Counting and Probability

Combinations, Pascal's Triangle, the Binomial Theorem

Lecture Notes #12

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Counting Combinations — Introduction

Consider drawing a poker hand (five cards, *e.g.* $\{\mathbf{10}\heartsuit, \mathbf{J}\heartsuit, \mathbf{Q}\heartsuit, \mathbf{K}\heartsuit, \mathbf{A}\clubsuit\}$) from a deck of cards. How many possibilities are there?

Last time we introduced the concept of an r-permutation

> ***Definition***: An ***r-permutation*** of a set of n elements is an ordered selection of r elements taken from the set. The number of r-permutations of a set of n elements is denoted $\mathbf{P(n, r)}$.

But a poker hand is ***not an ordered selection*** — it does not matter in what order you draw the cards!

Next, we introduce $\mathbf{r}$-***combinations*** — an ***unordered selection*** of r elements from a set of n elements...

Counting Subsets — r-combinations

***Definition:* r*-combination* —**
Let n and r be non-negative integers with $r \leq n$. An r-combination of a set of n elements is a subset of r of the n elements. The symbol $\binom{n}{r}$, read "n choose r," denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

Selection Type	**Ordered**	**Unordered**
Name	r**-permutation**	r**-combination**
Symbol	$P(n, r)$	$\binom{n}{r}$
# of Possibilities	$\frac{n!}{(n-r)!}$	**???**

Table: Summary of ordered (permutations) and unordered (combinations) selection of r elements from a set containing n elements.

Example

Example #1: A 3-combination of S, where $n(S) = 4$. Let $S = \{$**Math, Physics, Chemistry, Biology**$\}$ — next semester you must take 3 of these subjects, what are your options?

$\{$**Physics, Chemistry, Biology**$\}$ $\{$**Math, Chemistry, Biology**$\}$

$\{$**Math, Physics, Biology**$\}$ $\{$**Math, Physics, Chemistry**$\}$

Example #2: A 2-combination of S, where $n(S) = 4$. Let $S = \{0, 1, 2, 3\}$, how many subsets are there?

$$\{0,1\},\ \{0,2\},\ \{0,3\},\ \{1,2\},\ \{1,3\},\ \{2,3\}$$

We notice that the number of combinations is given by

$$\binom{4}{3} = \frac{4!}{3!} = 4, \qquad \binom{4}{2} = \frac{4!}{2! \cdot 2!} = \frac{24}{4} = 6$$

We can think of ***ordered selection*** as a 2-step process:

1. Select r (**unordered**) elements from the set of n elements.
2. Assign an ordering to the r elements.

If there are n_1 ways to perform step 1 and n_2 ways to perform step 2, then by the ***multiplication rule*** there are $n_1 \cdot n_2$ ways to perform the two-step process.

We know we can perform the two-step process (generating an r-combination) in $n_1 \cdot n_2 = P(n, r)$ ways, where $n_1 = \binom{n}{r}$, and $n_2 = r!$ by the following theorem (from last lecture)

> ***Theorem:*** For any integer $r \geq 1$, the number of permutations of a set with r elements is $r!$ (r-factorial).

We now have the following relationship

$$P(n,r) = \binom{n}{r} \cdot r! \quad \Leftrightarrow \quad \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)! \cdot r!}$$

We summarize in a theorem:

Theorem: The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

where n and r are non-negative integers with $r \leq n$.

Summary: Combinations, Set Combinations, and r-Permutations

Type	Ordering	Ordered Selection	Unordered Selection
Name	Permutation	r-permutation	r-combination
Symbol (count)	—	$P(n, r)$	$\binom{n}{r}$
# of Possibilities	$n!$	$\frac{n!}{(n-r)!}$	$\frac{n!}{(n-r)! \cdot r!}$

Table: Summary of permutations of n elements, ordered selection and unordered selection of r elements from a set containing n elements.

Examples: Corporate Layoffs

Problem: You are a middle-manager of MegaCorp Inc., there are 12 employees in your department. You have been charged with the task of selecting 5 of them for termination — how many ways can this be done?

Solution: The number of ways this can be done is the number of subsets of size 5 of a set of 12 elements (a 5-combination). The number is given by

$$\binom{12}{5} = \frac{12!}{(12-5)! \cdot 5!} = \frac{12!}{7! \cdot 5!}$$

We cancel common factors before evaluating...

$$\frac{12!}{7! \cdot 5!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot \underbrace{4 \cdot 3}_{12} \cdot 2} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 2} = 11 \cdot 9 \cdot 8 = \mathbf{792}.$$

Examples: Basketball Teams

Problem: We are to form a 5-person team out of 12 players. Two of them are a "dynamic duo" and must either both be on the team, or off. — How many ways can this be done?

Solution: The problem splits into two cases

1. The duo is on the team, and we have

$$\binom{10}{3} = \frac{10!}{3! \cdot 7!} = 120$$

ways to select the remaining 3 players from a pool of 10.

2. The duo is off the team, and we have

$$\binom{10}{5} = \frac{10!}{5! \cdot 5!} = 252$$

ways to select the 5 players from a pool of 10.

Clearly, the cases are disjoint, so the ***addition rule*** applies and we have $120 + 252 = \mathbf{372}$ combinations.

Suppose a group consists of five men and seven women.

Problems:

(a) How many 3M+2W teams are there?

(b) How many 5-person team contain at least 1M?

(c) How many 5-person team contain at most 1M?

Solutions:

Part (a) is straight-forward. We can think of this selection as a 2-step process. First select 3 out of 5 men, then 2 out of 7 women:

$$\binom{5}{3} \cdot \binom{7}{2} = \frac{5!}{3! \cdot 2!} \cdot \frac{7!}{5! \cdot 2!} = 10 \cdot 21 = 210.$$

For part (b) we use the ***difference rule***

$\{\geq 1$**-man 5-person teams**$\} = \{$**All 5-person teams**$\} - \{$**All-Women 5-person teams**$\}$

We get

$$\binom{12}{5} - \binom{7}{5} = \frac{12!}{7!5!} - \frac{7!}{5!2!} = 792 - 21 = \mathbf{771}$$

For part (c) we use the ***addition rule***

$\{$**0-man 5-person teams**$\} \cup \{$**1-man 5-person teams**$\}$

We get

$$\binom{5}{0}\binom{7}{5} + \binom{5}{1}\binom{7}{4} = 1 \cdot 21 + 5 \cdot 35 = \mathbf{196}$$

Problems:

(a) How many 5-card poker hands contain two pairs?

(b) What is the probability that a 5-card hand dealt at random contains two pairs?

Solutions:

(a) We can view this as a 4-step process

1. Choose the denomination for the pairs
2. Choose two cards from the smaller denomination
3. Choose two cards from the larger denomination
4. Choose one card from the remaining cards

Since there are 13 denominations $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$ there are $\binom{13}{2}$ ways to perform step 1.

There are 4 cards of each denomination $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, so therefore each of steps 2 and 3 can be performed in $\binom{4}{2}$ ways.

There are 44 allowable cards remaining (if we pick any of the 4 cards which have the same denomination we end up with a "full house," *e.g.* $\{\mathbf{8}\heartsuit, 8\clubsuit, \mathbf{A}\diamondsuit, \mathbf{A}\heartsuit, A\spadesuit\}$), hence step 4 can be performed in $\binom{44}{1}$ ways.

The steps are independent, hence the ***multiplication rule applies***

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1} = 78 \cdot 6 \cdot 6 \cdot 44 = \mathbf{123,552}$$

so, $123,552$ poker hands contain two pairs.

Part (b):

There are a total of $\binom{52}{5}$ 5-card hands from an ordinary deck of cards. If all hands are equally likely, the probability of obtaining a hand with two pairs is

$$P(\textbf{two pairs}) = \frac{n(\textbf{two-pair hands})}{n(\textbf{all hands})} = \frac{123,552}{2,598,960} = \frac{198}{4165} = \mathbf{0.0475}$$

i.e. just shy of 5%.

To think about: How many poker hands beat (all) hands with two pairs?

Permutations of a Set with Repeated Elements

Problem: How many ***distinguishable orderings*** are there of the letters in the word "MISSISSIPPI"?

Solution: Copies of the same letter cannot be distinguished from one another... We can view the ordering as a 4-step process

1. Choose a subset of four positions for the S's
2. Choose a subset of four positions for the I's
3. Choose a subset of two positions for the P's
4. Choose a subset of one position for the M.

There are 11 positions, so step 1 can be performed in $\binom{11}{4}$ ways, step 2 in $\binom{7}{4}$ ways, step 3 in $\binom{3}{2}$ ways, and step 4 in $\binom{1}{1}$ ways, for a grand total of

$$\binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1} = 330 \cdot 35 \cdot 3 \cdot 1 = 34,650$$

Question: Does the order in which we place the letters change the answer???

Permutations of a Set with Repeated Elements

Theorem: Suppose a collection consists of n objects of which:

n_1 are of type 1 and are indistinguishable from each other

n_2 are of type 2 and are indistinguishable from each other

$\vdots$

n_k are of type k and are indistinguishable from each other

and suppose $n = n_1 + n_2 + \ldots + n_k$. Then the number of distinct permutations of the n objects are

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\ldots-n_{k-1}}{n_k}$$

this expression simplifies to

$$\frac{\mathbf{n!}}{\mathbf{n_1! \cdot n_2! \cdot n_3! \cdots n_k!}}$$

(Epp-v3.0)

Epp-6.4.6, Epp-6.4.16 , Epp-6.4.19

(Epp-v2.0)

Epp-6.4.6, Epp-6.4.16 , Epp-6.4.19

r-Combinations with Repetition Allowed

Definition: An **$\mathbf{r}$-combination with repetition allowed**, or a **multi-set of size $\mathbf{r}$**, chosen from a set S of n elements is an unordered selection of elements taken from S with repetition allowed. If $S = \{s_1, s_2, \ldots, s_n\}$, we write a multi-set of size r as $[\mathbf{x_{i_1}}, \mathbf{x_{i_2}}, \ldots, \mathbf{x_{i_r}}]$ where each $x_{i_j} \in S$ and it is allowed for some (or all) of the x_{i_j} to equal each other.

Example: Let $S = \{1, 2, 3, 4\}$ then some of the 5-combinations are

$$[1,1,1,1,1],\ [1,2,3,3,5],\ [1,2,3,4,5]$$

Note that since a multi-set is unordered, the following are considered equivalent

$$[1,1,1,1,2] \equiv [1,1,2,1,1]$$

How many r-combinations with repetition allowed are there?

If we view each element of S as a category, and view the construction of the multi-set as a selection from these categories with repetition allowed... We can write down a table like this:

Cat#1		Cat#2		Cat#3		Cat#4		Cat#5	Multi-set
x	|	xx	|	x	|	x	|		[1,2,2,3,4]
xxxx	|		|		|		|	x	[1,1,1,1,5]
	|	xx	|	x	|		|	xx	[2,2,3,5,5]

We notice that we can describe each multi-set with a 9-digit string containing 5 x's and 4 -'s, *e.g.* "x-xx-x-x-" corresponds to [1,2,2,3,4], and "-xx-x--xx" corresponds to [2,2,3,5,5].

With this description of the multi-set, we notice that we need $(n-1)$ -'s to separate the n categories (elements), and r x's to symbolize the choices.

We have a total of $(r+n-1)$ symbols.

Generation of the possible symbol combinations can be viewed as a 2-step process:

1. Choose a subset of r positions for the x's
2. Choose a subset of $(n-1)$ positions for the -'s

This can be done in

$$\binom{r+n-1}{r}\binom{n-1}{n-1} = \binom{r+n-1}{r} \cdot 1 = \binom{r+n-1}{r}$$

ways.

We summarize our finding in a theorem:

Theorem: The number of r-combinations with repetitions allowed (or multi-sets of size r) that can be selected from a set of n elements is

$$\binom{r+n-1}{r}$$

This equals the number of ways r objects can be selected from n categories of objects with repetition allowed.

Summary: Counting Formulas

	Order Matters	Order Does Not Matter
Repetition Allowed	n^k	$\binom{n+k-1}{k}$
Repetition Not Allowed	$P(n,k)$	$\binom{n}{k}$

Table: We have four different ways of choosing k elements from a set of n elements. The count is very different depending on whether order and/or repetition matters.

Example: Integer Solutions...

Problem: How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 10$$

if we require $x_1, x_2, x_3, x_4 \geq 0$?

Solution: Think of x_1, x_2, x_3, x_4 as 4 categories. Then this problem is equivalent to selecting 10 objects from 4 categories (with repetition allowed), the answer is given by

$$\binom{r+n-1}{r}, \text{ with } r = 10 \text{ and } n = 4 \ \Rightarrow \ \binom{13}{10} = \mathbf{286}.$$

In the last few lectures we have derived a number of counting formulas, *i.e.*

Type	Ordering	Ordered Selection	Unordered Selection
Name	**Permutation**	**r-permutation**	**r-combination**
Symbol	—	$P(n,r)$	$\binom{n}{r}$
# of Possibilities	$n!$	$\frac{n!}{(n-r)!}$	$\frac{n!}{(n-r)! \cdot r!}$

Table: Summary of permutations of n elements, ordered selection and unordered selection of r elements from a set containing n elements.

	Order Matters	Order Does Not Matter
Repetition Allowed	n^k	$\binom{n+k-1}{k}$
Repetition Not Allowed	$P(n,k)$	$\binom{n}{k}$

Table: We have four different ways of choosing k elements from a set of n elements. The count is very different depending on whether order and/or repetition matters.

Looking Forward...

Next, we will take a closer look at the properties of counting, and

1. Derive a number of useful formulas for $\binom{n}{r}$ for special values of n and r,
2. Find relations between different values of $\binom{n}{r}$
3. In particular we will discuss ***Pascal's Formula*** (Pascal's Triangle) which is perhaps one of the most used formulas in combinatorics (the study of counting combinations).
4. We wrap up our discussion of counting with a discussion of the ***Binomial Theorem***.

Some Values of $\binom{n}{r}$

$$\binom{\mathbf{n}}{\mathbf{n}} = \frac{n!}{n! \cdot (n-n)!} = \frac{n!}{n! \cdot 0!} = \frac{1}{0!} = \frac{1}{1} = \mathbf{1}$$

Hence, there is only one way of selecting all the elements (without repetition). [Here, $n \geq 0$]

$$\binom{\mathbf{n}}{\mathbf{n-1}} = \frac{n!}{(n-1)! \cdot (n-(n-1))!} = \frac{n!}{(n-1)! \cdot 1!} = \frac{n}{1!} = \frac{n}{1} = \mathbf{n}$$

Hence, there are only n ways to select all but 1 element. [Here, $n \geq 1$]

$$\binom{\mathbf{n}}{\mathbf{n-2}} = \frac{n!}{(n-2)! \cdot (n-(n-2))!} = \frac{n!}{(n-2)! \cdot 2!} = \frac{n(n-1)}{2!} = \frac{\mathbf{n(n-1)}}{\mathbf{2}}$$

[Here, $n \geq 2$]

$\binom{n}{r}$ represents the number of ways to select r elements from n elements. (*E.g.* selecting which 5 players of 12 who should be on the court.)

We can think of $\binom{n}{n-r}$ as the complementary action: selecting which $n-r$ elements we do not want from the n elements. (*E.g.* selecting which 7 players of 12 who should be on the bench.)

The resulting action (what elements are selected / what players are on the court) is the same — so the number of ways to perform the two actions should be the same... A bit of algebra and use of the definition of $\binom{n}{r}$ shows that this is indeed true:

$$\binom{\mathbf{n}}{\mathbf{r}} = \frac{n!}{(n-r)! \cdot r!} = \frac{n!}{r! \cdot (n-r)!} = \binom{\mathbf{n}}{\mathbf{n-r}}$$

New Formulas from Old by Substitution

We have established that

$$\binom{n}{n-2} = \frac{n(n-1)}{2}, \quad \forall n \in \mathbb{Z},\ n \geq 2$$

n is just dummy variable (place holder) which can be replaced by any other integer expression — as long as the integer expression is greater than or equal to 2, and each occurrence is n is replaced.

Examples:

1. $$\binom{m+1}{m-1} = \frac{(m+1)m}{2}, \quad m \geq 1$$

2. $$\binom{s-1}{s-3} = \frac{(s-1)(s-2)}{2}, \quad s \geq 3$$

3. $$\binom{k+2}{k} = \frac{(k+2)(k+1)}{2}, \quad k \geq 0$$

Pascal's Formula

Pascal's Formula relates the value of $\binom{n+1}{r}$ to the values of $\binom{n}{r-1}$ and $\binom{n}{r}$:

$$\binom{\mathbf{n+1}}{\mathbf{r}} = \binom{\mathbf{n}}{\mathbf{r-1}} + \binom{\mathbf{n}}{\mathbf{r}}$$

Usage: If we know all the values $\binom{n}{r}$, $r = 0, 1, 2 \ldots, n$ are known, we can immediately find the values for $\binom{n+1}{r}$, $r = 1, 2 \ldots, n$. — ***By one addition, per value!***

The "missing" values $\binom{n+1}{r}$, where $r = 0$, or $r = n+1$ are always **1**, since they correspond to selecting none/all of the $n+1$ elements.

Table: Pascal's Formula

$n \setminus r$	0	1	2	3	4	5	$\cdots$	$r-1$	r	$\cdots$
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
$\vdots$										
n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\cdots$	$\binom{n}{r-1}$	$\binom{n}{r}$	$\cdots$
$n+1$	$\binom{n+1}{0}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\binom{n+1}{3}$	$\binom{n+1}{4}$	$\binom{n+1}{5}$	$\cdots$	$\binom{n+1}{r-1}$	$\binom{n+1}{r}$	$\cdots$
$\vdots$										

Table: Illustration of Pascal's Formula. The arrows indicate how two previously computed values are combined to fill in a new value in the table.

Proving Pascal's Formula

There are two very different approaches to proving Pascal's Formula:

1. The first version is algebraic. It uses the formula for the number of r-combinations $\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$ and pure algebraic manipulation.

2. The second version is combinatorial. It uses the definition of the number of r-combinations as the number of subsets of size r taken from a set with n elements.

We look at both versions, since both approaches have applications in other situations.

Theorem: Let n and r be positive integers and suppose $r \leq n$, then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Proof: Let n and r be positive integers with $r \leq n$, from previously proved theorems we can write:

$$\binom{n}{r-1} + \binom{n}{r} = \frac{n!}{(r-1)! \cdot (n-r+1)!} + \frac{n!}{r! \cdot (n-r)!}$$

To add these fractions, we need a common denominator. The first fraction is "missing" an r, and the second is "missing" a factor of $(n-r+1)$. We get...

$$\frac{n!}{(r-1)! \cdot (n-r+1)!} \cdot \frac{r}{r} + \frac{n!}{r! \cdot (n-r)!} \cdot \frac{(n-r+1)}{(n-r+1)}$$

We can now combine the terms:

$$\frac{n!}{(r-1)! \cdot (n-r+1)!} \cdot \frac{r}{r} + \frac{n!}{r! \cdot (n-r)!} \cdot \frac{(n-r+1)}{(n-r+1)}$$

and get

$$\frac{r \cdot n! + (n-r+1) \cdot n!}{r! \cdot (n-r+1)!} = \frac{(n+1) \cdot n!}{r! \cdot ((n+1)-r))!} = \frac{(n+1)!}{r! \cdot ((n+1)-r)!}$$

Finally, we identify

$$\frac{(n+1)!}{r! \cdot ((n+1)-r)!} = \binom{n+1}{r}$$

which proves the theorem. □

Theorem: Let n and r be positive integers and suppose $r \leq n$, then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

Proof: Let n and r be positive integers with $r \leq n$. Suppose S is a set with $n+1$ elements. The number of subsets of size r can be calculated by thinking of S as the union of the set with n elements $\{x_1, x_2, \ldots, x_n\}$ and the set $\{x_{n+1}\}$ containing one element.

Any subset of S either contains x_{n+1} or it does not:

1. If a subset of size r contains x_{n+1} then it also contains $r-1$ elements from $\{x_1, x_2, \ldots, x_n\}$. There are $\binom{n}{r-1}$ of these.
2. If a subset of size r does not contain x_{n+1} then it contains r elements from $\{x_1, x_2, \ldots, x_n\}$. There are $\binom{n}{r}$ of these.

Since the subsets of type#1 (containing x_{n+1}) and type#2 (not containing x_{n+1}) are disjoint, the ***addition rule*** applies, and we have:

#subsets of $\{x_1, x_2, \ldots, x_n, x_{n+1}\} =$

#subsets of $\{x_1, x_2, \ldots, x_n\}$ **of size** $(r-1)+$

#subsets of $\{x_1, x_2, \ldots, x_n\}$ **of size** r

Which means,

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}.$$

as was to be shown. $\square$

(Epp-v3.0)

Epp-6.4.6, Epp-6.4.16 , Epp-6.4.19

Epp-6.5.1, Epp-6.5.3, Epp-6.5.5, Epp-6.5.10,

Epp-6.5.11, Epp-6.6.11, Epp-6.6.14

(Epp-v2.0)

Epp-6.4.6, Epp-6.4.16 , Epp-6.4.19

Epp-6.5.1, Epp-6.5.3, Epp-6.5.5, Epp-6.5.10,

Epp-6.5.11, Epp-6.6.11, Epp-6.6.14

***Definition:* Binomial —**
A **binomial** is a sum of two terms $\mathbf{a} + \mathbf{b}$.

The binomial theorem gives an expression for the powers of a binomial $(\mathbf{a} + \mathbf{b})^{\mathbf{n}}$ $\forall n \in \mathbb{Z}^+$ and $a, b \in \mathbb{R}$.

We know (the distributive law of algebra) that the answer is the sum of the product of all individual terms, *e.g.*

$$\begin{aligned}
(a+b)^2 &= (a+b)(a+b) \\
&= aa + ab + ba + bb \\
&= a^2 + 2ab + b^2 \\
(a+b)^3 &= (a+b)(a+b)(a+b) \\
&= aaa + \mathbf{aab} + \mathbf{aba} + \mathbf{abb} + \mathbf{baa} + \mathbf{bab} + \mathbf{bba} + bbb \\
&= a^3 + \mathbf{3a^2b} + \mathbf{3ab^2} + b^3
\end{aligned}$$

Consider

$$(a+b)^4 = \underbrace{(a+b)}_{\textbf{1st factor}}\underbrace{(a+b)}_{\textbf{2nd factor}}\underbrace{(a+b)}_{\textbf{3rd factor}}\underbrace{(a+b)}_{\textbf{4th factor}}$$

$$= aaaa + aaab + aaba + aabb + abaa + abab + abba + abbb$$
$$+baaa + baab + baba + babb + bbaa + bbab + bbba + bbbb$$

Each term on the right-hand-side is a built by

1. Selecting one of $\{a, b\}$ from the first factor (2 possibilities)
2. Selecting one of $\{a, b\}$ from the second factor (2 possibilities)
3. Selecting one of $\{a, b\}$ from the third factor (2 possibilities)
4. Selecting one of $\{a, b\}$ from the fourth factor (2 possibilities)
5. Multiplying the selected terms together ($2^4 = 16$ total possibilities)

In particular (selections **high-lighted**)

$$(\mathbf{a}+b)(\mathbf{a}+b)(a+\mathbf{b})(a+\mathbf{b}) \quad \longrightarrow \quad aabb$$
$$(\mathbf{a}+b)(a+\mathbf{b})(\mathbf{a}+b)(a+\mathbf{b}) \quad \longrightarrow \quad abab$$
$$(\mathbf{a}+b)(a+\mathbf{b})(a+\mathbf{b})(\mathbf{a}+b) \quad \longrightarrow \quad abba$$
$$(a+\mathbf{b})(\mathbf{a}+b)(\mathbf{a}+b)(a+\mathbf{b}) \quad \longrightarrow \quad baab$$
$$(a+\mathbf{b})(\mathbf{a}+b)(a+\mathbf{b})(\mathbf{a}+b) \quad \longrightarrow \quad baba$$
$$(a+\mathbf{b})(a+\mathbf{b})(\mathbf{a}+b)(\mathbf{a}+b) \quad \longrightarrow \quad bbaa$$

This shows that the coefficient for the a^2b^2-term is

$$\binom{4}{2}\binom{2}{2} = 6.$$

In general the coefficient for the term $a^{4-k}b^k$ $(0 \leq k \leq 4)$ corresponds to

1. Selecting k of 4 positions for the b's
   — $\binom{4}{k}$ possibilities.
2. Selecting $4-k$ of $(4-k)$ positions for the a's
   — $\binom{4-k}{4-k} = 1$ possibilities.

Hence, the coefficient for $a^{4-k}b^k$ $(0 \leq k \leq 4)$ is $\binom{4}{k}$, and we have

$$(a+b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3b^1 + \binom{4}{2}a^2b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

We are now ready to state the binomial theorem:

Theorem: — Given any real numbers a and b and any non-negative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

We will look at the algebraic and combinatorial versions of the proof.

We need the following definitions for our algebraic version of the proof:

Definition: For any real number a and any non-negative integer n, the **non-negative integer powers of a** are defined as follows:

$$a^n = \begin{cases} 1 & \text{if } n = 0 \\ a \cdot a^{n-1} & \text{if } n > 0 \end{cases}$$

Notice that here we are defining:

$$\mathbf{0^0 = 1}$$

This is convenient here, but not always desirable in other mathematical applications...

Suppose a and b are real numbers. We prove that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \text{ for all integers } n \geq 0,$$

by induction on n...

Base When $n = 0$ the binomial theorem states that

$$(a+b)^0 = \sum_{k=0}^{0} \binom{n}{k} a^{n-k} b^k$$

The left-hand-side is 1 (by the definition of power), and the right-hand side is

$$\sum_{k=0}^{0} \binom{n}{k} a^{n-k} b^k = \binom{0}{0} a^0 b^0 = 1$$

Inductive Step — Assume true for $n = m$, show true for $n = m + 1$

Let $m \geq 1$ be a given integer, and suppose the equality holds for $n = m$, *i.e.*

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$$

We must show that

$$(a+b)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} a^{(m+1)-k} b^k$$

We use the definition of the $(m+1)$st power and the inductive hypothesis:

$$(a+b)^{m+1} = (a+b)(a+b)^m = (a+b) \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$$

Now,

$$\begin{aligned}(a+b)^{m+1} &= (a+b)\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k \\ &= a\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k + b\sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^k \\ &= \sum_{k=0}^{m}\binom{m}{k}a^{(m+1)-k}b^k + \sum_{k=0}^{m}\binom{m}{k}a^{m-k}b^{k+1}\end{aligned}$$

We make a change of variables in the second summation $j = k+1$:

$$(a+b)^{m+1} = \sum_{k=0}^{m}\binom{m}{k}a^{(m+1)-k}b^k + \sum_{j=1}^{m+1}\binom{m}{j-1}a^{(m+1)-j}b^j$$

j is just a dummy variable, so we can rename it k (again)...

$$(a+b)^{m+1} = \sum_{k=0}^{m} \binom{m}{k} a^{(m+1)-k} b^k + \sum_{k=1}^{m+1} \binom{m}{k-1} a^{(m+1)-k} b^k$$

We can now combine the terms $1 \le k \le m$:

$$(a+b)^{m+1} = \binom{m}{0} a^{m+1} + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1} \right] a^{(m+1)-k} b^k + \binom{m}{m} b^{m+1}$$

We use the fact that $\binom{m+1}{m+1} = \binom{m+1}{0} = \binom{m}{m} = \binom{m}{0} = 1$ and ***Pascal's Formula*** to get

$$\begin{aligned}(a+b)^{m+1} &= a^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} a^{(m+1)-k} b^k + b^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} a^{(m+1)-k} b^k \end{aligned}$$

...and Bob's your uncle! $\square$

Let a and b be real numbers and n an integer $n \geq 1$. The expression $(a+b)^n$ can be expanded (using the distributive law) into products of n letters, where each letter is either a or b for each $k = 0, 1, 2, \ldots, n$, the product

$$a^{n-k}b^k = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n-k \textbf{ factors}} \cdot \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{k \textbf{ factors}}$$

occurs as a term in the sum the same number of times as there are orderings of $(n-k)$ a's and k b's.

The number of such orderings is $\binom{n}{k}$, the number of ways to choose k positions in which to place the b's. Hence, when like terms are combined, the coefficient of $a^{n-k}b^k$ in the sum is $\binom{n}{k}$. Thus,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k. \quad \square$$

Example: Estimating a Numerical Power

Which number is larger: $(1.01)^{1,000,000}$ **or** $10,000$**?**

Solution: By the binomial theorem

$$\begin{aligned}
(1.01)^{1,000,000} &= (1+0.01)^{1,000,000} \\
&= 1 + \binom{1,000,000}{1} 1^{999,999} 0.01^1 + \textbf{positive terms} \\
&= 1 + 1,000,000 \cdot 1 \cdot 0.01 + \textbf{positive terms} \\
&= 1 + 10,000 + \textbf{positive terms} \\
&> 10,001 \\
&> 10,000
\end{aligned}$$

Example: Deriving Another Combinatorial Identity

Problem: Use the binomial theorem to show that

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Solution: Since $2 = (1 + 1)$, $2^n = (1 + 1)^n$. We apply the binomial theorem with $a = b = 1$:

$$2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

Consequently,

$$2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}.$$

(Epp-v3.0)

Epp-6.4.6, Epp-6.4.16 , Epp-6.4.19

Epp-6.5.1, Epp-6.5.3, Epp-6.5.5, Epp-6.5.10,

Epp-6.5.11, Epp-6.6.11, Epp-6.6.14

Epp-6.7.1, Epp-6.7.17

(Epp-v2.0)

Epp-6.4.6, Epp-6.4.16 , Epp-6.4.19

Epp-6.5.1, Epp-6.5.3, Epp-6.5.5, Epp-6.5.10,

Epp-6.5.11, Epp-6.6.11, Epp-6.6.14

Epp-6.7.1, Epp-6.7.13