

Math 245: Discrete Mathematics

Relations on Sets

Reflexivity, Symmetry and Transitivity; Equivalence Relations

Lecture #14

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Relations: Introduction

Mathematical Relations — Examples:

- * Two logical expressions can be said to be related if they have the same truth tables.
- * A set A can be said to be related to a set B if $A \subseteq B$.
- * A real number x can be said to related to y if $x < y$.
- * An integer n can be said to related to m if $n|m$.
- * An integer n can be said to related to m if n and m are both odd.
- * Etc, etc, etc, ...

We are going to study *mathematical relations on sets*: their properties and representations.

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$.

The relation: Let an element $x \in A$ be related to an element $y \in B$ if and only if $x < y$.

Notation: $x R y \equiv$ “ x is related to y ”, $x \not R y \equiv$ “ x is not related to y ”

We have the following relations:

$0 R 1$ since $0 < 1$

$0 R 2$ since $0 < 2$

$0 R 3$ since $0 < 3$

$1 R 2$ since $1 < 2$

$1 R 3$ since $1 < 3$

$2 R 3$ since $2 < 3$

$1 \not R 1$ since $1 \not< 1$

$2 \not R 1$ since $2 \not< 1$

$2 \not R 2$ since $2 \not< 2$

Relations and Cartesian Products:

The Cartesian product ($A \times B$) of two sets A and B is the set of all ordered pairs whose first element is in A and second elements in B :

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

In our example

$$A \times B = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$$

The elements of some ordered pairs

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$$

are considered to be related (others are not).

Knowing which ordered pairs are in this set is equivalent to knowing which elements are related.

Relations: Formal Definition

Definition: Binary Relation —

Let A and B be sets. A **(binary) relation R from A to B** is a subset of $A \times B$. Given an ordered pair $(x, y) \in A \times B$, x is **related to y by R** , written $x R y$, if and only if $(x, y) \in R$.

Symbolic Notation

$$x R y \iff (x, y) \in R$$

$$x \not R y \iff (x, y) \notin R$$

The term ***binary*** is used in the definition to indicate that the relation is a subset of the Cartesian product of ***two*** sets.

Illustration: Relations

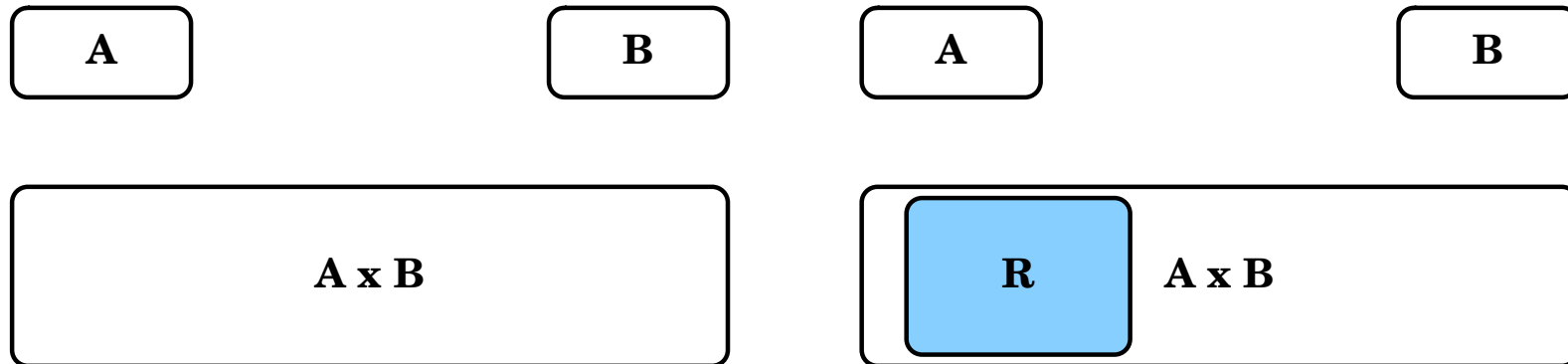


Figure: Given 2 sets A and B , we form the Cartesian product $A \times B$; $(x, y) \in A \times B \equiv (x \in A)$ **and** $(y \in B)$.

Figure: The Relation R is a subset of $A \times B$. If and only if $(x, y) \in R$ we say that x is related to y by R , symbolically $x R y$.

The subset $R \subseteq A \times B$ can be specified

1. Directly / Explicitly, by indicating what pairs $(x, y) \in R$. This is only feasible when A and B are finite (and small) sets.
2. By specifying a rule for what elements are in R , e.g. by saying that $(x, y) \in R$ if and only if $x = y^2$.

We generalize the previous example to the set of all integers \mathbb{Z} , *i.e.*

$$\text{for all } (m, n) \in \mathbb{Z} \times \mathbb{Z}, m R n \Leftrightarrow m - n \text{ is even}$$

Questions:

- (a) is $4 R 0$? $2 R 6$? $3 R (-3)$? $5 R 2$?
- (b) List 5 integers that are related by R to 1.
- (c) Prove that if n is odd, then $n R 1$.

Answers:

- (a-i) Yes, $4 R 0$, since $4 - 0 = 4$ is even.
- (a-ii) Yes, $2 R 6$, since $2 - 6 = -4$ is even.
- (a-iii) Yes, $3 R (-3)$, since $3 - (-3) = 6$ is even.
- (a-iv) No, $5 \not R 2$, since $5 - 2 = 3$ is odd.

(b) There are infinitely many examples, e.g.

$$1 \quad \text{since } 1 - 1 = 0 \quad \text{is even}$$

$$11 \quad \text{since } 11 - 1 = 10 \quad \text{is even}$$

$$111 \quad \text{since } 111 - 1 = 110 \quad \text{is even}$$

$$1111 \quad \text{since } 1111 - 1 = 1110 \quad \text{is even}$$

$$11111 \quad \text{since } 11111 - 1 = 11110 \quad \text{is even}$$

(c) **Proof:** Suppose n is any odd integer. Then $n = 2k + 1$ for some integer k . By substitution

$$n - 1 = 2k + 1 - 1 = 2k \text{ is even}$$

Hence

$$n R 1, \forall n \text{ odd. } \square$$

Representation: Arrow Diagrams for Relations

Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$

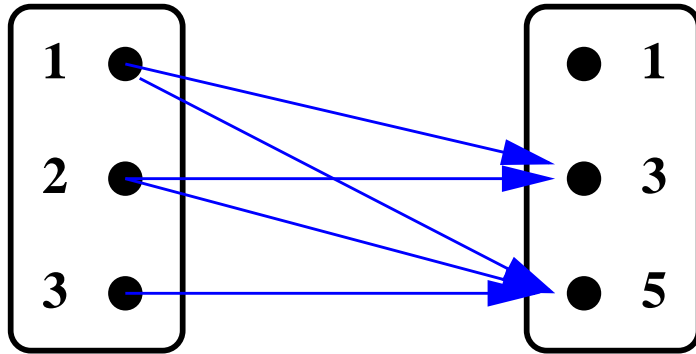


Figure: Arrow diagram representation of the relation

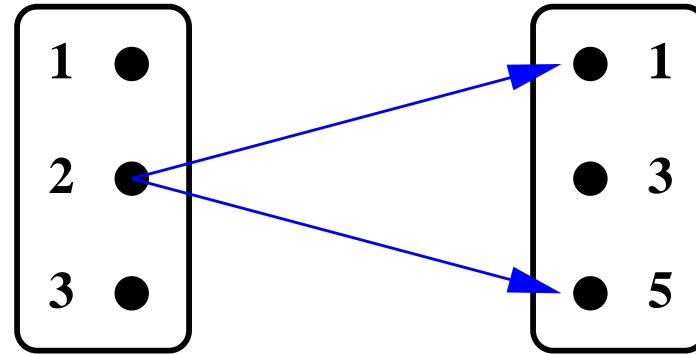


Figure: Arrow diagram representation of the relation

for all $(x, y) \in A \times B$,
 $(x, y) \in R \Leftrightarrow x < y$

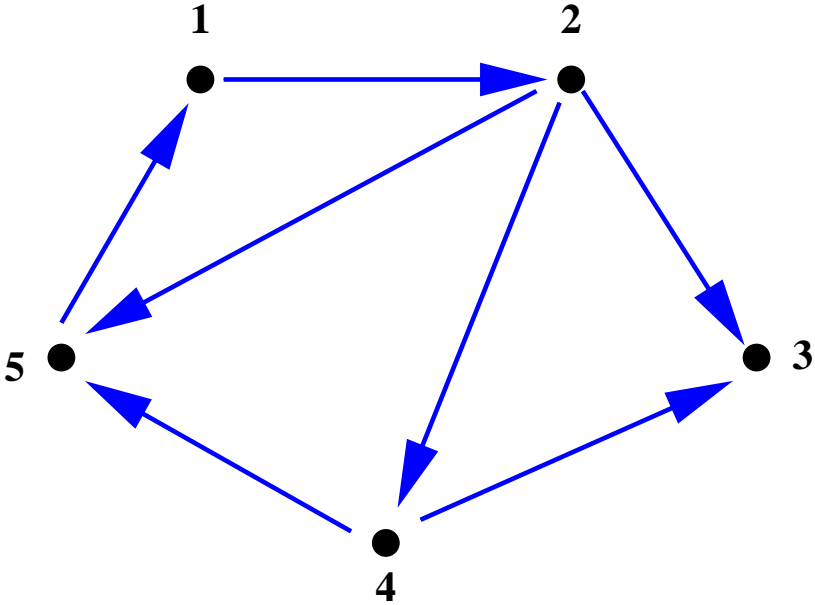
$$R = \{(2, 1), (2, 5)\}$$

Notes: (i) It is possible to have an element that does not have an arrow coming out of it; (ii) It is possible to have several arrows coming out of the same element of A pointing in different directions; (iii) It is possible to have an element in B that does not have an arrow pointing to it.

Relation from A to A Directed Graph of a Relation

Definition: A binary relation on a set A is a binary relation from A to A .

In this case, we can modify the arrow diagram to be a **directed graph** — instead of representing A twice, we only represent it once and draw arrows from each point of A to each related point, e.g.



there is an arrow from x to y $\Leftrightarrow x \mathbf{R} y \Leftrightarrow (x, y) \in \mathbf{R}$

Example: Directed Graph of a Relation

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a binary relation R on A :

$$R = \{(x, y) \in A \times A : 2 \mid (x - y)\}$$

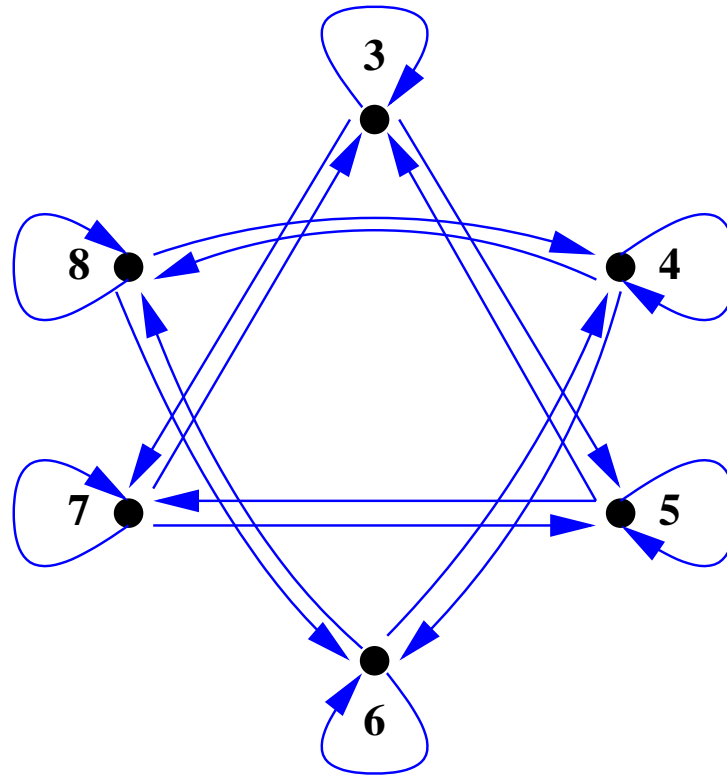


Figure: We notice that the graph must be symmetric, since if $2 \mid n$, then $2 \mid (-n)$. Since $2 \mid 0$, there is a loop at every node in the graph.

Properties of a Binary Relation on One Set A

Recall:

Definition: A binary relation on a set A is a binary relation from A to A .

In the context of a binary relation on a set, we can name 3 properties:

Definition: Let R be a binary relation on a set A

1. R is **Reflexive** if and only if $\forall x \in A, x R x$.
2. R is **Symmetric** if and only if $\forall x, y \in A$, if $x R y$ then $y R x$.
3. R is **Transitive** if and only if $\forall x, y, z \in A$, if $x R y$ and $y R z$ then $x R z$.

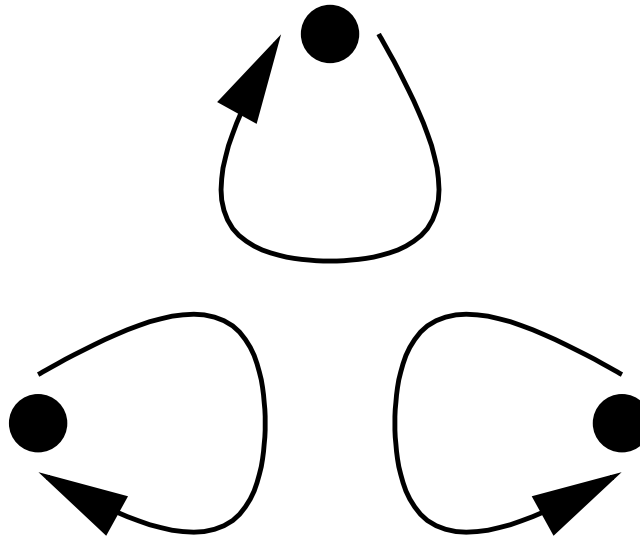
Reflexivity

Formal: R is **Reflexive** if and only if $\forall x \in A, x R x$.

Functional: R is **Reflexive** \Leftrightarrow for all $x \in A, (x, x) \in R$.

Informal: Each element is related to itself.

Graph: Each point of the graph has an arrow looping around back to itself.



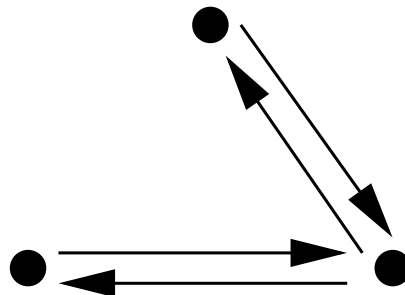
Symmetry

Formal: R is **Symmetric** if and only if $\forall x, y \in A$, if $x R y$ then $y R x$.

Functional: R is **Symmetric** \Leftrightarrow for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$.

Informal: If one element is related to a second element, then the second element is related to the first.

Graph: In all cases where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.



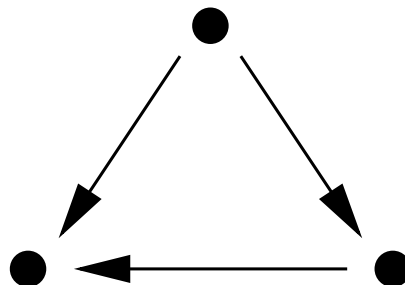
Transitivity

Formal: R is **Transitive** if and only if $\forall x, y, z \in A$, if $x R y$ and $y R z$ then $x R z$.

Functional: R is **Transitive** \Leftrightarrow for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Informal: If one element is related to a second element, and that second element is related to a third element, then the first element is related to the third element.

Graph: In all cases where there is an arrow going from one point to a second, and from the second point to a third, there is an arrow going from the first point to the third.



Non-Reflexivity, Non-Symmetry, and Non-Transitivity

If R is a binary relation defined on a set A , then

1. R is not reflexive \Leftrightarrow there is an element $x \in A$ such that $x \not R x$, i.e. $(x, x) \notin R$.
2. R is not symmetric \Leftrightarrow there are elements $x, y \in A$ such that $x R y$ but $y \not R x$, i.e. $(x, y) \in R$, but $(y, x) \notin R$.
3. R is not transitive \Leftrightarrow there are elements $x, y, z \in A$ such that $x R y$ and $y R z$ but $x \not R z$, i.e. $(x, y), (y, z) \in R$, but $(x, z) \notin R$.

To show that a binary relation does *not* have one of the properties, it is sufficient to find a counterexample.

Let $A = \{0, 1, 2, 3\}$ and define relations R , S , and T :

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$$

$$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$$

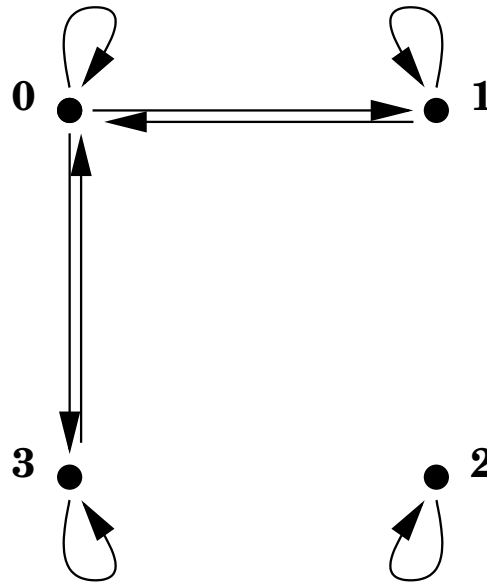
$$T = \{(0, 1), (2, 3)\}$$

Fill in the table:

	Reflexive	Symmetric	Transitive
R			
S			
T			

We have $A = \{0, 1, 2, 3\}$ and

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$$



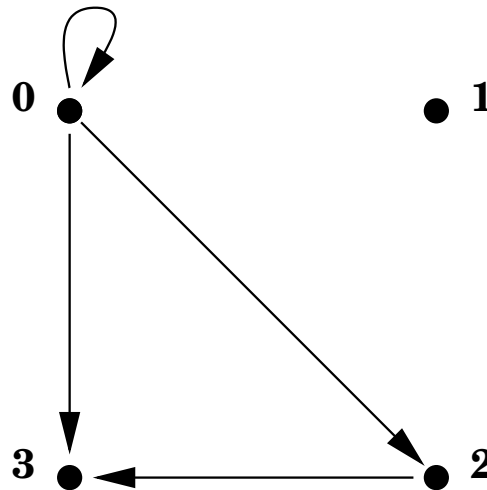
R is reflexive since there is a loop at each point in the directed graph.

R is symmetric since in for every arrow going from one point to another, there is another arrow going back.

R is not transitive since e.g. $1 R 0$ and $0 R 3$ but $1 \not R 3$ i.e. there is no “short-cut” arrow connecting 1 and 3.

We have $A = \{0, 1, 2, 3\}$ and

$$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$$



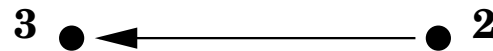
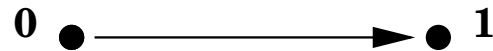
S is not reflexive since there are missing loops at 1, 2, and 3.

S is not symmetric, the arrows from 2-to-0, 3-to-0, and 3-to-2 are missing.

S is transitive since there is always a “short-cut” arrow so that if $(x, y) \in S$ and $(y, z) \in S$ then $(x, z) \in S$.

We have $A = \{0, 1, 2, 3\}$ and

$$T = \{(0, 1), (2, 3)\}$$



T is not reflexive since there are missing loops at 0, 1, 2, and 3.

T is not symmetric, the arrows from 1-to-0, and 3-to-2 are missing.

T is transitive since it is *not not transitive*.

Let $A = \{0, 1, 2, 3\}$ and define relations R , S , and T :

$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$$

$$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$$

$$T = \{(0, 1), (2, 3)\}$$

Fill in the table:

	Reflexive	Symmetric	Transitive
R	Yes	Yes	No
S	No	No	Yes
T	No	No	Yes

Irreflexivity, Anti-Symmetry, and Intransitivity

Definition: Let R be a binary relation on a set A

1. R is **Irreflexive** if and only if $\forall x \in A, x \not R x$.
2. R is **Anti-symmetric** if and only if $\forall x, y \in A$, if $x R y$ then $y \not R x$.
3. R is **Intransitive** if and only if $\forall x, y, z \in A$, if $x R y$ and $y R z$ then $x \not R z$.

- R can be reflexive, non-reflexive, or irreflexive,
- R can be symmetric, non-symmetric, or anti-symmetric
- R can be transitive, non-transitive, or intransitive.

Think about these definitions!!!

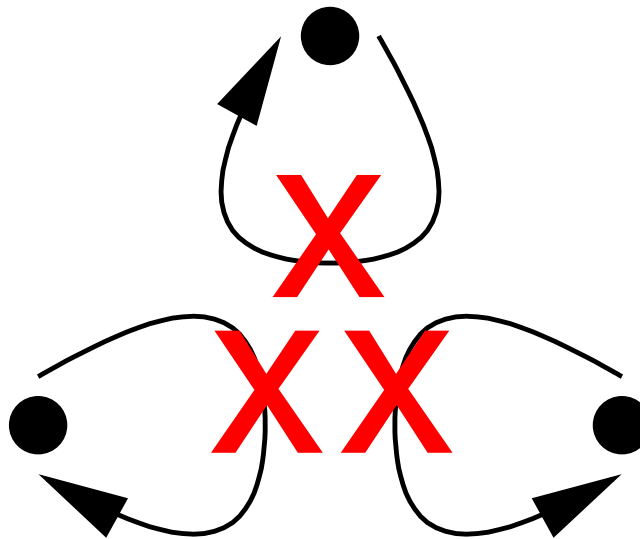
Irreflexivity

Formal: R is **Irreflexive** if and only if $\forall x \in A, x \not R x$.

Functional: R is **Irreflexive** \Leftrightarrow for all $x \in A, (x, x) \notin R$.

Informal: No element is related to itself.

Graph: No point of the graph has an arrow looping around back to itself.



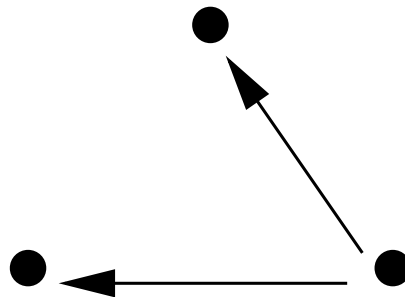
Anti-Symmetry

Formal: R is **Anti-Symmetric** if and only if $\forall x, y \in A$, if $x R y$ then $y \not R x$.

Functional: R is **Anti-Symmetric** \Leftrightarrow for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \notin R$.

Informal: If one element is related to a second element, then the second element is **NOT** related to the first.

Graph: In all cases where there is an arrow going from one point to a second, there is **no** arrow going from the second point back to the first.



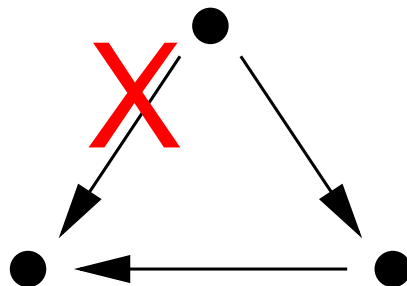
Intransitivity

Formal: R is **Intransitive** if and only if $\forall x, y, z \in A$, if $x R y$ and $y R z$ then $x \not R z$.

Functional: R is **Intransitive** \Leftrightarrow for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \notin R$.

Informal: If one element is related to a second element, and that second element is related to a third element, then the first element is **not** related to the third element.

Graph: In all cases where there is an arrow going from one point to a second, and from the second point to a third, there is **never** an arrow going from the first point to the third (no shortcut exist, anywhere.).



Example: Equality (=) on \mathbb{R}

Let $A = \mathbb{R}$ (the set of real numbers), and define the relation R

$$x R y \iff x = y$$

Properties:

R is reflexive: R is reflexive if and only if $\forall x \in \mathbb{R}, x R x$. Here, this means $x = x$, i.e. $\forall x \in \mathbb{R} x = x$. This statement is certainly true; every real number equals itself.

R is symmetric: This is true since if $x = y$ then $y = x$, hence $(x, y) \in R$ and $(y, x) \in R$.

R is transitive: This is true since if $x = y$ and $y = z$, then $x = z$.

Example: Less Than ($<$) on \mathbb{R}

Let $A = \mathbb{R}$ (the set of real numbers), and define the relation R

$$x R y \iff x < y$$

Properties:

R is irreflexive: If $x R x$ then $x < x$, but that is never true, hence $x \not R x$
 $\forall x \in \mathbb{R}$.

R is anti-symmetric: If $x R y$ then $x < y$, which means $y \not< x$ i.e. $y \not R x$.

R is transitive: This is true since if $x < y$ and $y < z$, then $x < z$.

Example: Congruence Modulo 3 on \mathbb{Z}

We define a relation R on \mathbb{Z} as follows

$$\forall m, n \in \mathbb{Z} : m R n \iff 3|(m - n)$$

R is reflexive: Suppose m is an integer. Now, $m - m = 0$ and $3|0$ since $0 = 3 \cdot 0$, so by definition of R we have $m R m$. \square

R is symmetric: Suppose $m, n \in \mathbb{Z}$ such that $m R n$. By definition of R we have $3|(m - n) \iff m - n = 3 \cdot k$, for some $k \in \mathbb{Z}$. Multiplying both sides by (-1) gives $n - m = 3 \cdot (-k)$, which shows $3|(n - m)$, hence $n R m$. \square

R is transitive: Suppose $m, n, p \in \mathbb{Z}$ such that $m R n$ and $n R p$. We have $3|(m - n)$ and $3|(n - p)$, and we can write $(m - n) = 3r$ and $(n - p) = 3s$ for some $r, s \in \mathbb{Z}$. Adding the two gives $(m - n) + (n - p) = (m - p) = 3(r + s)$ which shows that $3|(m - p)$. Hence $m R p$, and it follows that R is transitive. \square

Equivalence Relations: Different, but the Same...

Idea: We are going to group elements that look different, but really are the same...

Example: Think about the rational numbers, there are several ways of writing the same fraction, e.g.

$$\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{4711}{9422}$$

We can define a relation on $\mathbb{Q} \times \mathbb{Q}$, where \mathbb{Q} is the set of all rational numbers

$$R = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x = y\}$$

now $(\frac{1}{2}, \frac{2}{4}) \in R$, $(\frac{4711}{9422}, \frac{2}{4}) \in R$, $(\frac{1}{3}, \frac{2}{6}) \in R$, etc...

A Relation Induced by a Partition

Recall:

Definition: A collection of non-empty sets $\{A_1, A_2, \dots, A_n\}$ is a **partition** of a set A if and only if

1. $A = A_1 \cup A_2 \cup \dots \cup A_n$.
2. A_1, A_2, \dots, A_n are mutually disjoint.

Definition: Given a partition of a set A the **binary relation induced by the partition**, R , is defined on A as follows

$$\forall x, y \in A, \quad x R y \iff \text{there is a set } A_i \text{ of the partition such that both } x \in A_i \text{ and } y \in A_i.$$

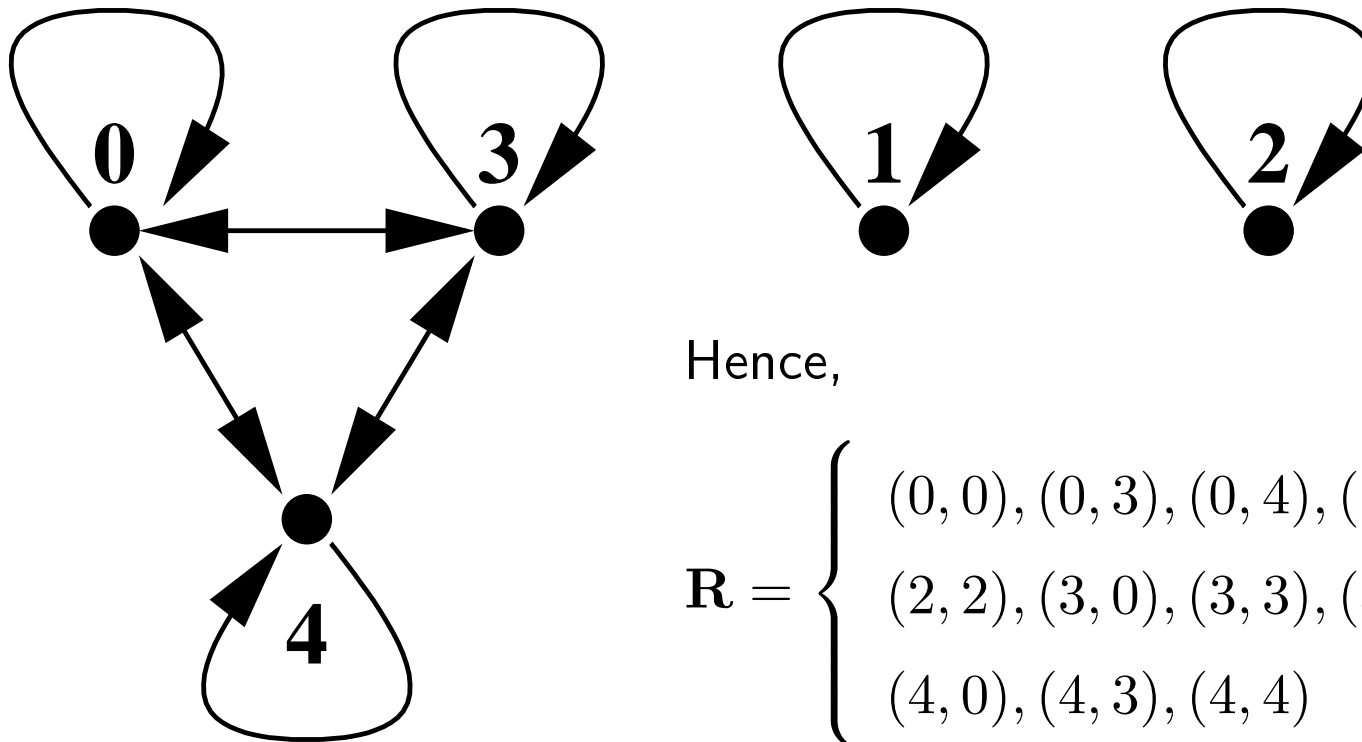
We need an example to make sense out of this definition...

Example: Relation Induced by a Partition

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A :

$$A_1 = \{0, 3, 4\}, \quad A_2 = \{1\}, \quad A_3 = \{2\}$$

Now, two elements $x, y \in A$ are related if and only if they belong to the same subset of the partition...



Equivalence Relations

Theorem: Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Definition: Equivalence Relation —

Let A be a non-empty set and R a binary relation on A . R is an **equivalence relation** if and only if R is reflexive, symmetric, and transitive.

Example: By the theorem the relation induced by a partition is an equivalence relation.

Notation: Congruence Modulo n

Notation: Let $m, n, d \in \mathbb{Z}$ with $d > 0$. The notation

$$m \equiv n \pmod{d}$$

is read “ m is congruent to n modulo d ” and means that

$$d \mid (m - n)$$

Symbolically,

$$m \equiv n \pmod{d} \quad \Leftrightarrow \quad d \mid (m - n)$$

Recall the ***Quotient-Remainder Theorem***:

Theorem: Given any integer n and a positive integer d , there exist unique integers q (the quotient) and r (the remainder) such that

$$n = d \cdot q + r, \quad \text{and} \quad 0 \leq r < d$$

Equivalence Relation: Congruence Modulo 3

Let R be the relation $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m = n \pmod{3}\}$. We show that this is an equivalence relation.

[Reflexivity] Let $m \in \mathbb{Z}$, then $3|(m - m)$ since $0 = 3 \cdot 0$, and it follows that $m R m$.

[Symmetry] Let $m, n \in \mathbb{Z}$, so that $m R n$. We have $3|(m - n) \Leftrightarrow (m - n) = 3 \cdot k$ for some $k \in \mathbb{Z} \Leftrightarrow (n - m) = 3 \cdot (-k) \Leftrightarrow 3|(n - m) \Leftrightarrow n R m$.

[Transitivity] Let $m, n, p \in \mathbb{Z}$, so that $m R n$ and $n R p$. We have

$$3|(m - n) \Leftrightarrow (m - n) = 3 \cdot r, \quad r \in \mathbb{Z}$$

$$3|(n - p) \Leftrightarrow (n - p) = 3 \cdot s, \quad s \in \mathbb{Z}$$

$$\mathbf{add} \quad (m - p) = 3 \cdot (r + s)$$

Hence $3|(m - p)$ and we have $m R p$. \square

Equivalence Classes

Suppose we have a set A and an equivalence relation R on A . Given a particular element $x \in A$ it is natural to ask the question “*what elements are related to x ?*”

All the elements that are related to x form a subset of A and this subset is called *the equivalence class of x* :

Definition: Suppose A is a set and R is an equivalence relation on A . For each element $x \in A$, the **equivalence class of x** , denoted $[x]$ and called the **class of x** for short, is the set of all elements $y \in A$ such that $y R x$.

Symbolically,

$$[x] = \{y \in A \mid y R x\}$$

Let $A = \{0, 1, 2, 3, 4\}$ and define a binary relation R on A

$$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$$

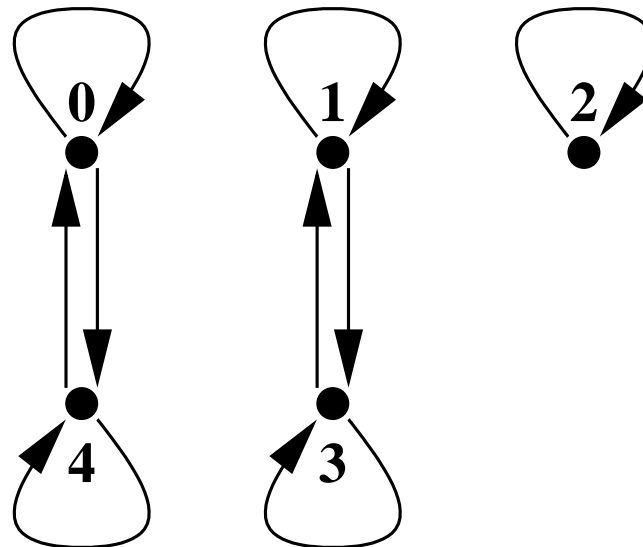
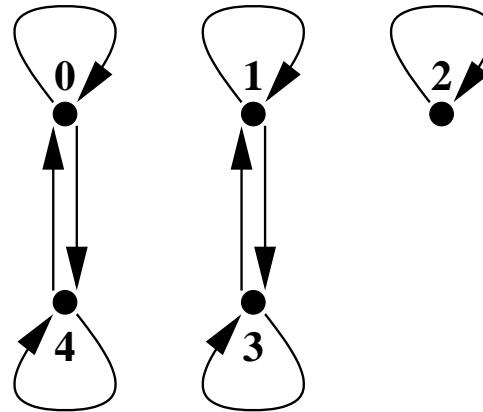


Figure: The array diagram (directed graph) corresponding to the relation.

By quick inspection we see that R is reflexive, symmetric, and transitive, hence an equivalence relation.



The equivalence classes are:

$$\begin{aligned}
 [0] &= \{x \in A \mid x R 0\} = \{0, 4\} \\
 [1] &= \{x \in A \mid x R 1\} = \{1, 3\} \\
 [2] &= \{x \in A \mid x R 2\} = \{2\} \\
 [3] &= \{x \in A \mid x R 3\} = \{1, 3\} \\
 [4] &= \{x \in A \mid x R 4\} = \{0, 4\}
 \end{aligned}$$

Note that $[0] = [4]$ and $[1] = [3]$, hence the *distinct* equivalence classes are: $\{0, 4\}$, $\{1, 3\}$, $\{2\}$.

Equivalence Classes: A Theorem

The following theorem tells us that an equivalence relation induces a partition:

Theorem: If A is a non-empty set and R is an equivalence relation on A , then the distinct equivalence classes of R form a partition of A ; *i.e.* the union of the equivalence classes is all of A and the intersection of any two distinct classes is empty.

Example: Equivalence Classes of Congruence Modulo 3 1 of 3

Let R be the relation of congruence modulo 3 on the set \mathbb{Z} , *i.e.*

$$\forall m, n \in \mathbb{Z}$$

$$m R n \Leftrightarrow 3|(m - n) \Leftrightarrow m \equiv n \pmod{3}$$

We describe the equivalence classes: For each integer a , the class of a is

$$\begin{aligned} [a] &= \{x \in \mathbb{Z} \mid x R a\} \\ &= \{x \in \mathbb{Z} \mid 3|(x - a)\} \\ &= \{x \in \mathbb{Z} \mid x - a = 3 \cdot k, k \in \mathbb{Z}\} \\ &= \{x \in \mathbb{Z} \mid x = 3 \cdot k + a, k \in \mathbb{Z}\} \end{aligned}$$

In particular

$$[0] = \{x \in \mathbb{Z} \mid x = 3 \cdot k, k \in \mathbb{Z}\} = \{0, 3, -3, 6, -6, 9, -9, \dots\}$$

$$[1] = \{x \in \mathbb{Z} \mid x = 3 \cdot k + 1, k \in \mathbb{Z}\} = \{1, 4, -2, 7, -5, 10, -8, \dots\}$$

$$[2] = \{x \in \mathbb{Z} \mid x = 3 \cdot k + 2, k \in \mathbb{Z}\} = \{2, 5, -1, 8, -4, 11, -7, \dots\}$$

We have

$$[0] = \{x \in \mathbb{Z} \mid x = 3 \cdot k, k \in \mathbb{Z}\} = \{0, 3, -3, 6, -6, 9, -9, \dots\}$$

$$[1] = \{x \in \mathbb{Z} \mid x = 3 \cdot k + 1, k \in \mathbb{Z}\} = \{1, 4, -2, 7, -5, 10, -8, \dots\}$$

$$[2] = \{x \in \mathbb{Z} \mid x = 3 \cdot k + 2, k \in \mathbb{Z}\} = \{2, 5, -1, 8, -4, 11, -7, \dots\}$$

By lemma#1

$$[0] = [3] = [-3] = [6] = [-6] = [9] = [-9] = \dots$$

$$[1] = [4] = [-2] = [7] = [-5] = [10] = [-8] = \dots$$

$$[2] = [5] = [-1] = [8] = [-4] = [11] = [-7] = \dots$$

Hence the distinct equivalence classes are

$$\{x \in \mathbb{Z} \mid x = 3 \cdot k, k \in \mathbb{Z}\}, \quad \{x \in \mathbb{Z} \mid x = 3 \cdot k + 1, k \in \mathbb{Z}\},$$

$$\{x \in \mathbb{Z} \mid x = 3 \cdot k + 2, k \in \mathbb{Z}\}$$

The distinct equivalence classes are

$$\{x \in \mathbb{Z} \mid x = 3 \cdot k, k \in \mathbb{Z}\}, \quad \{x \in \mathbb{Z} \mid x = 3 \cdot k + 1, k \in \mathbb{Z}\},$$

$$\{x \in \mathbb{Z} \mid x = 3 \cdot k + 2, k \in \mathbb{Z}\}$$

The class of $[0]$ can also be called the class of $[3]$ or the class of $[96]$, but the class *is* the set $\{x \in \mathbb{Z} \mid x = 3 \cdot k, k \in \mathbb{Z}\}$.

Definition: Suppose R is an equivalence relation on a set A and S is an equivalence class of R . A **representative** of the class S is any element $a \in A$ such that $[a] = S$.

Notes

- It is possible to define multiplication and addition of the equivalence classes corresponding to the rational numbers (previous example).
- The rational numbers can be defined as equivalence classes of ordered integers.
- It can be shown that all integers — negative, zero, and positive — can be defined as equivalence classes of ordered pairs of positive integers.
- Frege and Peano showed (late 19th century) that the positive integers can be defined entirely in terms of sets.
- Dedekind (1848–1916) showed that all real numbers can be defined as sets of rational numbers.
- All together, these results show that the real numbers can be defined using logic and set theory alone!

(Epp-v3.0)

*10.1.1, 10.1.5, 10.1.7, 10.1.15, 10.1.23, 10.1.25, 10.2.3, 10.2.4,
10.2.12, 10.2.14, 10.2.37, 10.3.3, 10.3.17, 10.3.19, 10.3.40*

(Epp-v2.0)

*10.1.1, 10.1.5, 10.1.7, 10.1.15, 10.1.23, 10.1.25, 10.2.3, 10.2.4,
10.2.12, 10.2.14, 10.2.37, 10.3.2, 10.3.14, 10.3.16, 10.3.35*