The Logic of Quantified Statements

Predicates and Quantified Statement; Statements Containing Multiple Quantifiers; Arguments with Quantified Statements

Lecture Notes #4

Peter Blomgren
Department of Mathematics and Statistics
San Diego State University
San Diego, CA 92182-7720
blomgren@terminus.SDSU.EDU
http://terminus.SDSU.EDU

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So far we have discussed **statement calculus** (or **propositional calculus**) — *i.e.* symbolic analysis of compound statements.

We have introduced ***logical connectives*** such as $\wedge$, $\vee$, $\sim$, $\rightarrow$, and $\leftrightarrow$.

We have created quite a useful toolbox — it is quite sufficient if you want to build microchips for a living... You can indeed live quite large if you build microchips!

We cannot however determine if the following is a valid statement:

All humans need logic
Peter is human
$\therefore$ Peter needs logic

In order to study intuitively valid arguments

All humans need logic

Peter is human

$\therefore$ Peter needs logic

We must understand (in the logic sense) the meaning of words like ***all***, ***some***, etc...

Further, we must separate our statements into parts in much the same way we separate declarative statements into subject and predicates.

The symbolic analysis of predicates and quantified statements is called **predicate calculus.**

Predicates

In grammar, the **predicate** refers to the part of the sentence which gives information about the subject, *e.g.*

$$\underbrace{\text{Peter}}_{\text{subject}}\ \underbrace{\text{is a professor at SDSU}}_{\text{predicate}}$$

The predicate is the part of the sentence from which the subject has been removed.

In logic, predicates are obtained by removing any (some) nouns from a statement.

$$\begin{aligned} P &= \text{“\ldots is a professor at SDSU”} \\ Q &= \text{“\ldots is a professor at \ldots”} \\ R &= \text{“\ldots is a \ldots at \ldots”} \end{aligned}$$

Example: P, Q and R are predicate symbols.

Predicates

We use the **predicate variables** x and y to define predicates $P(x)$ and $Q(x,y)$:

$$\begin{aligned} \mathrm{P}(x) &= \text{“}x\text{ is a professor at SDSU”} \\ \mathrm{Q}(x,y) &= \text{“}x\text{ is a professor at }y\text{”} \\ \mathrm{R}(x,y,z) &= \text{“}x\text{ is a }z\text{ at }y\text{”} \end{aligned}$$

Example: P(x), Q(x,y) and R(x,y,z) are predicate symbols, x, y and z are predicate variables.

***Definition*: Predicate —**
A **predicate** is sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

We must specify the domains of our predicate variables.

If A is a set, we say that x ***is a member of*** the set A — denoted $\mathbf{x} \in \mathbf{A}$. If x ***is not a member of*** the set A we write $\mathbf{x} \notin \mathbf{A}$.

A could be the set of all students at SDSU.

The following sets are so common, they have their own reserved symbols:

Symbol	**Set of...**
$\mathbb{C}$	Set of all Complex Numbers
$\mathbb{N}$	Set of all Non-negative Integers
$\mathbb{Q}$	Set of all Rational Numbers
$\mathbb{R}$	Set of all Real Numbers
$\mathbb{Z}$	Set of all Integers

Truth Set

When an element in the domain of a variable of a one-variable predicate is substituted for the variable, the resulting statement ***is either* `true` *or* `false`*.**

> ***Definition*: Truth Set —**
>
> If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ **`true`** when substituted for x. The truth set of $P(x)$ is denoted
>
> $$\{x \in D \,|\, P(x)\}$$
>
> which is read "the set of all x in D such that $P(x)$."

Examples: Truth Sets

Example #1:

Let $P(x) =$ "x is a senior" and suppose the domain, D, of x is the set of all SDSU students. Then the truth set of $P(x)$ is the set of all SDSU students in senior standing.

Example #2:

Let $P(x) =$ "3 is a factor of x" and $D = \mathbb{N}$. Then $\{x \in D \,|\, P(x)\}$ $= \{3, 6, 9, 12, 15, \ldots\}$.

Example #3:

Let $P(x) =$ "x is a factor of 8", and $D = \mathbb{N}$. Then the truth set of $P(x)$ is $\{1, 2, 4, 8\}$.

More symbols: $\Rightarrow$, and $\Leftrightarrow$

Let $P(x)$ and $Q(x)$ predicates and suppose they have a common domain $x \in D$.

The notation $P(x) \Rightarrow Q(x)$ means that every element in the truth set of $P(x)$ is in the truth set of $Q(x)$.

The notation $P(x) \Leftrightarrow Q(x)$ means that $P(x)$ and $Q(x)$ have identical truth sets.

Example: $\Rightarrow$, and $\Leftrightarrow$

Example:

Let

$P(x) =$ "x is a factor of 8"

$Q(x) =$ "x is a factor of 4"

$R(x) =$ "$x < 5$ and $x \neq 3$"

$D = \mathbb{Z}^+$ (the set of positive integers).

The truth sets are

$\{x \in D \mid P(x)\} = \{1, 2, 4, 8\}$

$\{x \in D \mid Q(x)\} = \{1, 2, 4\}$

$\{x \in D \mid R(x)\} = \{1, 2, 4\}$

We have the following

$Q(x) \Rightarrow P(x), \quad R(x) \Rightarrow P(x)$

$Q(x) \Leftrightarrow R(x).$

The Universal Quantifier — "For all" Symbol: $\forall$

The symbol $\forall$ denotes "for all" and is called the **universal quantifier**.

If we let S be the set of all humans beings, we can write

$$\forall x \in S, \quad x \text{ is mortal}$$

The following phrases translate to $\forall$:

"for all"	"for every"
"for arbitrary"	"for any"
"for each"	"given any"

The Existential Quantifier — "There exists" Symbol: $\exists$

The symbol $\exists$ denotes "there exists" and is called the **existential quantifier**.

If we let S be the set of all humans beings, we can write

$$\exists\, x \in S, \text{ such that } x \text{ is student in Math 245}$$

The following phrases translate to $\exists$:

"there exists"	"there is a"
"we can find a"	"there is at least a"
"for some"	"for at least one"

Formal Definitions: Universal and Existential Statements

Definition: Universal Statement —
Let $Q(x)$ be a predicate and D the domain of x. A **universal statement** is a statement in the form "$\forall x \in D, Q(x)$." It is defined to be **`true`** if, and only if, $Q(x)$ is **`true`** for every x in D. A value for which $Q(x)$ is **`false`** is called a **counterexample** to the universal statement.

Definition: Existential Statement —
Let $Q(x)$ be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists\, x \in D$ such that $Q(x)$." It is defined to be **`true`** if, and only if, $Q(x)$ is **`true`** for at least one x in D. If is false if, and only if, $Q(x)$ is **`false`** for all x in D.

Translating to Formal (Symbolic) Language

Example: Rewrite the following statements formally.

1. "All triangles have three sides."

2. "No dogs have wings."

3. "Some programs are structured."

Solutions:

1a. Let T be the set of triangles; $\forall t \in T$, t has three sides.

1b. $\forall$ triangles t, t has three sides.

2a. Let D be the set of all dogs; $\forall d \in D$, d does not have wings.

2b. $\forall$ dogs d, d does not have wings.

3a. Let P be the set of all programs; $\exists p \in P$ such that p is structured.

3b. $\exists$ a program p such that p is structured.

Universal Conditional Statements

One of the most important forms of statements in mathematics (in proofs and theorems) is the **universal conditional statement**

$$\forall x, \text{ if } P(x) \text{ then } Q(x)$$

Example #1

Let the domain of x be the positive integers $\mathbb{Z}^+$,
$P(x) =$ "x is prime",
$Q(x) =$ "x cannot be factored".

We make the statement
$\forall x \in \mathbb{Z}^+$, if $P(x)$ then $Q(x)$.

Example #2

The definition of a valid argument form is a universal conditional statement: $\forall$ combinations of truth values for the component statements, **if** the premises are all true, **then** the conclusion is also true.

Consider the two statements:

"$\forall$ real numbers x, if x is an integer, then x is a rational"

"$\forall$ integers x, x is a rational"

They mean the same thing!

In general given a statement of the form

$$\forall x \in U, \text{ if } P(x) \text{ then } Q(x)$$

and the truth set D for $P(x)$:

$$D = \{x \in U \mid P(x)\}$$

the statement can be rewritten as

$$\forall x \in D, \; Q(x)$$

Consider the two statements:

"$\exists$ a number n such that n is prime and n is even"

"$\exists$ a prime n such that n is even"

They mean the same thing!

In general given a statement of the form

$$\exists\, x \in U \text{ such that } P(x) \text{ and } Q(x)$$

and the truth set D for $P(x)$:

$$D = \{x \in U \mid P(x)\}$$

the statement can be rewritten as

$$\exists\, x \in D \text{ such that } Q(x)$$

Implicit Quantification **"There Be Dragons Here!"**

The statement ***"if a number is an integer, then it is a rational number."*** is equivalent to a universal statement (see slide 16).

However, it does not contain any of the telltale $\forall$-phrases (see slide 11).

The only indication of universal quantification is the indefinite article — ***"a"***.

This is an example of ***implicit quantification***.

The quantification of a statement crucially determines both how the statement can be applied and what method must be used to establish its truth. Thus is is important to be alert to the presence of hidden (implicit) quantifiers when reading mathematics so that statements are interpreted in a logically correct way.

Examples: Implicit Quantification

The informal statement

"24 can be written as a sum of two integers"

Formally means

"$\exists$ integers n, m such that $24 = m + n$."

Consider:

a. $(x+1)^2 = x^2 + 2x + 1$.

b. Solve $(x+2)^2 = 25$.

a. is ***implicitly universally quantified***, and

b. ***implicitly existentially quantified***:

a. $\forall x \in \mathbb{R}$, $(x+1)^2 = x^2 + 2x + 1$.

b. Show that $\exists\, x \in \mathbb{R}$ such that $(x+2)^2 = 25$.

Homework #2 — Due Friday 9/22/2006

version $\frac{1}{3}$

3rd Edition	**2nd Edition**
Problems	
2.1: 12, 14, 22	***2.1:*** 7, 9, 16

Please use the ***3rd Edition*** numbering when handing in your solutions.

Negations

The negation of "All mathematicians are strange" is **not** "No mathematicians are strange" it is...

$\sim$ "All mathematicians are strange"

"Some mathematicians are not strange"

***Theorem*: Negation of a Universal Statement —**
The negation of a statement of the form

$$\forall x \in D,\ Q(x)$$

is logically equivalent to a statement of the form

$$\exists\, x \in D \text{ such that } \sim Q(x).$$

Symbolically:

$$\sim (\forall x \in D,\ Q(x)) \quad \equiv \quad \exists\, x \in D \text{ such that } \sim Q(x).$$

Negations

The negation of "Some mathematicians are strange" is **not** "Some mathematicians are not strange" it is...

$\sim$ "Some mathematicians are strange"

"No mathematicians are strange"

***Theorem*: Negation of an Existential Statement —**

The negation of a statement of the form

$$\exists\, x \in D \text{ such that } Q(x)$$

is logically equivalent to a statement of the form

$$\forall\, x \in D,\ \sim Q(x).$$

Symbolically:

$$\sim (\exists\, x \in D \text{ such that } Q(x)) \quad \equiv \quad \forall\, x \in D,\ \sim Q(x).$$

Negations — Notes

The negation of a universal statement ("$\forall$" / "all are") is logically equivalent to an existential statement ("$\exists \cdots \sim$" / "some are not").

The negation of an existential statement ("$\exists$" / "some are") is logically equivalent to a universal statement ("$\forall \cdots \sim$" / "all are not").

Negation of Universal Conditional Statements

The negation of a Universal Conditional Statement

$$\sim (\forall x,\, P(x) \to Q(x))$$

is very important in mathematical arguments.

We already know how to negate a forall-statement:

$$\exists\, \mathbf{x} \text{ such that } \sim(P(x) \to Q(x))$$

And we know how to negate an implication, thus

$$\begin{aligned} & \sim (\forall x,\, P(x) \to Q(x)) \\ \equiv \quad & \exists\, x \text{ such that } \mathbf{P}(\mathbf{x}) \wedge (\sim \mathbf{Q}(\mathbf{x})) \end{aligned}$$

or

$$\sim (\forall x,\ \text{if } P(x) \text{ then } Q(x)) \equiv \exists\, x \text{ such that } P(x) \text{ and } (\sim Q(x)).$$

We have seen that the negation of a ***for all*** statement is a ***there is*** statement, and the other way around.

This is analogous to De Morgan's Laws, where the negation of an ***and*** statement is and ***or*** statement, and vice versa.

A universal statement is a generalization of the ***and*** statement: If $Q(x)$ is a predicate, and the domain D of the predicate variable x is the set $\{x_1, x_2, \ldots, x_n\}$, then the statements

$$\forall \mathbf{x} \in \mathbf{D}, \mathbf{Q}(\mathbf{x})$$

and

$$\mathbf{Q}(\mathbf{x_1}) \wedge \mathbf{Q}(\mathbf{x_2}) \wedge \cdots \wedge \mathbf{Q}(\mathbf{x_n})$$

are logically equivalent!

An existential statement is a generalization of the ***or*** statement:

If $Q(x)$ is a predicate, and the domain D of the predicate variable x is the set $\{x_1, x_2, \ldots, x_n\}$, then the statements

$$\exists \mathbf{x} \in \mathbf{D} \text{ such that } \mathbf{Q}(\mathbf{x})$$

and

$$\mathbf{Q}(\mathbf{x_1}) \vee \mathbf{Q}(\mathbf{x_2}) \vee \cdots \vee \mathbf{Q}(\mathbf{x_n})$$

are logically equivalent!

Now,

$$\sim (\forall x \in D, Q(x)) \quad \equiv \quad \sim (Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n))$$

is logically equivalent to

$$\exists x \in D, \sim Q(x) \quad \equiv \quad \sim Q(x_1) \vee \sim Q(x_2) \vee \ldots \vee \sim Q(x_n)$$

And,

$$\sim (\exists x \in D, Q(x)) \quad \equiv \quad \sim (Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n))$$

is logically equivalent to

$$\forall x \in D, \sim Q(x) \quad \equiv \quad \sim Q(x_1) \wedge \sim Q(x_2) \wedge \ldots \wedge \sim Q(x_n)$$

Contrapositive, Converse, and Inverse Universal C.S.

We know that a conditional statement has a contrapositive, a converse, and an inverse. We can generalize these definition to universal conditional statements:

***Definition*: Contrapositive, Converse, Inverse —**

Consider a statement of the form

$$\forall x \in D, \text{ if } P(x) \text{ then } Q(x)$$

1. Its **contrapositive** is the statement

$$\forall x \in D, \text{ if } (\sim Q(x)) \text{ then } (\sim P(x)).$$

2. Its **converse** is the statement

$$\forall x \in D, \text{ if } Q(x) \text{ then } P(x).$$

3. Its **inverse** is the statement

$$\forall x \in D, \text{ if } (\sim P(x)) \text{ then } (\sim Q(x)).$$

Example: CC&I of Universal Conditional Statements

Write the contrapositive, converse and inverse of the following statement:

"If a real number is greater than 2,
then its square is greater than 4."

Formal: $\forall r \in \mathbb{R}$, if $r > 2$, then $r^2 > 4$.

Contrapositive: $\forall r \in \mathbb{R}$, if $r^2 \leq 4$, then $r \leq 2$.

Converse: $\forall r \in \mathbb{R}$, if $r^2 > 4$ then $r > 2$.

Inverse: $\forall r \in \mathbb{R}$, if $r \leq 2$ then $r^2 \leq 4$.

Formal $\equiv$ **Contrapositive** $\not\equiv$ **Converse** $\equiv$ **Inverse**

More Extensions to Universal Conditional Statements

Further, we can extend the definitions of ***necessary***, ***sufficient*** and ***only if*** to apply to universal conditional statements:

Definition: **Sufficient, Necessary, Only If —**

1. "$\forall x$, $r(x)$ is a **sufficient condition** for $s(x)$" means "$\forall x$, if $r(x)$ then $s(x)$."

2. "$\forall x$, $r(x)$ is a **necessary condition** for $s(x)$", means "$\forall x$, if $(\sim r(x))$ then $(\sim s(x))$" or, equivalently, "$\forall x$, if $s(x)$ then $r(x)$."

3. "$\forall x$, $r(x)$ **only if** $s(x)$" means "$\forall x$, if $(\sim s(x))$, then $(\sim r(x))$" or equivalently, "$\forall x$, if $r(x)$ then $s(x)$."

Example: Necessary and Sufficient Conditions

Rewrite the following statements as quantified conditional statements without using the words *necessary* or *sufficient*:

1. Squareness is a sufficient condition for rectangularity.

2. Being at least 35 years old is a necessary condition for being President of the United States.

1. Let S be the set of shapes. $\forall x \in S$ if x is a square, then x is a rectangle. — ***"If a shape is a square, then it is a rectangle."***

2. Let H be the set of human beings. $\forall x \in H$, if x is younger than 35 years old, then x cannot be the President of the United States. **Using the contrapositive:** $\forall x \in H$**, if** x **is the President of the United States, then** x **is at least 35 years old.**

Example: Only If

Rewrite the following as a universal conditional statement:

"A product of two numbers is 0 only if one of the numbers is 0."

Solution:

"If neither of the two numbers is 0,
then the product of the numbers is not 0."

"Let $r_1 \in \mathbb{R}$, and $r_2 \in \mathbb{R}$. If $r_1 \neq 0$ and $r_2 \neq 0$, then $r_1 \cdot r_2 \neq 0$."

"If a product of two numbers is 0,
then [at least] one of the numbers is 0."

"Let $r_1 \in \mathbb{R}$, and $r_2 \in \mathbb{R}$. If $r_1 \cdot r_2 = 0$, then $r_1 = 0$ or $r_2 = 0$."

Here, we have used the equivalent contrapositive form.

Homework #2 — Due Friday 9/22/2006 **version $\frac{2}{3}$**

3rd Edition	2nd Edition
Problems	
2.2: 15, 17, 29, 40	***2.1:*** 28, 23; ***2.2:*** —, 33
2.1: 12, 14, 22	***2.1:*** 7, 9, 16

Please use the ***3rd Edition*** numbering when handing in your solutions.

Multiply Quantified Statements

In the previous section we expanded our "logic vocabulary" to include quantifiers, *e.g.* $\forall$ (for all) and $\exists$ (there exists).

Next, we are going to construct more complicated statements using these quantifiers — in particular we look at statement which contain more than one quantifier.

Examples of Multiply Quantified Statement

First off, lets translate the following informal statements to formal (symbolic) statements:

1. Everybody loves somebody.
2. Somebody loves everybody.

1. Let H be the set of human beings.
   $\forall x \in H$, $\exists y \in H$, such that x loves y.

2. Let H be the set of human beings.
   $\exists x \in H$ such that $\forall y \in H$, x loves y.

Examples of Multiply Quantified Statement

In calculus we define the limit of a sequence... *"The limit of the sequence a_n as n goes to infinity equals to L,*

$$\lim_{n \to \infty} a_n = L$$

if, and only if, the values of a_n become arbitrarily close to L as n grows. More precisely, this means that for any positive number ϵ, we can find and integer N such that whenever n is larger than N, then the number a_n is in the interval between $L-\epsilon$ and $L+\epsilon$."

Symbolically:

$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, **such that** $\forall n$,
if $n > N$, **then** $L - \epsilon < a_n < L + \epsilon$.

Once you know the symbols, this is a very effective way of communicating!

Negations of Multiply Quantified Statements

How do we negate a statement like "Everybody loves somebody."

> Let H be the set of human beings.
> $\forall x \in H$, $\exists y \in H$, such that x loves y.

The negation of the statement must be false when the statement is true.

Since the statement talks about a property assumed to be true for all people, all we need is the existence of a ***counterexample***:

> Let H be the set of human beings.
> $\exists x \in H$ such that $\sim(\exists y \in H$, such that x loves y.)
> $\Leftrightarrow \exists x \in H$ such that ($\forall y \in H$, x does not love y.)

That is

$$\sim \text{“Everybody loves somebody”}$$

is logically equivalent to

“There is somebody who does not love anybody.”

The argument we made can be generalized:

> The negation of
>
> $$\forall x,\ \exists y \text{ such that } P(x, y)$$
>
> is logically equivalent to
>
> $$\exists x \text{ such that } \forall y,\ \sim P(x, y)$$

In our example, $P(x, y) =$“x loves y.”

Negation Rules

We have

The negation of

$$\forall x,\ \exists y \text{ such that } P(x,y)$$

is logically equivalent to

$$\exists x \text{ such that } \forall y,\ \sim P(x,y)$$

Similarly,

The negation of

$$\exists x \text{ such that } \forall y,\ P(x,y)$$

is logically equivalent to

$$\forall x,\ \exists y, \text{ such that } \sim P(x,y)$$

Example: Negating Multiply Quantified Statements

Negate each of the following statements:

1. $\forall n \in \mathbb{Z}$, $\exists k \in \mathbb{Z}$ such that $n = 2k$. — "All integers are even."

2. $\exists$ a person x such that $\forall$ people y, x loves y. — "Somebody loves everybody."

1.1 $\exists n \in \mathbb{Z}$ such that $\sim$($\exists k \in \mathbb{Z}$ such than $n = 2k$).

1.2 $\exists n \in \mathbb{Z}$ such that $\forall k \in \mathbb{Z}$, $n \neq 2k$. — "There is some integer which is not equal to twice any other integer."

2.1 $\forall$ people x, $\sim$($\forall$ people y, x loves y).

2.2 $\forall$ people x, $\exists$ person y such that x does not love y. — "Nobody loves everybody."

A Quick Recap

We have introduced ***predicates***, $P(x)$ — sentences with a finite number of variables which become statements when specific values are substituted for the variables.

We talked about the ***truth set*** of a predicate — all the values of the variable(s) which make the predicate **true**.

We added the concepts of ***universal statements*** ($\forall$ / "for all"), ***existential statements*** ($\exists$ / "there exists"), and **conditional statements** ("if ... then ...") to our vocabulary.

To enable us to express more complex statements, we studied ***multiply quantified statements***.

Things we do to all our statements: finding truth values, writing the contrapositive, the converse, the inverse, and negating the statement.

Universal Instantiation

The Rule of Universal Instantiation:

> If some property is true of ***everything*** in the domain, then
>
> it is true of ***any particular*** thing in the domain.

Example:

All students want to graduate

Jane is a student

$\therefore$ Jane wants to graduate

Universal instantiation is ***the*** fundamental tool of deductive reasoning: it is used mathematical formulas, definitions, and theorems as well as in all kinds of everyday, legal, etc., arguments.

Recall **Modus Ponens** ("The method of affirming"):

If **p**, then **q**.
p.
$\therefore$ **q**

If we combine the rule of universal instantiation with modus ponens, we get universal modus ponens:

Universal Modus Ponens

	Formal version		***Informal version***
	$\forall x$, If $P(x)$, then $Q(x)$		If x makes $P(x)$ true, then x makes $Q(x)$ true
	$P(a)$ for a particular a		a makes $P(x)$ true
$\therefore$	$Q(a)$	$\therefore$	a makes $Q(x)$ true

Universal Modus Ponens

	Formal version		***Informal version***
	$\forall x$, If $P(x)$, then $Q(x)$		If x makes $P(x)$ true, then x makes $Q(x)$ true
	$P(a)$ for a particular a		a makes $P(x)$ true
$\therefore$	$Q(a)$	$\therefore$	a makes $Q(x)$ true

Universal Modus Ponens consists of two premises:

$\forall x$, If $P(x)$, then $Q(x)$ ***premise-1***

$P(a)$ for a particular a ***premise-2***

one of which (***1***) is quantified. An argument of this form is called a **syllogism** (rule of inference).

The first and second premise are called the **major** and **minor** premises, respectively.

Part of the reason we are building our "logic toolbox" is that we are gearing up to discussing methods of proving quantified statements — one of the most important activities in mathematical research.

For illustration, let us break the proof that ***"the sum of two even integers is even"*** into its smallest parts, and show how universal modus ponens guides us...

> Suppose m and n are particular, but arbitrarily chosen even integers. Then $m = 2r$ for some integer r (***ump-1***), and $n = 2s$ for some integer s (***ump-2***). Hence
>
> $$\begin{aligned} m+n &= 2r+2s && \text{by substitution} \\ &= 2(r+s) && \text{by factoring out the 2 } (\textbf{\textit{ump-3}}) \end{aligned}$$
>
> Now, $(r+s)$ is an integer (***ump-4***), and so is $2(r+s)$ (***ump-5***). Thus $(m+n)$ is even.

ump-1
If an integer is even, then it equals twice some integer.
m is a particular even integer.
$\therefore$ m equals twice some integer r.

ump-2
If an integer is even, then it equals twice some integer.
n is a particular even integer.
$\therefore$ n equals twice some integer s.

ump-3
If a quantity is an integer, then it is a real number.
r and s are integers.
$\therefore$ r and s are real numbers, and $2r + 2s = 2(r + s)$.

ump-4
For all m and n, if m and n are integers, then $(m + n)$ is an integer.
$m = r$ and $n = s$ are two particular integers.
$\therefore$ $(r + s)$ is an integer.

ump-5
If a number equals twice some integer, then that number is even.
$2(r + s)$ equals twice the integer $(r + s)$.
$\therefore$ $2(r + s)$ is even.

Recall **Modus Tollens** ("The method of denying"):

If **p**, then **q**.
$\sim$**q**.
$\therefore$ $\sim$**p**

If we combine the rule of universal instantiation with modus tollens, we get universal modus tollens:

Universal Modus Tollens

	Formal version		***Informal version***
	$\forall x$, If $P(x)$, then $Q(x)$		If x makes $P(x)$ true, then x makes $Q(x)$ true
	$\sim Q(a)$ for a particular a		a does not make $Q(x)$ true
$\therefore$	$\sim P(a)$	$\therefore$	a does not make $P(x)$ true

Universal Modus Tollens

	Formal version		***Informal version***
	$\forall x$, If $P(x)$, then $Q(x)$		If x makes $P(x)$ true, then x makes $Q(x)$ true
	$\sim Q(a)$ for a particular a		a does not make $Q(x)$ true
$\therefore$	$\sim P(a)$	$\therefore$	a does not make $P(x)$ true

Universal modus tollens is the key to mathematical proofs of contradiction — one of the most important mathematical arguments.

Example

All human beings are mortal

Zeus is not mortal

$\therefore$ Zeus is not human.

Proving Validity of Arguments with Quantified Statements

Definition:

To say that and ***argument form*** is **valid** means the following: No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

An ***argument*** is called **valid** if, and only if, its form is valid.

Note: If you think this looks familiar... it is a straight-forward generalization of the validity for statements with compound statements.

Note: We have to use the laws of logic to prove that the laws of logic are valid!

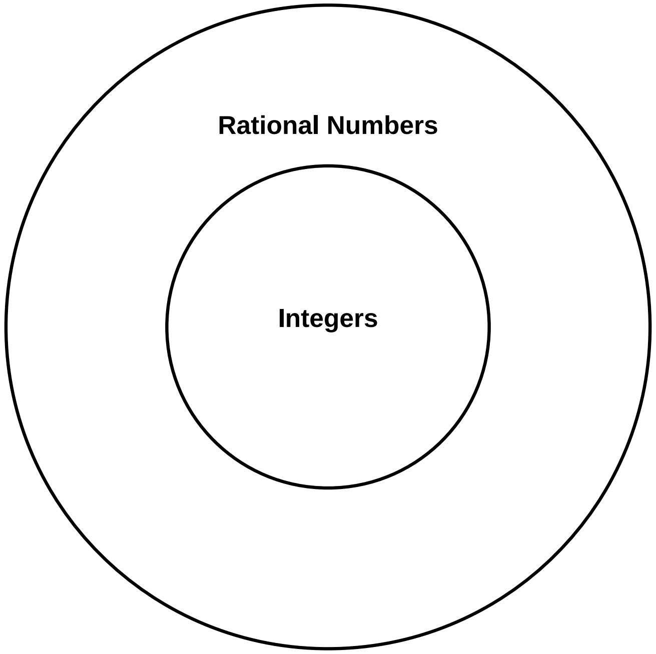
Note: Proving that a general quantified statement form is valid is beyond the scope of this class.

Finally, Something Less Abstract — "Proof by Diagram"

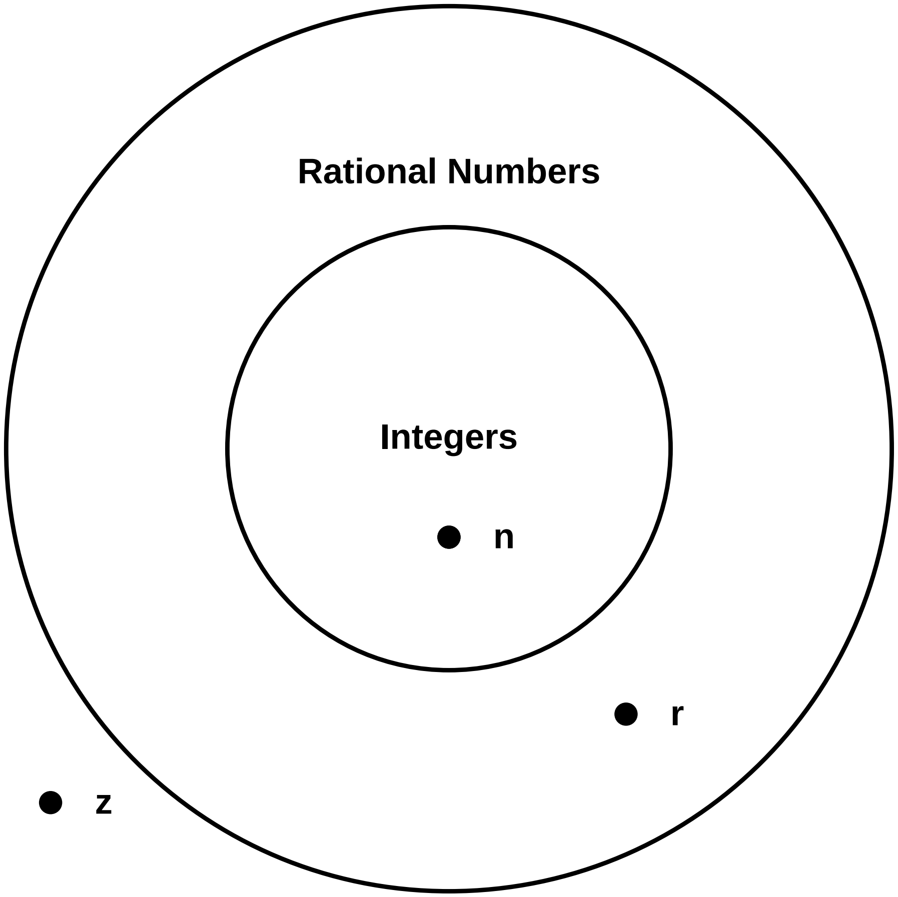
Consider the following statement:

Informal	All integers are rational number
Formal	$\forall$ integers n, n is a rational number.

Picture the sets of rational numbers and integers as disks. The truth of the statement means that the integer disk must be contained inside the disk of rational numbers.



Finally, Something Less Abstract — “Proof by Diagram”



Consider the following statements:

s1. n is an integer.

s2. r is a rational.

s3. z is not a rational.

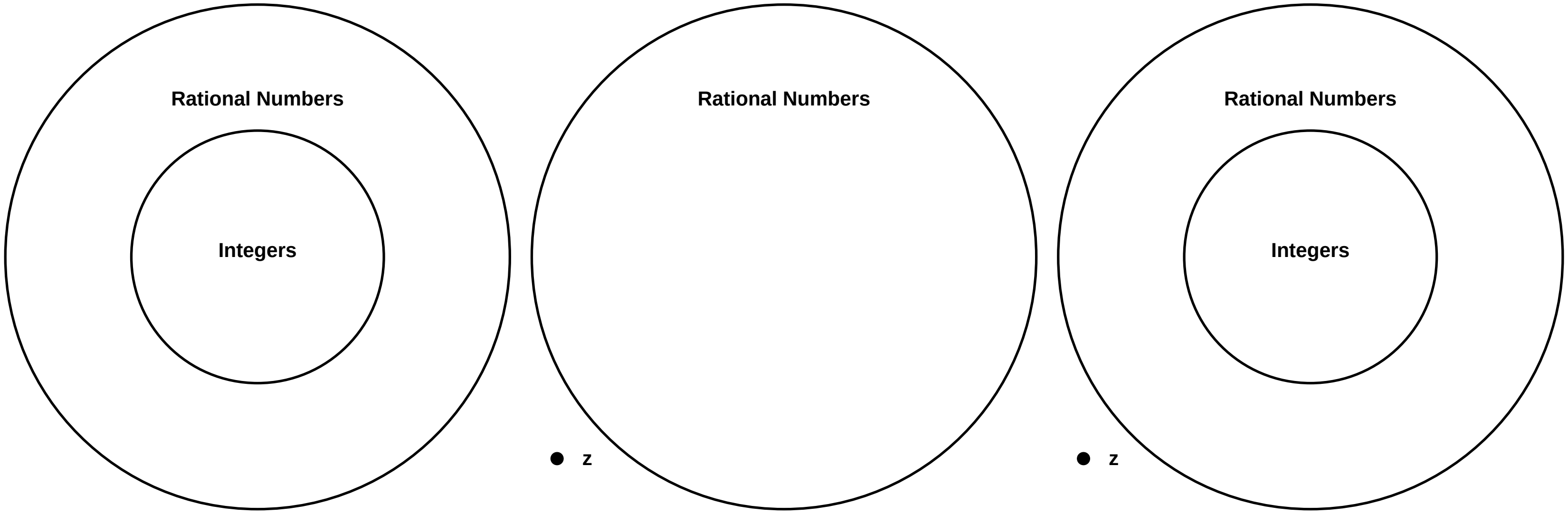
From the diagram we see that

s1. $\Rightarrow n$ is a rational.

s2. $\not\Rightarrow r$ is an integer. — There are rationals that are not integers.

s3. $\Rightarrow z$ is not an integer.

Showing the Validity of an Argument using a Diagram

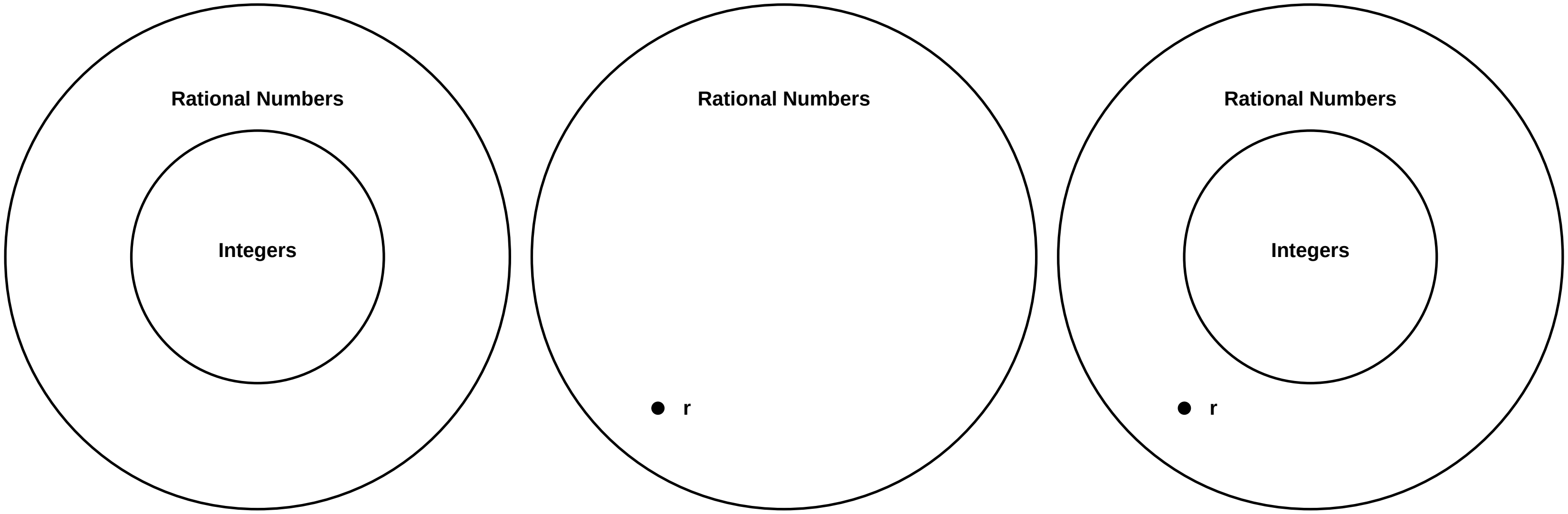


Major premise | Minor premise | Conclusion

	All integers are rational numbers	**Major premise**
	z is not a rational number	**Minor premise**
$\therefore$	z is not an integer	**Conclusion**

Example of a valid argument

Showing the Invalidity of an Argument using a Diagram

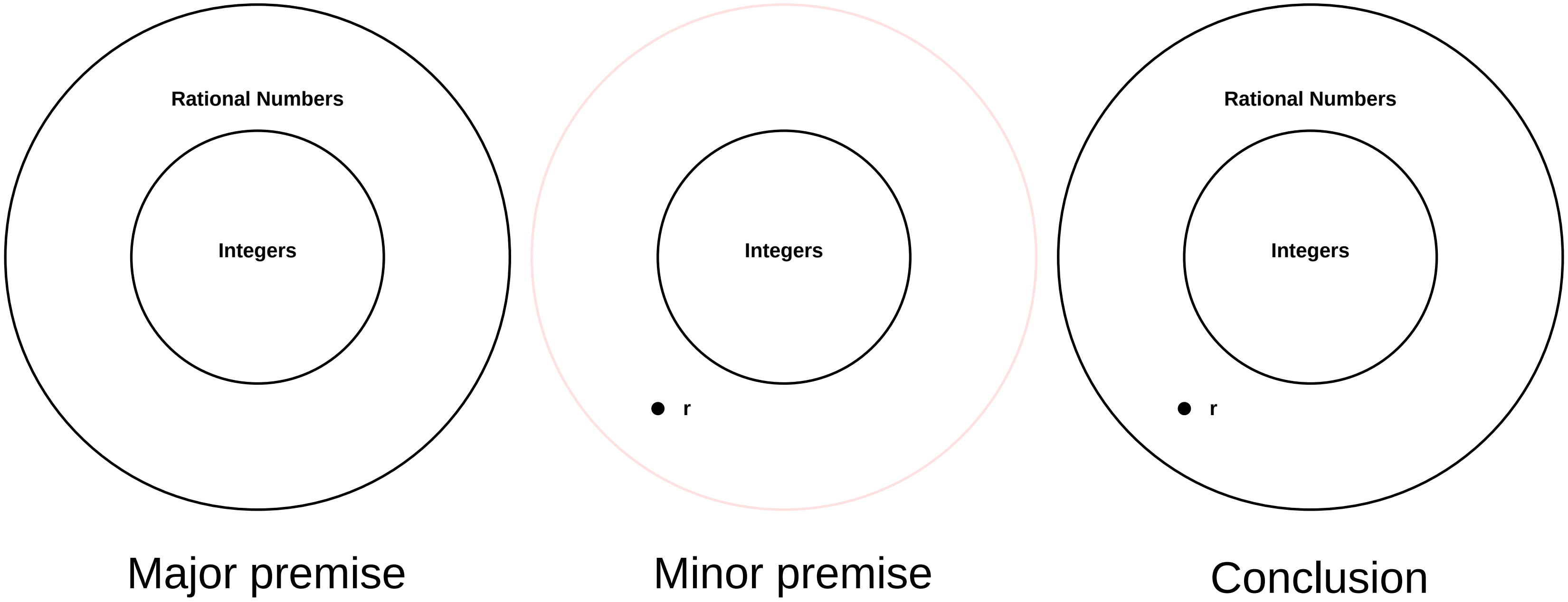


Major premise | Minor premise | Conclusion

	All integers are rational numbers	**Major premise**
	r is a rational number	**Minor premise**
$\therefore$	r is an integer	**Conclusion**

Example of an invalid argument — Converse Error

Showing the Invalidity of an Argument using a Diagram



	All integers are rational numbers	**Major premise**
	r is not an integer	**Minor premise**
$\therefore$	r is not a rational number	**Conclusion**

Example of an invalid argument — Inverse Error

Comments — Food for Thought

The reason inverse and converse errors are common is that the conclusions would be true if the major premise was a bi-conditional (“if, and only if” / $\Leftrightarrow$ / $\leftrightarrow$).

Call it “fuzzy logic,” “artificial intelligence,” or “abduction” — if you have a major premise:

“for all x, if $P(x)$ then $Q(x)$”

then if,

$Q(a)$ is true, for a particular a.

then is it a good idea to ***check*** if $P(a)$ is true! — This kind of reasoning is used by criminal investigators, doctors, auto mechanics, etc, etc... “$Q(a) =$ ***true***” is ***not evidence*** (of “$P(a) =$ ***true***”), but possibly ***a clue***.

Homework #2 — Due Friday 9/22/2006 **Final Version**

3rd Edition	2nd Edition
Problems	
2.4: 19, 20, 25, 31	***2.3:*** 19, 20, 24, 27
2.3: 37	***2.2:*** 15
2.2: 15, 17, 29, 40	***2.1:*** 28, 23; ***2.2:*** —, 33
2.1: 12, 14, 22	***2.1:*** 7, 9, 16

Please use the ***3rd Edition*** numbering when handing in your solutions.