# Math 245: Discrete Mathematics

**Counting and Probability**

**Permutations, Addition Rule, Inclusion/Exclusion**

**Lecture Notes #11**

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## Quick Recap

Last lecture we started talking about ***Counting and Probability***.

We introduced the concepts: **random process**, **sample space** ($S$) (all the possible outcomes of a random process), **event** ($E$) (a subset of the sample space), and **probability** (the relative size of the event vs. the sample space):

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{\# element is the event}}{\text{\# elements in the sample space}}$$

this formula is valid *if and only if* all outcomes are ***equally likely***.

We counted elements in a list, looked at the probability of outcomes when tossing coins, introduced the concept of a ***possibility tree*** (which shows all possible combinations of sequential events), and introduced the ***multiplication rule*** for independent events.



## Recap: Key Concepts — Possibility Tree

The previous figure shows all the possible ways the world series can play out, ***but*** there are multiple ways to reach some (most) states; *e.g.* the scenario "A wins, B wins" and "B wins, A wins" end up in the same **state** (one win for each team).

In a possibility tree, these two paths are differentiated; the possibility tree for the first 5 games looks like this:

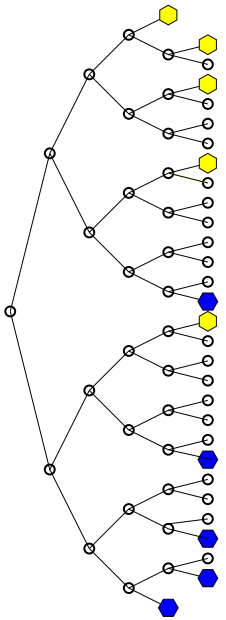


**Figure (to the right:)** The possibility tree for the first 5 games of the world series. Note that 2 (out of 16) paths terminate after 4 games. An additional 8 paths terminate after 5 games...



## Recap: Key Concepts — Independence / Multiplication Rule

If we have a sequence of events which are ***independent*** (note that this does not apply to the world series, since depending on the outcome of previous games, games #5, #6, and #7 may not be played) the multiplication rule applies:

> **Theorem:** ***Multiplication Rule*** —
> If an operation consists of $k$ steps and step $\#i$ can be performed in $n_i$ ways $i = 1, 2, \ldots, k$, then the entire operation can be performed in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways.

If all 7 games of the world series were played no matter what the outcome of the previously played games:

$$k = 7, \quad n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = 2$$

$$\mathbf{2^7 = 128} \text{ possibilities.}$$

## Permutations

Set + Order

A **permutation** of a set of objects is an **ordering** of the objects. For example, the set **{a,b,c}** has six permutations:

**abc**, **acb**, **bac**, **bca**, **cab**, **cba**

**Question:** How many permutations does a set with $n$ elements have???

The first element can be selected in $n$ ways, the second in $(n-1)$ ways, the third in $(n-2)$ ways, ...

$$\#\textbf{Permutations}(n) \quad = \quad n \cdot (n-1) \cdot \ldots \cdot 1 \quad = \quad \mathbf{n!}$$

***Theorem:*** For any integer $n \geq 1$, the number of permutations of a set with $n$ elements is $n!$ ($n$-factorial).



## Example: Permuting at the Dinner Table

We are to seat six dinner guests around a table:

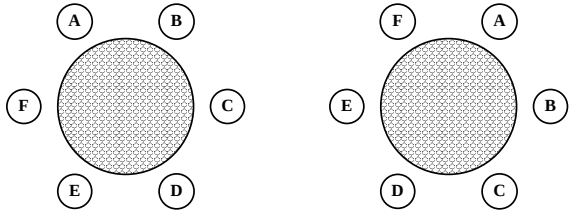



**Figure:** Two seating arrangements are considered the same if they are just a rotation of each other.

**Question:** How many seating arrangements are there, taking ***rotational symmetry*** into consideration?

**Solution:** We can take one guest and put him/her in a fixed position; — then the other five can be seated in $\mathbf{5! = 120}$ different ways relative to the first guest.



## Example: Permuting at the Dinner Table, Take #2

We are to seat six dinner guests around a table:

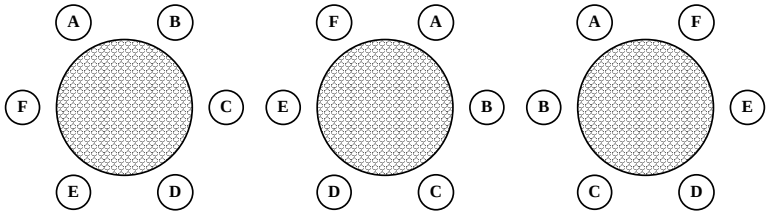



**Figure:** Two seating arrangements are considered the same if they are a rotation of each other and/or a reflection of each other.

**Question:** How many seating arrangements are there, taking ***rotational and reflective symmetry*** into consideration?

**Solution:** Since each seating arrangement has a mirror image, we now effectively have $\frac{\mathbf{1}}{\mathbf{2}} \cdot \mathbf{5!} = \mathbf{60}$ different seating arrangements.



## Permutations of Selected Elements

Given the set $\mathbf{S} = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ there are six ways to select two letters from $S$ and write them in order:

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

Each such ordering of 2 elements of $S$ is called a ***2-permutation*** of $S$.

***Definition: r-permutation*** —
An ***r-permutation*** of a set of $n$ elements is an ordered selection of $r$ elements taken from the set. The number of r-permutations of a set of $n$ elements is denoted $\mathbf{P(n, r)}$.

I wonder if we could create a game using a set with 52 elements, and consider the 5-permutations... ☺

## $\mathbf{P(n,r)}$ — Counting the $r$-permutations

***Theorem:*** If $n$ and $r$ are integers and $1 \leq r \leq n$, then the number of $r$-permutations of a set of $n$ elements is given by the formula

$$\mathbf{P(n,r) = n(n-1)(n-2)\cdots(n-(r-1)) = \frac{n!}{(n-r)!}}$$

**Proof:** There are $(n-0)$ ways to make the first choice, $(n-1)$ ways to make the second choice, ..., $(n-(r-1))$ ways to make the r$^{\text{th}}$ choice, therefore the number of combinations are

$$n \cdot (n-1) \cdot \ldots \cdot (n-(r-1)) = n \cdot (n-1) \cdot \ldots \cdot (n-r+1)$$

Now we notice

$$\frac{n!}{(n-r)!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-r+1) \cdot (n-r)!}{(n-r)!} = n \cdot (n-1) \cdot \ldots \cdot (n-r+1). \square$$



## Evaluating $\mathbf{P(n,r)}$

**Problem:** How many 4-permutations are there of a set of 15 elements?

**Solution #1:** We can just plug in and evaluate

$$P(15,4) = \frac{15!}{(15-4)!} = \frac{15!}{11!} = \frac{1,307,674,368,000}{39,916,800} = 32,760$$

However, this can become problematic if $n$ is large (my, quite ancient, calculator can only compute up to 69! $(1.711\ldots \times 10^{98})$...

**Solution #2:** Think about what the denominator does, *i.e.* canceling the "tail" of the factorial in the numerator:

$$P(15,4) = \frac{15!}{(15-4)!} = \frac{15!}{11!} = 15 \cdot 14 \cdot 13 \cdot 12 = 32,760$$



## Example: Proving $P(n,2) + P(n,1) = n^2$

***Proposition:*** For all integers $n \geq 2$,

$$P(n,2) + P(n,1) = n^2$$

**Proof:** Let $n$ be an integer $\geq 2$, and use the theorem on slide 9, *i.e.*

$$\begin{aligned} P(n,2) &= \frac{n!}{(n-2)!} = n(n-1) \\ P(n,1) &= \frac{n!}{(n-1)!} = n \end{aligned}$$

and therefore

$$P(n,2) + P(n,1) = n(n-1) + n = n^2 - n + n = n^2$$

$\forall n \geq 2$. $\square$



## The Addition Rule — Counting Elements of Disjoint Sets

***Theorem:*** If $\{\mathbf{A_1}, \mathbf{A_2}, \ldots, \mathbf{A_n}\}$ is a partition of $\mathbf{A}$, then

$$n(A) = n(A_1) + n(A_2) + \ldots + n(A_n)$$

The formal proof (exercise ***Epp-6.3.33***) uses mathematical induction. Intuitively it is clear: each element in $A$ is a member of exactly one of the sets $A_i$, so the element count on both the left- and right-hand-side must be the same.

## Example: Counting 4-6 Digit PINs

**Problem:** In order to use the Big Bank's ATMs the user must have a 4–6 digit PIN (each digit is an integer 0–9). How many such PINs are there?

**Solution:**

| | |
|---|---|
| **4-digit PINs** | $10^4 = 10,000$ |
| **5-digit PINs** | $10^5 = 100,000$ |
| **6-digit PINs** | $10^6 = 1,000,000$ |
| **4–6-digit PINs** | $1,110,000$ |



## The Difference Rule

***Theorem:*** If $A$ is a set with finitely many elements, and $B$ a subset of $A$, $B \subset A$, then

$$n(A - B) = n(A) - n(B)$$

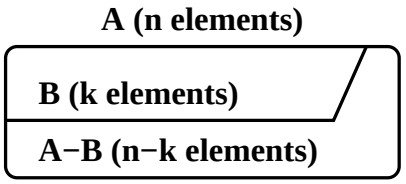



**Figure:** The Difference Rule — Illustration.



## Example: Counting 4-6 Digit PINs (with/without repetition)

**Problem:** In order to use the Big Bank's ATMs the user must have a 4–6 digit PIN (each digit is an integer 0–9). How many such PINs are there? — How many PINs have no repeating digits; How many PINs have repeating digits?)

**Solution:**

| PINs | Any Digits | No Repetition | With Repetition |
|---|---|---|---|
| **4-digit** | $10^4 = 10,000$ | $P(10,4) = 5,040$ | $4,960$ |
| **5-digit** | $10^5 = 100,000$ | $P(10,5) = 30,240$ | $69,760$ |
| **6-digit** | $10^6 = 1,000,000$ | $P(10,6) = 151,200$ | $848,800$ |
| **4–6-digit** | $1,110,000$ | $186,480$ | $923,520$ |

Thus requiring non-repeating passwords limits the password space quite severely.



## Example: Random PINs

**Problem:** When you get your ATM card from the Big Bank, you are assigned a random 4–6-digit PIN. What is the probability that the PIN will have repeated digits? What is the probability that it will not?

**Solution:**

$$P(\textbf{repeated}) = \frac{n(\textbf{repeated})}{n(\textbf{any})} = \frac{923,520}{1,110,000} = 0.832 = 83.2\%$$

$$P(\textbf{no repeated}) = \frac{n(\textbf{no repeated})}{n(\textbf{any})} = \frac{186,480}{1,110,000} = 0.168 = 16.8\%$$

We notice

$$P(\textbf{repeated}) + P(\textbf{no repeated}) = 1$$

This true in general for complementary events...

## Probability of the Complement of an Event

**Formula for the Probability of the Complement of an Event**

If $S$ is a finite sample space and $E$ is an event in $S$, then

$$\mathbf{P(E^c) = 1 - P(E)}$$

Since $S = E \cup E^c$, and $E \cap E^c = \emptyset$ we have

$$n(S) = n(E) + n(E^c) \quad \Leftrightarrow \quad n(E^c) = n(S) - n(E)$$

and

$$P(E) = \frac{n(E)}{n(S)}$$

$$P(E^c) = \frac{n(E^c)}{n(S)} = \frac{n(S) - n(E)}{n(S)} = \frac{n(S)}{n(S)} - \frac{n(E)}{n(S)} = 1 - P(E)$$



## The Inclusion/Exclusion Rule

***Theorem:*** If $A$ and $B$ are any finite sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

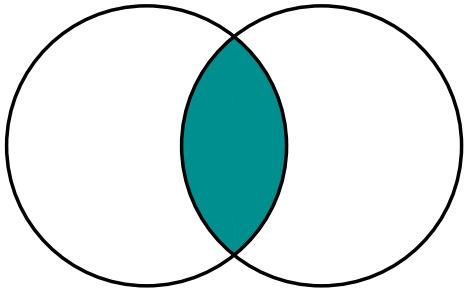


**Figure:** If we count the elements in A and add the elements in B, then the elements in the intersection $(A \cap B)$ are counted twice. The statement in the theorem subtracts one instance of the elements in the intersection, making the count correct.



## Inclusion/Exclusion for Three Sets

Given the Inclusion/Exclusion rule for two sets, we can find rules for more sets: Let $A$, $B$, and $C$ be any sets

$$\begin{aligned}
\mathbf{n(A \cup B \cup C)} &= n(A) + n(B \cup C) - n(A \cap (B \cup C)) \\
&= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap (B \cup C)) \\
&= n(A) + n(B) + n(C) - n(B \cap C) - n((A \cap B) \cup (A \cap C)) \\
&= n(A) + n(B) + n(C) - n(B \cap C) \\
&\quad - (n(A \cap B) + n(A \cap C) - n((A \cap B) \cap (A \cap C))) \\
&= \mathbf{n(A) + n(B) + n(C) - n(B \cap C)} \\
&\quad \mathbf{- n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)}
\end{aligned}$$

See exercise ***Epp-6.3.36*** for the general inclusion/exclusion rule for $n$ sets.



## Example: Knowledge of Computer Languages 1 of 3

**Problem:** 50 students replied to a survey of what computer programming languages they knew:

$$\begin{aligned}
A &= \{\textbf{Students that know Java}\}, & n(A) = 30 \\
B &= \{\textbf{Students that know Fortran}\}, & n(B) = 18 \\
C &= \{\textbf{Students that know C}\}, & n(C) = 26
\end{aligned}$$

Further the survey reveals

$$n(A \cap B) = 9, \quad n(A \cap C) = 16, \quad n(B \cap C) = 8, \quad n(A \cup B \cup C) = 47$$

Using the difference rule we find that the number of students that do not know any of the 3 languages:

$$n(U) - n(A \cup B \cup C) = 50 - 47 = 3$$

Here $U$, our "universe," is the set of all students who replied to the study.

## Example: Knowledge of Computer Languages 2 of 3

$$n(A) = 30,\ n(B) = 18,\ n(C) = 26,\ n(A \cup B \cup C) = 47,$$

$$n(A \cap B) = 9,\ n(A \cap C) = 16,\ n(B \cap C) = 8$$

Using our derived inclusion/exclusion formula for three set we find that the number of student that know all three languages are

$$\begin{aligned} n(A \cup B \cup C) &= n(A) + n(B) + n(C) - n(B \cap C) \\ &\quad -n(A \cap B) - n(A \cap C) + n(A \cap B \cap C) \\ 47 &= 30 + 18 + 26 - 9 - 16 - 8 + n(A \cap B \cap C) \end{aligned}$$

This gives us $n(A \cap B \cap C) = 6$.

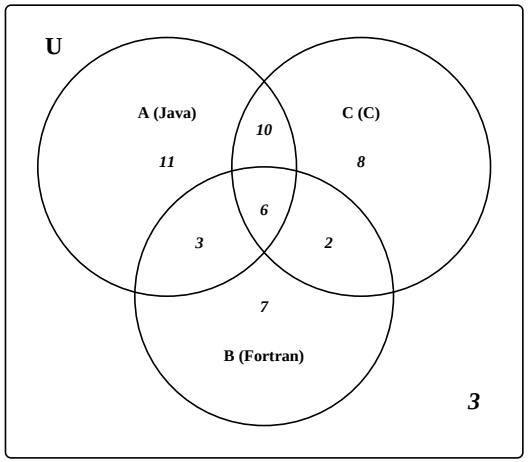
We now have a complete picture...



## Example: Knowledge of Computer Languages 3 of 3

$$n(A) = 30,\ n(B) = 18,\ n(C) = 26,\ n(A \cup B \cup C) = 47,$$

$$n(A \cap B) = 9,\ n(A \cap C) = 16,\ n(B \cap C) = 8,\ n(A \cap B \cap C) = 6$$






**Homework #10 — Due Wed. 11/22/2006, 12noon, GMCS-587 Final Version**

(Epp-v3.0)

***Epp-6.3.4, Epp-6.3.6, Epp-6.3.11, Epp-6.3.18, Epp-6.3.25, Epp-6.3.26, Epp-6.3.28***

Write down the inclusion/exclusion principle for 4 sets (hint: ***Epp-6.3.36***)

(Epp-v2.0)

***Epp-6.3.4, Epp-6.3.6, Epp-6.3.11, Epp-6.3.18, —, Epp-6.3.23, Epp-6.3.25***

Write down the inclusion/exclusion principle for 4 sets (hint: ***Epp-6.3.33***)