## Math 245: Discrete Mathematics

Relations on Sets

Reflexivity, Symmetry and Transitivity; Equivalence Relations

Lecture #14


**Peter Blomgren**
Department of Mathematics and Statistics
San Diego State University
San Diego, CA 92182-7720
**blomgren@terminus.SDSU.EDU**
**http://terminus.SDSU.EDU**



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## Relations: Introduction

**Mathematical Relations — Examples:**

* Two logical expressions can be said to be related if they have the same truth tables.
* A set $A$ can be said to be related to a set $B$ if $A \subseteq B$.
* A real number $x$ can be said to related to $y$ if $x < y$.
* An integer $n$ can be said to related to $m$ if $n|m$.
* An integer $n$ can be said to related to $m$ if $n$ and $m$ are both odd.
* Etc, etc, etc, ...

We are going to study ***mathematical relations on sets***: their properties and representations.



## Relations: Introductory Example 1 of 2

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$.

**The relation:** Let an element $x \in A$ be related to an element $y \in B$ if and only if $x < y$.

**Notation:** $x\,R\,y \equiv$ "**$x$ is related to $y$**", $x \not R\, y \equiv$ "**$x$ is not related to $y$**"

We have the following relations:

| | | | | | |
|---|---|---|---|---|---|
| $0\,R\,1$ | since | $0 < 1$ | $1 \not R\, 1$ | since | $1 \not< 1$ |
| $0\,R\,2$ | since | $0 < 2$ | $2 \not R\, 1$ | since | $2 \not< 1$ |
| $0\,R\,3$ | since | $0 < 3$ | $2 \not R\, 2$ | since | $2 \not< 2$ |
| $1\,R\,2$ | since | $1 < 2$ | | | |
| $1\,R\,3$ | since | $1 < 3$ | | | |
| $2\,R\,3$ | since | $2 < 3$ | | | |



## Relations: Introductory Example 2 of 2

**Relations and Cartesian Products:**

The Cartesian product $(A \times B)$ of two sets $A$ and $B$ is the set of all ordered pairs whose first element is in $A$ and second elements in $B$:

$$A \times B = \{(x, y) \,|\, x \in A \text{ and } y \in B\}$$

In our example

$$A \times B = \{(0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

The elements of some ordered pairs

$$\{(0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$$

are considered to be related (others are not).

Knowing which ordered pairs are in this set is equivalent to knowing which elements are related.

## Relations: Formal Definition

**Definition: Binary Relation —**
Let $A$ and $B$ be sets. A **(binary) relation** $R$ **from** $A$ **to** $B$ is a subset of $A \times B$. Given an ordered pair $(x, y) \in A \times B$, $x$ **is related to** $y$ **by** $R$, written $x\,R\,y$, if and only if $(x, y) \in R$.

**Symbolic Notation**

$$x\,R\,y \quad \Leftrightarrow \quad (x, y) \in R$$

$$x \not R\, y \quad \Leftrightarrow \quad (x, y) \notin R$$

The term ***binary*** is used in the definition to indicate that the relation is a subset of the Cartesian product of ***two*** sets.



## Illustration: Relations

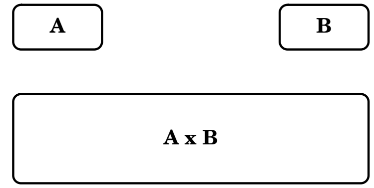



**Figure:** Given 2 sets $A$ and $B$, we form the Cartesian product $A \times B$; $(x, y) \in A \times B \equiv (x \in A)$ **and** $(y \in B)$.

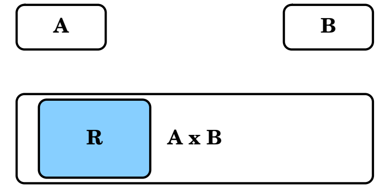



**Figure:** The Relation $R$ is a subset of $A \times B$. If and only if $(x, y) \in R$ we say that $x$ is related to $y$ by $R$, symbolically $x\,R\,y$.

The subset $R \subseteq A \times B$ can be specified

1. Directly / Explicitly, by indicating what pairs $(x, y) \in R$. This is only feasible when $A$ and $B$ are finite (and small) sets.
2. By specifying a rule for what elements are in $R$, *e.g.* by saying that $(x, y) \in R$ if and only if $x = y^2$.



## Example: Congruence Modulo 2 Relation 1 of 2

We generalize the previous example to the set of all integers $\mathbb{Z}$, *i.e.*

$$\text{for all } (m, n) \in \mathbb{Z} \times \mathbb{Z},\ m\,R\,n \ \Leftrightarrow\ m - n \text{ is even}$$

**Questions:**

(a) is $4\,R\,0$? $2\,R\,6$? $3\,R\,(-3)$? $5\,R\,2$?
(b) List 5 integers that are related by $R$ to $1$.
(c) Prove that if $n$ is odd, then $n\,R\,1$.

**Answers:**

(a-i) Yes, $4\,R\,0$, since $4 - 0 = 4$ is even.
(a-ii) Yes, $2\,R\,6$, since $2 - 6 = -4$ is even.
(a-iii) Yes, $3\,R\,(-3)$, since $3 - (-3) = 6$ is even.
(a-iv) No, $5 \not R\, 2$, since $5 - 2 = 3$ is odd.



## Example: Congruence Modulo 2 Relation 2 of 2

(b) There are infinitely many examples, *e.g.*

| | | |
|---|---|---|
| $1$ | since $1 - 1 = 0$ | is even |
| $11$ | since $11 - 1 = 10$ | is even |
| $111$ | since $111 - 1 = 110$ | is even |
| $1111$ | since $1111 - 1 = 1110$ | is even |
| $11111$ | since $11111 - 1 = 11110$ | is even |

(c) **Proof:** Suppose $n$ is any odd integer. Then $n = 2k + 1$ for some integer $k$. By substitution

$$n - 1 = 2k + 1 - 1 = 2k \text{ is even}$$

Hence

$$n\,R\,1,\ \forall n \text{ odd.} \quad \square$$

## Representation: Arrow Diagrams for Relations

Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$

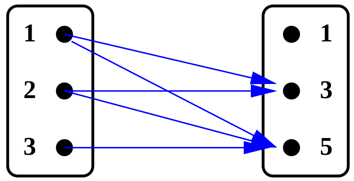



**Figure:** Arrow diagram representation of the relation

for all $(x, y) \in A \times B$,
$(x, y) \in R \Leftrightarrow x < y$

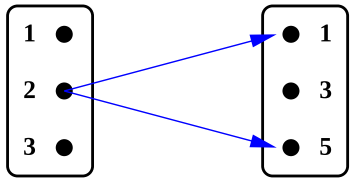



**Figure:** Arrow diagram representation of the relation

$R = \{(2, 1), (2, 5)\}$

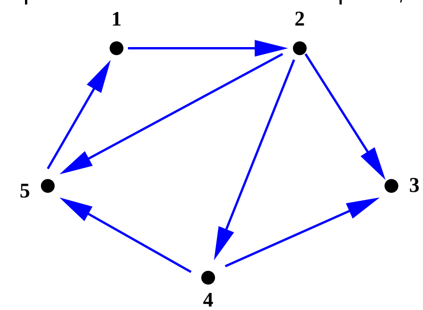
**Notes:** *(i)* It is possible to have an element that does not have an arrow coming out of it; *(ii)* It is possible to have several arrows coming out of the same element of $A$ pointing in different directions; *(iii)* It is possible to have an element in $B$ that does not have an arrow pointing to it.



## Relation from $A$ to $A$ Directed Graph of a Relation

> ***Definition:*** A **binary relation on a set A** is a binary relation from $A$ to $A$.

In this case, we can modify the arrow diagram to be a **directed graph** — instead of representing $A$ twice, we only represent it once and draw arrows from each point of $A$ to each related point, *e.g.*




**there is an arrow from x to y** $\Leftrightarrow$ $\mathbf{x\,R\,y}$ $\Leftrightarrow$ $\mathbf{(x, y) \in R}$



## Example: Directed Graph of a Relation

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a binary relation $R$ on $A$:

$$R = \{(x, y) \in A \times A : \ 2|(x - y)\}$$

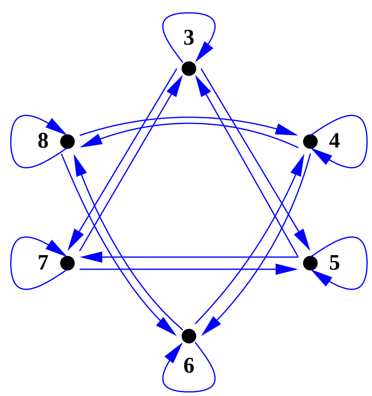



**Figure:** We notice that the graph must be symmetric, since if $2|n$, then $2|(-n)$. Since $2|0$, there is a loop at every node in the graph.



## Properties of a Binary Relation on One Set $A$

**Recall:**

> ***Definition:*** A **binary relation on a set A** is a binary relation from $A$ to $A$.

In the context of a binary relation on a set, we can name 3 properties:

> ***Definition:*** Let $R$ be a binary relation on a set $A$
>
> 1. $R$ is **Reflexive** if and only if $\forall x \in A$, $x\,R\,x$.
> 2. $R$ is **Symmetric** if and only if $\forall x, y \in A$, if $x\,R\,y$ then $y\,R\,x$.
> 3. $R$ is **Transitive** if and only if $\forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

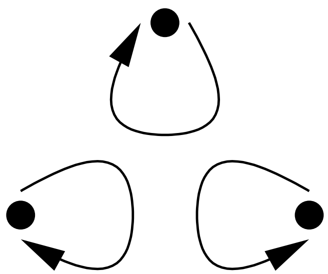
## Reflexivity

**Formal:** $R$ is **Reflexive** if and only if $\forall x \in A$, $x\,R\,x$.

**Functional:** $R$ is **Reflexive** $\Leftrightarrow$ for all $x \in A$, $(x, x) \in R$.

**Informal:** Each element is related to itself.

**Graph:** Each point of the graph has an arrow looping around back to itself.





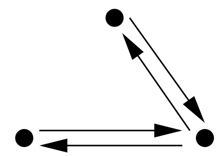
## Symmetry

**Formal:** $R$ is **Symmetric** if and only if $\forall x, y \in A$, if $x\,R\,y$ then $y\,R\,x$.

**Functional:** $R$ is **Symmetric** $\Leftrightarrow$ for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$.

**Informal:** If one element is related to a second element, then the second element is related to the first.

**Graph:** In all cases where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.





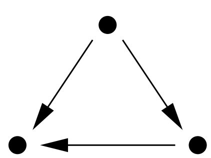
## Transitivity

**Formal:** $R$ is **Transitive** if and only if $\forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

**Functional:** $R$ is **Transitive** $\Leftrightarrow$ for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

**Informal:** If one element is related to a second element, and that second element is related to a third element, then the first element is related to the third element.

**Graph:** In all cases where there is an arrow going from one point to a second, and from the second point to a third, there is an arrow going from the first point to the third.





## Non-Reflexivity, Non-Symmetry, and Non-Transitivity

If $R$ is a binary relation defined on a set $A$, then

1. $R$ **is not reflexive** $\Leftrightarrow$ there is an element $x \in A$ such that $x \not\!R\, x$, *i.e.* $(x, x) \notin R$.
2. $R$ **is not symmetric** $\Leftrightarrow$ there are elements $x, y \in A$ such that $x\,R\,y$ but $y \not\!R\, x$, *i.e.* $(x, y) \in R$, but $(y, x) \notin R$.
3. $R$ **is not transitive** $\Leftrightarrow$ there are elements $x, y, z \in A$ such that $x\,R\,y$ and $y\,R\,z$ but $x \not\!R\, z$, *i.e.* $(x, y), (y, z) \in R$, but $(x, z) \notin R$.

To show that a binary relation does ***not*** have one of the properties, it is sufficient to find a counterexample.

## Example



Let $A = \{0, 1, 2, 3\}$ and define relations $R$, $S$, and $T$:

$$R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$$
$$S = \{(0,0), (0,2), (0,3), (2,3)\}$$
$$T = \{(0,1), (2,3)\}$$

Fill in the table:

| | **Reflexive** | **Symmetric** | **Transitive** |
|---|---|---|---|
| **R** | | | |
| **S** | | | |
| **T** | | | |

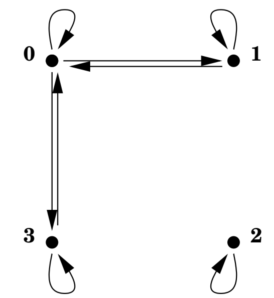


## Example



We have $A = \{0, 1, 2, 3\}$ and

$$R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$$




**R is reflexive** since there is a loop at each point in the directed graph.

**R is symmetric** since in for every arrow going from one point to another, there is another arrow going back.

**R is not transitive** since *e.g.* $1\,R\,0$ and $0\,R\,3$ but $1 \not R\, 3$ *i.e.* there is no "short-cut" arrow connecting 1 and 3.

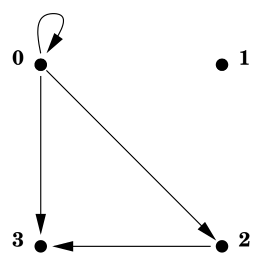


## Example



We have $A = \{0, 1, 2, 3\}$ and

$$S = \{(0,0), (0,2), (0,3), (2,3)\}$$




**S is not reflexive** since there are missing loops at 1, 2, and 3.

**S is not symmetric**, the arrows from 2-to-0, 3-to-0, and 3-to-2 are missing.

**S is transitive** since there is always a "short-cut" arrow so that if $(x, y) \in S$ and $(y, z) \in S$ then $(x, z) \in S$.

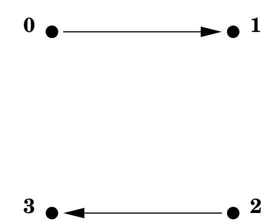


## Example



We have $A = \{0, 1, 2, 3\}$ and

$$T = \{(0,1), (2,3)\}$$




**T is not reflexive** since there are missing loops at 0, 1, 2, and 3.

**T is not symmetric**, the arrows from 1-to-0, and 3-to-2 are missing.

**T is transitive** since it is ***not*** *not transitive*.

## Example 

Let $A = \{0, 1, 2, 3\}$ and define relations $R$, $S$, and $T$:

$$R = \{(0,0),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\}$$
$$S = \{(0,0),(0,2),(0,3),(2,3)\}$$
$$T = \{(0,1),(2,3)\}$$

Fill in the table:

| | Reflexive | Symmetric | Transitive |
|---|---|---|---|
| R | Yes | Yes | No |
| S | No | No | Yes |
| T | No | No | Yes |



## Irreflexivity, Anti-Symmetry, and Intransitivity

***Definition:*** Let $R$ be a binary relation on a set $A$

1. $R$ is **Irreflexive** if and only if $\forall x \in A$, $x \not{R}\, x$.
2. $R$ is **Anti-symmetric** if and only if $\forall x, y \in A$, if $x\,R\,y$ then $y \not{R}\, x$.
3. $R$ is **Intransitive** if and only if $\forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$ then $x \not{R}\, z$.

- $R$ can be reflexive, non-reflexive, or irreflexive,
- $R$ can be symmetric. non-symmetric, or anti-symmetric
- $R$ can be transitive, non-transitive, or intransitive.

***Think about these definitions!!!***



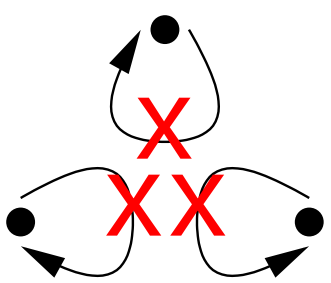
## Irreflexivity

**Formal:** $R$ is **Irreflexive** if and only if $\forall x \in A$, $\mathbf{x} \not{\mathbf{R}}\, \mathbf{x}$.

**Functional:** $R$ is **Irreflexive** $\Leftrightarrow$ for all $x \in A$, $(\mathbf{x}, \mathbf{x}) \notin \mathbf{R}$.

**Informal:** **No** element is related to itself.

**Graph:** **No** point of the graph has an arrow looping around back to itself.





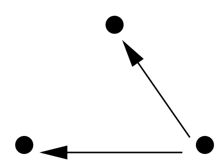
## Anti-Symmetry

**Formal:** $R$ is **Anti-Symmetric** if and only if $\forall x, y \in A$, if $x\,R\,y$ then $\mathbf{y} \not{\mathbf{R}}\, \mathbf{x}$.

**Functional:** $R$ is **Anti-Symmetric** $\Leftrightarrow$ for all $x, y \in A$, if $(x, y) \in R$ then $(\mathbf{y}, \mathbf{x}) \notin \mathbf{R}$.

**Informal:** If one element is related to a second element, then the second element is **NOT** related to the first.

**Graph:** In all cases where there is an arrow going from one point to a second, there is **no** arrow going from the second point back to the first.



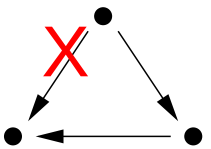
## Intransitivity

**Formal:** $R$ is **Intransitive** if and only if $\forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$ then $\mathbf{x} \not{\mathbf{R}}\, \mathbf{z}$.

**Functional:** $R$ is **Intransitive** $\Leftrightarrow$ for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(\mathbf{x}, \mathbf{z}) \notin \mathbf{R}$.

**Informal:** If one element is related to a second element, and that second element is related to a third element, then the first element is **not** related to the third element.

**Graph:** In all cases where there is an arrow going from one point to a second, and from the second point to a third, there is **never** an arrow going from the first point to the third (no shortcut exist, anywhere.).






## Example: Equality ($=$) on $\mathbb{R}$

Let $A = \mathbb{R}$ (the set of real numbers), and define the relation $R$

$$x\,R\,y \quad \Leftrightarrow \quad x = y$$

**Properties:**

**R is reflexive:** $R$ is reflexive if and only if $\forall x \in \mathbb{R}$, $x\,R\,x$. Here, this means $x = x$, *i.e.* $\forall x \in \mathbb{R}\; x = x$. This statement is certainly true; every real number equals itself.

**R is symmetric:** This is true since if $x = y$ then $y = x$, hence $(x, y) \in R$ and $(y, x) \in R$.

**R is transitive:** This is true since if $x = y$ and $y = z$, then $x = z$.



## Example: Less Than ($<$) on $\mathbb{R}$

Let $A = \mathbb{R}$ (the set of real numbers), and define the relation $R$

$$x\,R\,y \quad \Leftrightarrow \quad x < y$$

**Properties:**

**R is irreflexive:** If $x\,R\,x$ then $x < x$, but that is never true, hence $x \not{R}\, x$ $\forall x \in \mathbb{R}$.

**R is anti-symmetric:** If $x\,R\,y$ then $x < y$, which means $y \not< x$ *i.e.* $y \not{R}\, x$.

**R is transitive:** This is true since if $x < y$ and $y < z$, then $x < z$.



## Example: Congruence Modulo 3 on $\mathbb{Z}$

We define a relation $R$ on $\mathbb{Z}$ as follows

$$\forall m, n \in \mathbb{Z}: \quad m\,R\,n \quad \Leftrightarrow \quad 3|(m - n)$$

**R is reflexive:** Suppose $m$ is an integer. Now, $m - m = 0$ and $3|0$ since $0 = 3 \cdot 0$, so by definition of $R$ we have $m\,R\,m$. $\square$

**R is symmetric:** Suppose $m, n \in \mathbb{Z}$ such that $m\,R\,n$. By definition of $R$ we have $3|(m - n) \Leftrightarrow m - n = 3 \cdot k$, for some $k \in \mathbb{Z}$. Multiplying both sides by $(-1)$ gives $n - m = 3 \cdot (-k)$, which shows $3|(n - m)$, hence $n\,R\,m$. $\square$

**R is transitive:** Suppose $m, n, p \in \mathbb{Z}$ such that $m\,R\,n$ and $n\,R\,p$. We have $3|(m - n)$ and $3|(n - p)$, and we can write $(m - n) = 3r$ and $(n - p) = 3s$ for some $r, s \in \mathbb{Z}$. Adding the two gives $(m - n) + (n - p) = (m - p) = 3(r + s)$ which shows that $3|(m - p)$. Hence $m\,R\,p$, and it follows that $R$ is transitive. $\square$

## Equivalence Relations: Different, but the Same...

**Idea:** We are going to group elements that look different, but really are the same...

**Example:** Think about the rational numbers, there are several ways of writing the same fraction, *e.g.*

$$\frac{1}{2} \quad = \quad \frac{-1}{-2} \quad = \quad \frac{2}{4} \quad = \quad \frac{4711}{9422}$$

We can define a relation on $\mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers

$$R = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x = y\}$$

now $\left(\frac{1}{2}, \frac{2}{4}\right) \in R$, $\left(\frac{4711}{9422}, \frac{2}{4}\right) \in R$, $\left(\frac{1}{3}, \frac{2}{6}\right) \in R$, etc...



## A Relation Induced by a Partition

**Recall:**

> ***Definition:*** A collection of non-empty sets $\{A_1, A_2, \ldots, A_n\}$ is a **partition** of a set $A$ if and only if
>
> ***1.*** $A = A_1 \cup A_2 \cup \ldots \cup A_n$.
>
> ***2.*** $A_1, A_2, \ldots, A_n$ are mutually disjoint.

> ***Definition:*** Given a partition of a set $A$ the **binary relation induced by the partition**, $R$, is defined on $A$ as follows
>
> $\forall x, y \in A, \quad x\,R\,y \quad \Leftrightarrow$ there is a set $A_i$ of the partition such that both $x \in A_i$ and $y \in A_i$.

We need an example to make sense out of this definition...

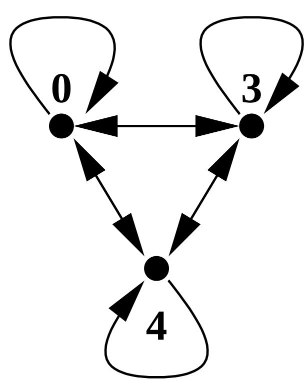


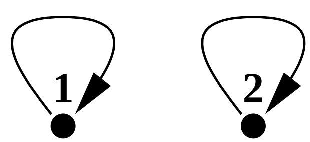
## Example: Relation Induced by a Partition

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of $A$:

$$A_1 = \{0, 3, 4\}, \quad A_2 = \{1\}, \quad A_3 = \{2\}$$

Now, two elements $x, y \in A$ are related if and only if they belong to the same subset of the partition...




Hence,

$$\mathbf{R} = \left\{ \begin{array}{l} (0,0), (0,3), (0,4), (1,1), \\ (2,2), (3,0), (3,3), (3,4), \\ (4,0), (4,3), (4,4) \end{array} \right\}$$



## Equivalence Relations

> ***Theorem:*** Let $A$ be a set with a partition and let $R$ be the relation induced by the partition. Then $R$ is reflexive, symmetric, and transitive.

> ***Definition: Equivalence Relation —***
> Let $A$ be a non-empty set and $R$ a binary relation on $A$. $R$ is an **equivalence relation** if and only if $R$ is reflexive, symmetric, and transitive.

**Example:** By the theorem the relation induced by a partition is an equivalence relation.

## Notation: Congruence Modulo $n$

**Notation:** Let $m, n, d \in \mathbb{Z}$ with $d > 0$. The notation

$$m \equiv n \pmod d$$

is read *"$m$ **is congruent to** $n$ **modulo** $d$"* and means that

$$d|(m-n)$$

Symbolically,

$$m \equiv n \pmod d \qquad \Leftrightarrow \qquad d|(m-n)$$

Recall the ***Quotient-Remainder Theorem:***

***Theorem:*** Given any integer $n$ and a positive integer $d$, there exist unique integers $q$ (the quotient) and $r$ (the remainder) such that

$$n = d \cdot q + r, \quad \text{and} \quad 0 \leq r < d$$



## Equivalence Relation: Congruence Modulo 3

Let $R$ be the relation $R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : \ m = n (\text{mod } 3)\}$. We show that this is an equivalence relation.

**[Reflexivity]** Let $m \in Z$, then $3|(m-m)$ since $0 = 3 \cdot 0$, and it follows that $m\,R\,m$.

**[Symmetry]** Let $m, n \in Z$, so that $m\,R\,n$. We have $3|(m-n) \Leftrightarrow (m-n) = 3 \cdot k$ for some $k \in \mathbb{Z} \Leftrightarrow (n-m) = 3 \cdot (-k)$ $\Leftrightarrow 3|(n-m) \Leftrightarrow n\,R\,m$.

**[Transitivity]** Let $m, n, p \in Z$, so that $m\,R\,n$ and $n\,R\,p$. We have

$$\begin{array}{rcl} 3|(m-n) & \Leftrightarrow & (m-n) = 3 \cdot r, \ r \in \mathbb{Z} \\ 3|(n-p) & \Leftrightarrow & (n-p) = 3 \cdot s, \ s \in \mathbb{Z} \\ \hline & \textbf{add} & (m-p) = 3 \cdot (r+s) \end{array}$$

Hence $3|(m-p)$ and we have $m\,R\,p$. $\square$



## Equivalence Classes

Suppose we have a set $A$ and an equivalence relation $R$ on $A$. Given a particular element $x \in A$ it is natural to ask the question ***"what elements are related to $x$?"***

All the elements that are related to $x$ form a subset of $A$ and this subset is called ***the equivalence class of*** $x$:

**Definition:** Suppose $A$ is a set and $R$ is an equivalence relation on $A$. For each element $x \in A$, the **equivalence class of** $x$, denoted $[\mathbf{x}]$ and called the **class of** $x$ for short, is the set of all elements $y \in A$ such that $y\,R\,x$.

Symbolically,

$$[x] = \{y \in A | \ y\,R\,x\}$$



## Example: Equivalence Classes 1 of 2

Let $A = \{0, 1, 2, 3, 4\}$ and define a binary relation $R$ on $A$

$$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$$

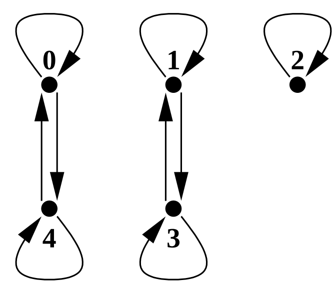
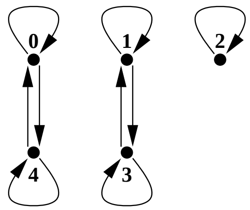



**Figure:** The array diagram (directed graph) corresponding to the relation.

By quick inspection we see that $R$ is reflexive, symmetric, and transitive, hence an equivalence relation.

## Example: Equivalence Classes 2 of 2




The equivalence classes are:

$$
\begin{aligned}
[0] &= \{x \in A \,|\, x\,R\,0\} &= \{0,4\}\\
[1] &= \{x \in A \,|\, x\,R\,1\} &= \{1,3\}\\
[2] &= \{x \in A \,|\, x\,R\,2\} &= \{2\}\\
[3] &= \{x \in A \,|\, x\,R\,3\} &= \{1,3\}\\
[4] &= \{x \in A \,|\, x\,R\,4\} &= \{0,4\}
\end{aligned}
$$

Note that $[0] = [4]$ and $[1] = [3]$, hence the ***distinct*** equivalence classes are: $\{0,4\}$, $\{1,3\}$, $\{2\}$.



## Equivalence Classes: A Theorem

The following theorem tells us that an equivalence relation induces a partition:

> **Theorem:** If $A$ is a non-empty set and $R$ is an equivalence relation on $A$, then the distinct equivalence classes of $R$ form a partition of $A$; *i.e.* the union of the equivalence classes is all of $A$ and the intersection of any two distinct classes is empty.



## Example: Equivalence Classes of Congruence Modulo 3 1 of 3

Let $R$ be the relation of congruence modulo 3 on the set $\mathbb{Z}$, *i.e.* $\forall m, n \in \mathbb{Z}$

$$m\,R\,n \;\Leftrightarrow\; 3|(m-n) \;\Leftrightarrow\; m \equiv n (\mathrm{mod}\, 3)$$

We describe the equivalence classes: For each integer $a$, the class of $a$ is

$$
\begin{aligned}
[a] &= \{x \in \mathbb{Z} \,|\, x\,R\,a\}\\
&= \{x \in \mathbb{Z} \,|\, 3|(x-a)\}\\
&= \{x \in \mathbb{Z} \,|\, x - a = 3 \cdot k,\ k \in \mathbb{Z}\}\\
&= \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + a,\ k \in \mathbb{Z}\}
\end{aligned}
$$

In particular

$$
\begin{aligned}
[0] &= \{x \in \mathbb{Z} \,|\, x = 3 \cdot k,\ k \in \mathbb{Z}\} &= \{0, 3, -3, 6, -6, 9, -9, \ldots\}\\
[1] &= \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 1,\ k \in \mathbb{Z}\} &= \{1, 4, -2, 7, -5, 10, -8, \ldots\}\\
[2] &= \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 2,\ k \in \mathbb{Z}\} &= \{2, 5, -1, 8, -4, 11, -7, \ldots\}
\end{aligned}
$$



## Example: Equivalence Classes of Congruence Modulo 3 2 of 3

We have

$$
\begin{aligned}
[0] &= \{x \in \mathbb{Z} \,|\, x = 3 \cdot k,\ k \in \mathbb{Z}\} &= \{0, 3, -3, 6, -6, 9, -9, \ldots\}\\
[1] &= \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 1,\ k \in \mathbb{Z}\} &= \{1, 4, -2, 7, -5, 10, -8, \ldots\}\\
[2] &= \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 2,\ k \in \mathbb{Z}\} &= \{2, 5, -1, 8, -4, 11, -7, \ldots\}
\end{aligned}
$$

By lemma#1

$$
\begin{aligned}
&[0] = [3] = [-3] = [6] = [-6] = [9] = [-9] = \ldots\\
&[1] = [4] = [-2] = [7] = [-5] = [10] = [-8] = \ldots\\
&[2] = [5] = [-1] = [8] = [-4] = [11] = [-7] = \ldots
\end{aligned}
$$

Hence the distinct equivalence classes are

$$\{x \in \mathbb{Z} \,|\, x = 3 \cdot k,\ k \in \mathbb{Z}\}, \quad \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 1,\ k \in \mathbb{Z}\},$$

$$\{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 2,\ k \in \mathbb{Z}\}$$

The distinct equivalence classes are

$$\{x \in \mathbb{Z} \,|\, x = 3 \cdot k,\ k \in \mathbb{Z}\}, \quad \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 1,\ k \in \mathbb{Z}\},$$

$$\{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 2,\ k \in \mathbb{Z}\}$$

The class of $[0]$ can also be called the class of $[3]$ or the class of $[96]$, but the class ***is*** the set $\{x \in \mathbb{Z} \,|\, x = 3 \cdot k,\ k \in \mathbb{Z}\}$.

**Definition:** Suppose $R$ is an equivalence relation on a set $A$ and $S$ is an equivalence class of $R$. A **representative** of the class $S$ is any element $a \in A$ such that $[a] = S$.



## Notes

- It is possible to define multiplication and addition of the equivalence classes corresponding to the rational numbers (previous example).
- The rational numbers can be defined as equivalence classes of ordered integers.
- It can be shown that all integers — negative, zero, and positive — can be defined as equivalence classes of ordered pairs of positive integers.
- Frege and Peano showed (late 19th century) that the positive integers can be defined entirely in terms of sets.
- Dedekind (1848–1916) showed that all real numbers can be defined as sets of rational numbers.
- All together, these results show that the real numbers can be defined using logic and set theory alone!





(Epp-v3.0)

***10.1.1, 10.1.5, 10.1.7, 10.1.15, 10.1.23, 10.1.25, 10.2.3, 10.2.4, 10.2.12, 10.2.14, 10.2.37, 10.3.3, 10.3.17, 10.3.19, 10.3.40***

(Epp-v2.0)

***10.1.1, 10.1.5, 10.1.7, 10.1.15, 10.1.23, 10.1.25, 10.2.3, 10.2.4, 10.2.12, 10.2.14, 10.2.37, 10.3.2, 10.3.14, 10.3.16, 10.3.35***