Counting and Probability

Gambling Like a Mathematican

Lecture Notes #13

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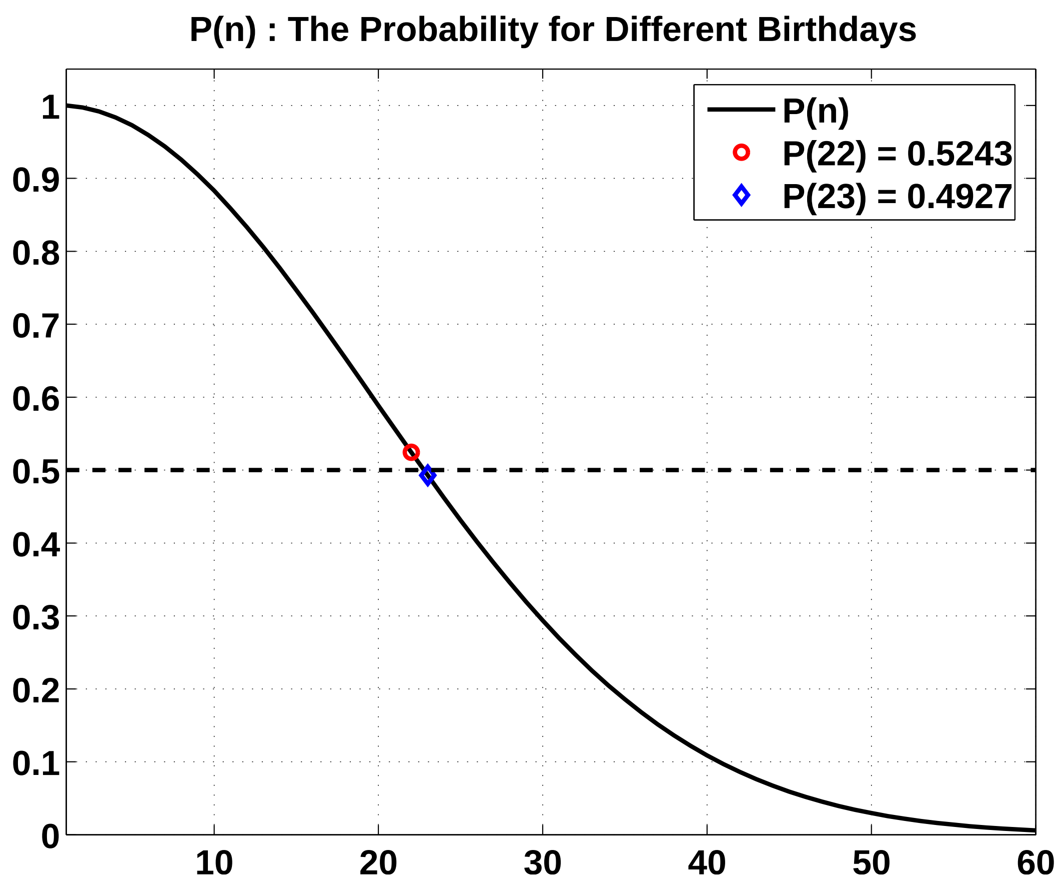
Consider the ***Birthday problem***: The probability that n people have different birthdays is given by

$$P(n) = \frac{365-0}{365} \cdot \frac{365-1}{365} \cdots \frac{365-(n-1)}{365} = \frac{365!}{(365-n)! \cdot 365^n}$$



This means that in a group of 23 or more people, the probability that two persons have the same birthday is greater than 50%; this is somewhat surprising!

In a class of 42 students, the probability of 42 unique birthdays is 0.0860 (8.6%).

Since the result above is somewhat surprising, we can find some unlucky fool who has not taken this class and make ***"a friendly wager."***

In a room of 23 people (selected at random from the population) you will win the bet "at least two people have the same birthday" 50.73% of the time. If you bet with 1-to-1 odds, your average (or ***expected***) payoff (per dollar) is

$$P_{\text{win}} * 2 + P_{\text{lose}} * 0 = \$1.0146$$

so, in the long run you make money!

Expected Value

Formally, the expected value is defined as:

Definition: Expected Value —

Suppose the possible outcomes of an experiment, or random process, are real numbers a_1, a_2, ..., a_n, which occur with probabilities p_1, p_2, ..., p_n. The ***expected value*** of the process is

$$\sum_{k=1}^{n} a_k p_k$$

The "Super Lotto Plus" draw consists of a "Mega" number in the range 1–27, and five numbers in the range 1–47. Hence, there are

$$27 \cdot \binom{47}{5} = 41,416,353 \quad \text{possible combinations}$$

The following combinations constitute winning tickets

Combination	**Possibilities**	**Odds***
5 of 5 + Mega	1	1 in 41,416,353
5 of 5	26	1 in 1,592,936
4 of 5 + Mega	$\binom{5}{4} \cdot 42 = 210$	1 in 197,221
4 of 5	$\binom{5}{4} \cdot 42 \cdot 26 = 5,460$	1 in 7,585
3 of 5 + Mega	$\binom{5}{3} \cdot \binom{42}{2} = 8,610$	1 in 4,810
2 of 5 + Mega	$\binom{5}{2} \cdot \binom{42}{3} = 114,800$	1 in 361
3 of 5	$\binom{5}{3} \cdot \binom{42}{2} \cdot 26 = 223,860$	1 in 185
1 of 5 + Mega	$\binom{5}{1} \cdot \binom{42}{4} = 559,650$	1 in 74
0 of 5 + Mega	$\binom{5}{0} \cdot \binom{42}{5} = 850,668$	1 in 49

In a draw #1837 (11/10/2004) the winning amounts where

Combination	**Odds*** (p_k)	**Value** (a_k)	$\mathbf{a_k \cdot p_k}$
5 of 5 + Mega	1 in 41,416,353	39,000,000	0.941657
5 of 5	1 in 1,592,936	35,844	0.022502
4 of 5 + Mega	1 in 197,221	1,378	0.006987
4 of 5	1 in 7,585	104	0.013711
3 of 5 + Mega	1 in 4,810	49	0.010187
2 of 5 + Mega	1 in 361	9	0.024930
3 of 5	1 in 185	10	0.054054
1 of 5 + Mega	1 in 74	1	0.013513
0 of 5 + Mega	1 in 49	1	0.020408
Expected Value:			$\mathbf{1.107950}$

Hence, in this particular draw, the expected “return of investment” per dollar was 1.10. The jackpot starts at \$7,000,000, in such a draw the expected value is (approximately) only 0.20.

The "Mega Millions" draw consists of a "Mega" number in the range 1–46, and five numbers in the range 1–56. Hence, there are

$$46 \cdot \binom{56}{5} = 175,711,536 \quad \text{possible combinations}$$

The following combinations constitute winning tickets

Combination	**Possibilities**	**Odds***
5 of 5 + Mega	1	1 in 175,711,536
5 of 5	45	1 in 3,904,701
4 of 5 + Mega	$\binom{5}{4} \cdot 51 = 255$	1 in 689,065
4 of 5	$\binom{5}{4} \cdot 51 \cdot 45 = 11,475$	1 in 15,313
3 of 5 + Mega	$\binom{5}{3} \cdot \binom{51}{2} = 12,750$	1 in 13,781
3 of 5	$\binom{5}{3} \cdot \binom{51}{2} \cdot 45 = 573,750$	1 in 306
2 of 5 + Mega	$\binom{5}{2} \cdot \binom{51}{3} = 208,250$	1 in 844
1 of 5 + Mega	$\binom{5}{1} \cdot \binom{51}{4} = 1,249,500$	1 in 141
0 of 5 + Mega	$\binom{5}{0} \cdot \binom{51}{5} = 2,349,060$	1 in 75

Expected Value: Comments

From a mathematical standpoint it makes "sense" to play the California lottery (or any other state lottery, *e.g.* "powerball") when the value of the jackpot approaches the number of combinations possible in the lottery. Note that ***your chances of winning do not improve***, but ***the expected payoff is greater than the price of the ticket!***

In all ***Las Vegas*** style gambling, the expected payoff is less than one-to-one; so in the long run the "house" always wins.

In some games, *e.g.* ***Blackjack***, the odds (and therefore expected payoff) change as the game progresses. This is where "counting cards" (keeping track of what cards have been played) can help the player achieve an expected value greater than 1. Casinos combat this by shuffling as many as 8 decks of cards together, and only playing half of the cards before the next reshuffle.

See: Edward O. Thorp, *"Beat the Dealer A Winning Strategy for the Game of Twenty-One,"* Vintage; Revised edition (April 12, 1966).

Poker: 5-Card Draw, 52-card Deck

There are $\binom{52}{5} = 2,598,960$ possible poker hands. In decreasing order of "value," the following are the hands of interest:

Hand	**Count**	**Count**	**Odds***	**Probability**
Royal Flush	4	4	1 in 649,740	$1.5391 \cdot 10^{-6}$
Straight Flush	$4 \cdot 10 - 4$	36	1 in 72,193	$1.3852 \cdot 10^{-5}$
Four-of-a-Kind	$13 \cdot 48$	624	1 in 4,165	$2.4010 \cdot 10^{-4}$
Full House	$13\binom{4}{3} \cdot 12\binom{4}{2}$	3,744	1 in 694	0.0014
Flush	$4\binom{13}{5} - 36 - 4$	5,108	1 in 509	0.0020
Straight	$10 \cdot 4^5 - 36 - 4$	10,200	1 in 255	0.0039
Three-of-a-Kind	$13\binom{4}{3}\frac{48 \cdot 44}{2}$	54,912	1 in 47	0.0211
Two Pairs	$\binom{13}{2}\binom{4}{2}\binom{4}{2}44$	123,552	1 in 21	0.0475
One Pair	$13\binom{4}{2}\binom{12}{3}4^3$	1,098,240	1 in 2	0.4226

* Odds rounded to closest integer.

Reference: Poker Hands

Royal Flush: A♣-K♣-Q♣-J♣-10♣, or A♠-K♠-Q♠-J♠-10♠, or **A**♡-**K**♡-**Q**♡-**J**♡-**10**♡, or **A**♢-**K**♢-**Q**♢-**J**♢-**10**♢.

Straight Flush: Sequence of five cards in either suit, except A-K-Q-J-10; *e.g.* K♣-Q♣-J♣-10♣-9♣

Four-of-a-Kind: *E.g.* A♠-**A**♡-A♣-**A**♢-**X**

Full House: Three-of-a-kind and a pair, *e.g.* A♠-**A**♡-A♣-**K**♢-**K**♡

Flush: All five cards of same suit, but not in sequence.

Straight: All five cards in sequence, but not of the same suit.

Three-of-a-kind: Three-of-a-kind, and two other (non-paired) cards *e.g.* A♠-**A**♡-A♣-**K**♢-**10**♡

Two pairs: Two (different) pairs, and one "other" card*e.g.* A♠-**A**♡-K♣-**K**♢-**10**♡

Pair: One pair, and three "other" cards *e.g.* A♠-**A**♡-K♣-**Q**♢-**J**♡

Texas Hold'em

Texas Hold'em is played:

(1) ***Preflop:*** each player gets two "pocket" cards (face down),

(2) Betting,

(3) ***The Flop:*** 3 cards are put face-up on the table,

(4) Betting,

(5) ***The Turn:*** one card is put face up on the table,

(6) Betting,

(7) ***The River:*** one card is put face up on the table,

(8) Betting.

Each player creates a poker hand out of five cards from his/her two "pocket" cards and the "community cards" on the table. The strongest hand wins.

Here, order (and definitely psychology) matters!

There are $\binom{52}{2} = 1,326$ possible pocket-card combinations.

The strongest preflop is a pair of aces, of which there are $\binom{4}{2} = 6$ possibilities, hence the probability for this event is $6/1326 = 0.0045$ (1 in 221).

The probability for any (specified) pair, is the same: $6/1326 = 0.0045$ (1 in 221), and the probability for any (some) pair is $(6 \cdot 13)/1326 = 0.0588$ (1 in 17).

In a ***3-player game***, the probability that you have pocket aces, and both the other players have pocket kings:

$$\frac{\binom{4}{2}\binom{4}{2}\binom{2}{2}}{\binom{52}{6}} = \frac{36}{20,358,520} = 1.7683 \cdot 10^{-6}$$

Here, you are almost guaranteed a win (how can you lose?)

Hand	Count	Count	Odds	Probability
Pair of Aces	$\binom{4}{2}$	6	1 in 221	0.0045
Some pair	$13\binom{4}{2}$	78	1 in 17	0.0588
Ace-King suited	4	4	1 in 332	0.0030
Ace-King off-suit*	4·3	4	1 in 111	0.0090
Two cards suited	$4\cdot\binom{13}{2}$	312	1 in 4	0.2352

* a.k.a "Big Slick"

This far, it's pretty straight forward. As long as you know that $\binom{4}{2} = 6$, $\binom{13}{2} = 78$, and $\binom{52}{2} = 1,326$ calculating the odds/probabilities can be done quickly in your head (with some practice).

The flop is something of a main event... There are $\binom{52}{3} = 22,100$ three-card combinations that can flop.

From the point of view of the TV-audience, there are $\binom{52-2n}{3}$ possible flops in an n-player game (since $2n$ cards are already "known.")

n	2	3	4	5	6	7
# flops	17,296	15,180	13,244	11,480	9,880	8,436

From the point of view of a player, holding two cards from the pre-flop, there are now $\binom{50}{3} = 19,600$ possible flops. From now on, unless otherwise specified, we take the stance of one player:

You, the player, hold A♠-K♠, from the preflop.

Flop	Count	Count	Odds	Probability	Hand
Q♠-J♠-10♠	1	1	1 in 19,600	$5.1020 \cdot 10^{-5}$	Royal Flush
A♣-**A**♡-**A**♢	1	1	1 in 19,600	$5.1020 \cdot 10^{-5}$	Four-of-a-kind
K♣-**K**♡-**K**♢	1	1	1 in 19,600	$5.1020 \cdot 10^{-5}$	Four-of-a-kind
A-A-K	$\binom{3}{2}\binom{3}{1}$	9	1 in 2,178	$4.5918 \cdot 10^{-4}$	Full House (A-K)
A-K-K	$\binom{3}{1}\binom{3}{2}$	9	1 in 2,178	$4.5918 \cdot 10^{-4}$	Full House (K-A)
A-A-(non A/K)	$\binom{3}{2}\binom{44}{1}$	132	1 in 148	0.0067	Three-of-a-Kind
K-K-(non A/K)	$\binom{3}{2}\binom{44}{1}$	132	1 in 148	0.0067	Three-of-a-Kind
Any 3 ♠	$\binom{11}{3}$	165	1 in 119	0.0084	Flush
A-K-(non A/K)	$\binom{3}{1}\binom{3}{1}\binom{44}{1}$	396	1 in 49	0.0202	2 Pairs
Any 2 ♠ + ♡/♢/♣	$\binom{11}{2}\binom{39}{1}$	2145	1 in 9	0.1094	Flush Draw^{*1}
No ♠, A or K	$\binom{33}{3}$	5,456	1 in 5	0.2784	Mostly Nothing^{*2}

^{*1} Includes pair of aces, and [logical xor] pair of kings.

^{*2} Includes 27 A-K-Q-J-10 straights, and an untold number of straight-draws.

You, the player, hold K♠-K♣, from the preflop.

Flop	Count	Count	Odds	Probability	Hand
3 Aces	$\binom{4}{3}$	4	1 in 4900	$2.0408 \cdot 10^{-4}$	Full House (A-K)*[1]
K-A-A	$\binom{2}{1}\binom{4}{2}$	12	1 in 1633	$6.1237 \cdot 10^{-4}$	Full House (K-A)*[2]
K-K-other	48	48	1 in 408	0.0024	Four-of-a-Kind
K-(non A pair)	$\binom{2}{1}\binom{11}{1}\binom{4}{2}$	132	1 in 148	0.0068	Full House (K+other)
Any 3 ♠	$\binom{12}{3}$	220	1 in 89	0.0112	Flush Draw
K-(non pair)	$\binom{2}{1}\binom{12}{2}\binom{4}{1}\binom{4}{1}$	384	1 in 51	0.0196	Three-of-a-Kind
Pair-(non K)	$\binom{12}{2}\binom{4}{2}\binom{4}{1}$	1584	1 in 12	0.0808	2 Pairs*[3]
Any 2 ♠	$\binom{12}{2}\binom{39}{1}$	2,574	1 in 8	0.1313	Flush Draw*[4]
3 non-pair, non-K	$\binom{12}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}$	14,080	1 in 1.4	0.7184	Pair-of-K

*[1] Loses to a player holding the remaining ace (four-of-a-kind).

*[2] Loses to pocket aces (four-of-a-kind).

*[3] Loses to three-of-a-kind.

*[4] Long shot, results in flush 1 in 24 (0.0416).

You hold **A♠-A♢**, and the flop is **K♣-Q♠-J♢**

Final Hand	Count	Count	Odds	Probability	Comment
A♠-A♢-A♣-A♡	1	1	1 in 1081	$9.2507 \cdot 10^{-4}$	You win
A-A-K-K-K	$\binom{3}{2}$	3	1 in 360	0.0028	
A-A-A-K-K	$\binom{2}{1}\binom{3}{1}$	6	1 in 180	0.0056	
A-A-A-(K-K or Q-Q or J-J)	$\binom{2}{1}\binom{9}{1}$	18	1 in 60	0.0167	
A-A-(low pair)	$\binom{8}{1}\binom{4}{2}$	48	1 in 22	0.0444	
A-A-A	$\binom{2}{1}\binom{9\cdot 4}{1}$	72	1 in 15	0.0666	
A-A-K-K	$\binom{3}{1}\binom{38}{1}$	114	1 in 9	0.1055	
A-K-Q-J-10	$\binom{4}{1}\cdot 43+\binom{4}{2}$	178	1 in 6	0.1647	
A-A	$\binom{8}{2}\binom{4}{1}\binom{4}{1}$	448	1 in 2	0.4144	

Note that two pairs is quite vulnerable, since it loses to three-of-a-kind. With a pair on the table, there is a $(\binom{2}{1}\cdot 44+\binom{2}{2})/\binom{45}{2} = 89/990 = 0.0899$ probability that a particular opponent has at three- or four-of-a-kind.

You hold **A**$\diamondsuit$-**Q**$\diamondsuit$, and the flop is **7**$\diamondsuit$-**4**$\diamondsuit$-**2**$\clubsuit$

Final Hand	**Count**	**Count**	**Odds**	**Probability**	**Note**
$\diamondsuit$-Flush	$\binom{9}{2}+\binom{9}{1}\binom{38}{1}$	378	1 in 3	0.3497	
A-A-A	$\binom{3}{2}$	3	1 in 360	0.0028	
Q-Q-Q	$\binom{3}{2}$	3	1 in 360	0.0028	
A-A-Q-Q	$\binom{3}{1}\binom{3}{1}$	9	1 in 120	0.0083	
A-A-7-7	$\binom{3}{1}\binom{3}{1}$	9	1 in 120	0.0083	$*^1$
Q-Q-7-7	$\binom{3}{1}\binom{3}{1}$	9	1 in 120	0.0083	$*^1$
A-A or Q-Q	$\binom{6}{1}\binom{24}{1}$	144	1 in 8	0.1332	

$*^1$ When the second pair comes completely from the board you are vulnerable since a single 7 in an opponents hand gives him/her a three-of-a-kind.

What Are the Odds of Winning?!?

So far we have calculated the probabilities that you can form certain hands.

The ***real*** question is whether you will win or lose. At every stage of the game you can in a similar way compute the probabilities that your opponent(s) have stronger hands than you. After the flop, there are $\binom{47}{2} = 1,081$ two-card combinations (pocket cards) your opponent(s) may be holding.

If your hand is vulnerable to single cards, then you want to force other opponents out of the game by betting aggressively; this will reduce your risk of losing to a "dumb-luck hand."

In the long run, understanding the relative strengths of the hands and being able to quickly (in your head) compute rough estimates of the probabilities "on the fly" should give you an advantage over the average player. Just remember that in any given round of play dumb luck beats skill! (Never bet more than you can afford to lose!)