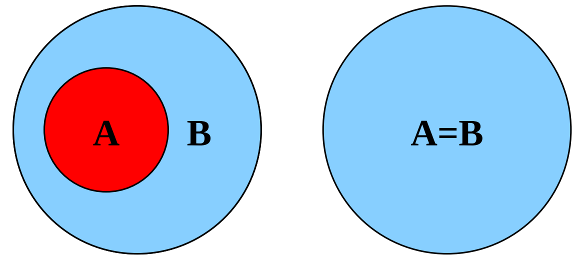



**Figure:** Two cases illustrating $A \subseteq B$. Note that in the left figure $A \subset B$ (proper/strict subset) also holds.



## Venn Diagrams $A \not\subseteq B$

There are three cases when $A \not\subseteq B$:

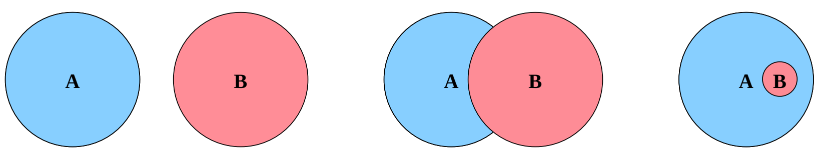



**Figure:** [**Left**] The sets $A$ and $B$ are completely disjoint (non-overlapping). [**Center**] The sets have some common elements. [**Right**] $B$ is contained in $A$.



## Venn Diagram for the Integers, Rationals, ...

We can express the relationship between the Integers ($\mathbb{Z}$), the Rational Numbers ($\mathbb{Q}$), the Real Numbers ($\mathbb{R}$), and the Complex Numbers ($\mathbb{C}$):

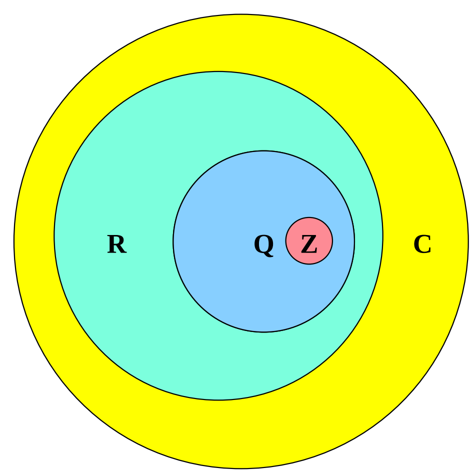



**Figure:** $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ — all proper subsets.

## The Difference between ∈ (element of) and ⊆ (subset of)

| ***true*** Statements | ***false*** Statements |
|---|---|
| $2 \in \{1,2,3\}$ | $\{2\} \in \{1,2,3\}$ |
| $\{2\} \subseteq \{1,2,3\}$ | $2 \subseteq \{1,2,3\}$ |
| $\{2\} \in \{\{1\},\{2\},\{3\}\}$ | $\{2\} \subseteq \{\{1\},\{2\},\{3\}\}$ |

**Discussion:** For $\{2\} \in \{1,2,3\}$ to be ***true***, the set $\{1,2,3\}$ would have to contain the element $\{2\}$, but the only elements are 1, 2, and 3.

For $2 \subseteq \{1,2,3\}$ to be ***true***, the number 2 would have to be a set and every element in the set 2 would have to be an element of $\{1,2,3\}$.

For $\{2\} \subseteq \{\{1\},\{2\},\{3\}\}$ to be ***true***, every element in the set containing only the number 2 would have to be an element of the set whose elements are $\{1\}$, $\{2\}$, and $\{3\}$. But 2 is not equal to $\{1\}$ nor $\{2\}$ nor $\{3\}$...



## Set Equality

**Definition: *Set Equality* —**
Given sets $A$ and $B$, $A$ **equals** $B$, written $A = B$ if and only if, every element of $A$ is in $B$ and every element of $B$ is in $A$.

Symbolically:

$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

To check two sets for equality:

***[1]*** Check $\forall x \in A$, $x \in B$

***[2]*** Check $\forall x \in B$, $x \in A$



## The Universal Set — Context

Mathematical discussions are carried out in some context — we may be talking about the properties of real numbers, integers, positive integers of the form $(2^{2^k} - 3)$, etc...

Say, we are discussing real numbers; in this situation the set of all real numbers ($\mathbb{R}$) would be called the **universal set** or a **universe of discourse** for the discussion.



## Operations on Sets

**Definition:** Let $A$ and $B$ be subsets of a universal set $U$:

***[1]*** The **union** of $A$ and $B$, denoted $\mathbf{A \cup B}$, is the set of all elements $x \in U$ such that $x \in A$ or $x \in B$.

***[2]*** The **intersection** of $A$ and $B$, denoted $\mathbf{A \cap B}$, is the set of all elements $x \in U$ such that $x \in A$ and $x \in B$.

***[3]*** The **difference** of $B$ minus $A$, denoted $\mathbf{B - A}$, is the set of all elements $x \in U$ such that $x \in B$ and $x \notin A$.

***[4]*** The **complement** of $A$, denoted $\mathbf{A^c}$, is the set of all elements $x \in U$ such that $x \notin A$.

Symbolically

$$\begin{aligned} A \cup B &= \{x \in U \,|\, (x \in A) \vee (x \in B)\} \\ A \cap B &= \{x \in U \,|\, (x \in A) \wedge (x \in B)\} \\ B - A &= \{x \in U \,|\, (x \in B) \wedge (x \notin A)\} \\ A^c &= \{x \in U \,|\, x \notin A\} \end{aligned}$$

## Venn Diagrams

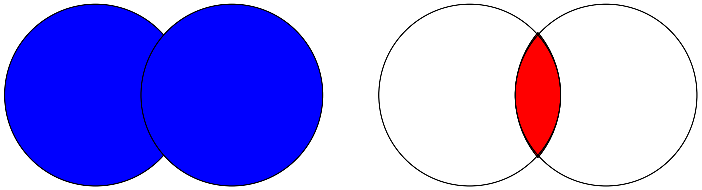


**Figure:** [**Left**] The Union of $A$ and $B$ ($A \cup B$).
[**Right**] The Intersection of $A$ and $B$ ($A \cap B$).

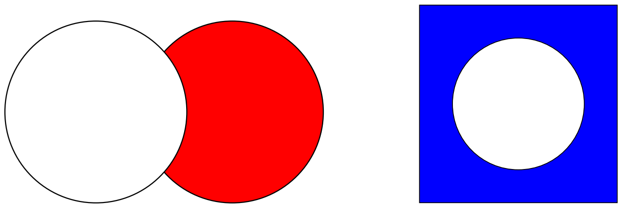


**Figure:** [**Left**] The difference of $B$ and $A$ ($B - A$).
[**Right**] The complement of $A$ ($A^c$).



## The Empty Set — "A Collection of Nothing"

Just as with numbers (where we have "0") we need a the concept of "nothing" in set theory.

**Definition: The Empty Set —**
The unique set with no elements is called **the empty set**. It is denoted by the symbol $\emptyset$.

***"The empty set is a subset of every set:"***

**Proof:** Suppose not [by contradiction] — then there exists a set $\emptyset$ with no elements, and a set $A$ such that $\emptyset \not\subseteq A$. [we must find a contradiction] Then there would be an element of $\emptyset$ which is not an element of $A$ [definition of subset]. But there can be no such element since $\emptyset$ has no elements. *This is a contradiction!* $\square$



## Partitions of Sets — Disjoint Sets

**Definition: Partition —**
In many applications of set theory sets are divided into non-overlapping (or ***disjoint***) pieces. Such a division is called a **partition**.

**Figure:** Partitioning tool?

**Definition: Disjoint —**
Two sets $A$ and $B$ are ***disjoint*** if and only if they have no elements in common — Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset$$



## Mutually Disjoint Sets and Partitions

**Definition: *Mutually Disjoint* —**
The sets $A_1$, $A_2$, ..., $A_n$ are **mutually disjoint** (or **pairwise disjoint** / **non-overlapping**) if and only if, no two sets $A_i$ and $A_j$ with distinct subscripts ($i \neq j$) have any elements in common —

$$\forall i, j \in \{1, 2, \ldots, n\},\ A_i \cap A_j = \emptyset, \text{ whenever } i \neq j.$$

**Definition: *Partition* —**
A collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set $A$ if and only if

***1.*** $A = A_1 \cup A_2 \cup \ldots \cup A_n$.
***2.*** $A_1, A_2, \ldots, A_n$ are mutually disjoint.

## Partitions — Illustration

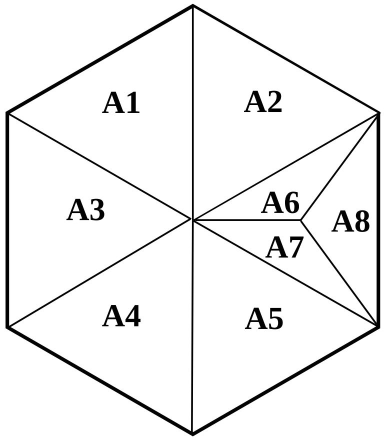



**Figure:** A partition of the set $A$ into the disjoint proper subsets $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, $A_6$, $A_7$, and $A_8$.



## Partitions — Examples

Let $\mathbb{Z}$ be the set of all integers, and let

$$T_0 = \{n \in \mathbb{Z} \,|\, n = 3k,\ k \in \mathbb{Z}\}$$
$$T_1 = \{n \in \mathbb{Z} \,|\, n = 3k+1,\ k \in \mathbb{Z}\}$$
$$T_2 = \{n \in \mathbb{Z} \,|\, n = 3k+2,\ k \in \mathbb{Z}\}$$

$\{T_0, T_1, T_2\}$ is a partition of $\mathbb{Z}$ — since by the Quotient-Remainder theorem every integer can be represented in exactly one of the three forms

$$n = 3k, \quad n = 3k+1, \quad n = 3k+2.$$



## Power Sets — The Set of all Subsets

**Definition:** ***Power Set*** —
Given a set $A$, the **power set** of $A$, denoted $\mathcal{P}(A)$, is the set of all subsets of $A$.

**Example:**
Let $A = \{1, 2\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

Note that $\forall A$:

$$\emptyset \in \mathcal{P}(A)$$
$$A \in \mathcal{P}(A)$$



## Ordered $n$-tuples

**Definition:** ***Ordered $n$-tuple*** —
Let $n$ be a positive integer and let $x_1$, $x_2$, ..., $x_n$ be (not necessarily distinct) elements. The **ordered $n$-tuple,** $(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n})$ consists of the elements $x_1$, $x_2$, ..., $x_n$ together with the ordering: sorted in ascending order with respect to the indices.

Two ordered $n$-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are **equal** if and only if $x_1 = y_1$, $x_2 = y_2$, ... $x_n = y_n$.

An ordered 2-tuple is called an **ordered pair**.
An ordered 3-tuple is called an **ordered triple**.

## Sets and $n$-tuples — Ordering

| Sets | $n$-tuples |
|---|---|
| Order of elements does not matter. | Order of elements matters. |
| Invariant to repetition of elements. | |
| {a,b} = {b,a,a,b} | (a,b,c) ≠ (a,c,b) |



## Cartesian Products

**Definition: Cartesian Product —**

Given two sets $A$ and $B$, the **Cartesian product** of $A$ and $B$, denoted $A \times B$ (read "A cross B"), is the set of all ordered pairs $(a, b)$ where $a \in A$ and $b \in B$.

Given the sets $A_1$, $A_2$, ..., $A_n$, the **Cartesian product** of $A_1$, $A_2$, ..., $A_n$ denoted $\mathbf{A_1} \times \mathbf{A_2} \times \ldots \times \mathbf{A_n}$ is the set of all ordered $n$-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$, $i = 1, 2, \ldots, n$.



## Cartesian Products: Example

**Example:** Any point on the earth's surface is uniquely determined by its latitude and longitude.

The Latitude is a measure (in degrees) of how far off the equator you are it is in the range $[90°N, 90°S]$.

The Longitude is a measure of far from the zero-meridian (which runs through Greenwich, UK) and is a value in the the range $[180°W, 180°E]$. If we let North and West be negative, and South and East be positive, then a position is given by the ordered pair

$$(\text{latitude}, \text{longitude}) \in [\mathbf{-90°}, \mathbf{90°}] \times [\mathbf{-180°}, \mathbf{180°}]$$

**Part II:** If you are drilling for oil, you also want to know how deep your well is... One description of a well is the location of the drill site, and the depth of the well — an ordered triple:

$$(\text{latitude}, \text{longitude}, \text{depth}) \in [\mathbf{-90°}, \mathbf{90°}] \times [\mathbf{-180°}, \mathbf{180°}] \times \mathbb{R}^+$$



**Homework #8 — Due Friday 11/10/2006, 12noon, GMCS-587 Final Version**

(Epp v3.0)

***Epp-5.1.8***, ***Epp-5.1.9***, ***Epp-5.1.10***, ***Epp-5.1.21***, ***Epp-5.1.30***

(Epp v2.0)

***Epp-5.1.6***, —, —, ***Epp-5.1.15***, ***Epp-5.1.18***