Relations on Sets

Reflexivity, Symmetry and Transitivity; Equivalence Relations

Lecture #14

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Relations: Introduction

Mathematical Relations — Examples:

* Two logical expressions can be said to be related if they have the same truth tables.
* A set A can be said to be related to a set B if $A \subseteq B$.
* A real number x can be said to related to y if $x < y$.
* An integer n can be said to related to m if $n|m$.
* An integer n can be said to related to m if n and m are both odd.
* Etc, etc, etc, ...

We are going to study ***mathematical relations on sets***: their properties and representations.

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$.

The relation: Let an element $x \in A$ be related to an element $y \in B$ if and only if $x < y$.

Notation: $x\,R\,y \equiv$ **"x is related to y"**, $x \not R\, y \equiv$ **"x is not related to y"**

We have the following relations:

$0\,R\,1$	since	$0 < 1$	$1 \not R\, 1$	since	$1 \not< 1$
$0\,R\,2$	since	$0 < 2$	$2 \not R\, 1$	since	$2 \not< 1$
$0\,R\,3$	since	$0 < 3$	$2 \not R\, 2$	since	$2 \not< 2$
$1\,R\,2$	since	$1 < 2$			
$1\,R\,3$	since	$1 < 3$			
$2\,R\,3$	since	$2 < 3$			

Relations and Cartesian Products:

The Cartesian product $(A \times B)$ of two sets A and B is the set of all ordered pairs whose first element is in A and second elements in B:

$$A \times B = \{(x, y) \,|\, x \in A \text{ and } y \in B\}$$

In our example

$$A \times B = \{(0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$$

The elements of some ordered pairs

$$\{(0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}$$

are considered to be related (others are not).

Knowing which ordered pairs are in this set is equivalent to knowing which elements are related.

Relations: Formal Definition

Definition: ***Binary Relation*** —
Let A and B be sets. A **(binary) relation** R **from** A **to** B is a subset of $A \times B$. Given an ordered pair $(x, y) \in A \times B$, x **is related to** y **by** R, written $x\,R\,y$, if and only if $(x, y) \in R$.

Symbolic Notation

$$x\,R\,y \quad \Leftrightarrow \quad (x, y) \in R$$

$$x \not{R}\, y \quad \Leftrightarrow \quad (x, y) \notin R$$

The term ***binary*** is used in the definition to indicate that the relation is a subset of the Cartesian product of ***two*** sets.

Illustration: Relations

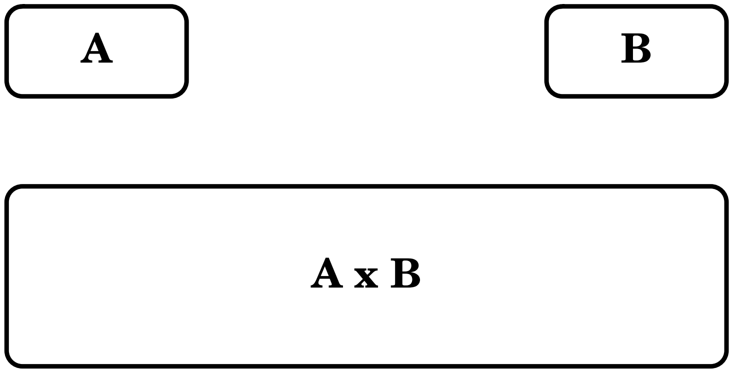


Figure: Given 2 sets A and B, we form the Cartesian product $A \times B$; $(x, y) \in A \times B \equiv (x \in A)$ **and** $(y \in B)$.

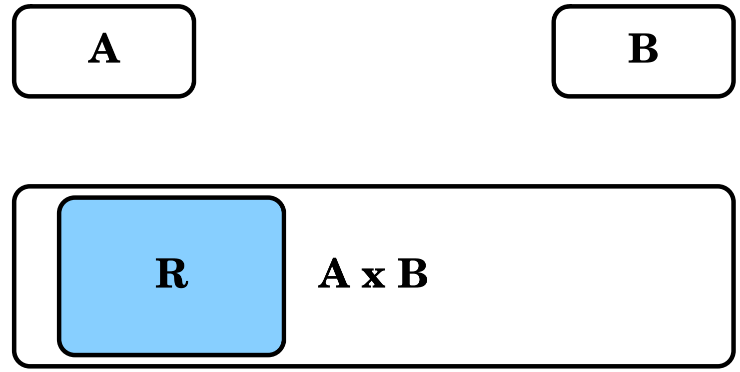


Figure: The Relation R is a subset of $A \times B$. If and only if $(x, y) \in R$ we say that x is related to y by R, symbolically $x\,R\,y$.

The subset $R \subseteq A \times B$ can be specified

1. Directly / Explicitly, by indicating what pairs $(x, y) \in R$. This is only feasible when A and B are finite (and small) sets.
2. By specifying a rule for what elements are in R, *e.g.* by saying that $(x, y) \in R$ if and only if $x = y^2$.

We generalize the previous example to the set of all integers $\mathbb{Z}$, *i.e.*

$$\text{for all } (m, n) \in \mathbb{Z} \times \mathbb{Z},\ m\,R\,n \ \Leftrightarrow\ m - n \text{ is even}$$

Questions:

(a) is $4\,R\,0$? $2\,R\,6$? $3\,R\,(-3)$? $5\,R\,2$?

(b) List 5 integers that are related by R to 1.

(c) Prove that if n is odd, then $n\,R\,1$.

Answers:

(a-i) Yes, $4\,R\,0$, since $4 - 0 = 4$ is even.

(a-ii) Yes, $2\,R\,6$, since $2 - 6 = -4$ is even.

(a-iii) Yes, $3\,R\,(-3)$, since $3 - (-3) = 6$ is even.

(a-iv) No, $5 \not R\ 2$, since $5 - 2 = 3$ is odd.

(b) There are infinitely many examples, *e.g.*

1	since $1 - 1 = 0$	is even
11	since $11 - 1 = 10$	is even
111	since $111 - 1 = 110$	is even
1111	since $1111 - 1 = 1110$	is even
11111	since $11111 - 1 = 11110$	is even

(c) **Proof:** Suppose n is any odd integer. Then $n = 2k + 1$ for some integer k. By substitution

$$n - 1 = 2k + 1 - 1 = 2k \text{ is even}$$

Hence

$$n\, R\, 1, \ \forall n \text{ odd.} \quad \square$$

Representation: Arrow Diagrams for Relations

Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$

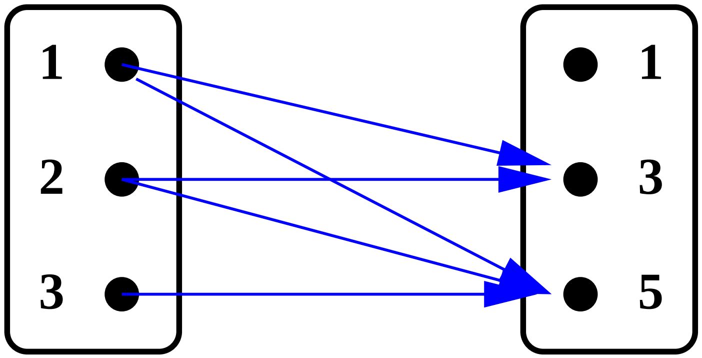


Figure: Arrow diagram representation of the relation

$$\text{for all } (x, y) \in A \times B,$$
$$(x, y) \in R \Leftrightarrow x < y$$

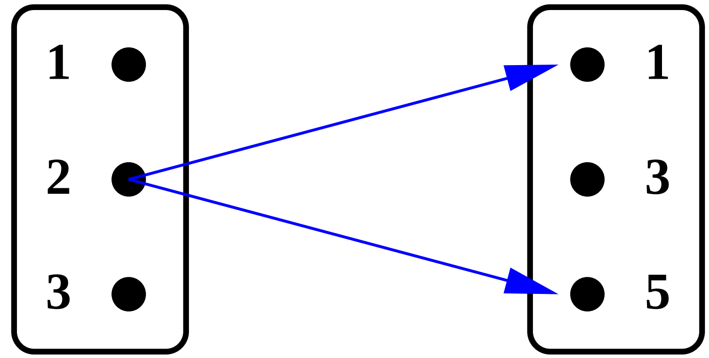


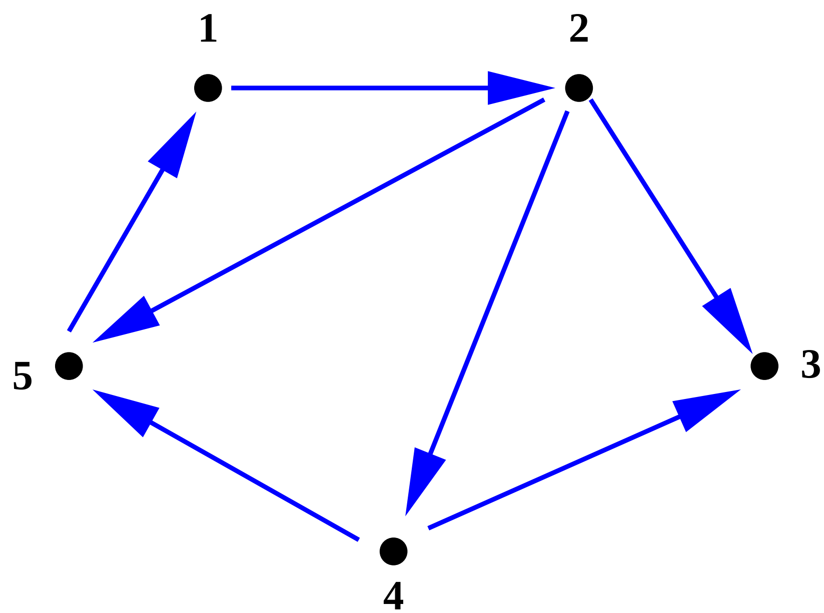
Figure: Arrow diagram representation of the relation

$$R = \{(2, 1),\ (2, 5)\}$$

Notes: *(i)* It is possible to have an element that does not have an arrow coming out of it; *(ii)* It is possible to have several arrows coming out of the same element of A pointing in different directions; *(iii)* It is possible to have an element in B that does not have an arrow pointing to it.

Definition: A **binary relation on a set $\mathbf{A}$** is a binary relation from A to A.

In this case, we can modify the arrow diagram to be a **directed graph** — instead of representing A twice, we only represent it once and draw arrows from each point of A to each related point, *e.g.*



there is an arrow from x to y $\Leftrightarrow \ \mathbf{x\,R\,y} \ \Leftrightarrow \ (\mathbf{x},\mathbf{y}) \in \mathbf{R}$

Example: Directed Graph of a Relation

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a binary relation R on A:

$$R = \{(x, y) \in A \times A : \; 2|(x - y)\}$$

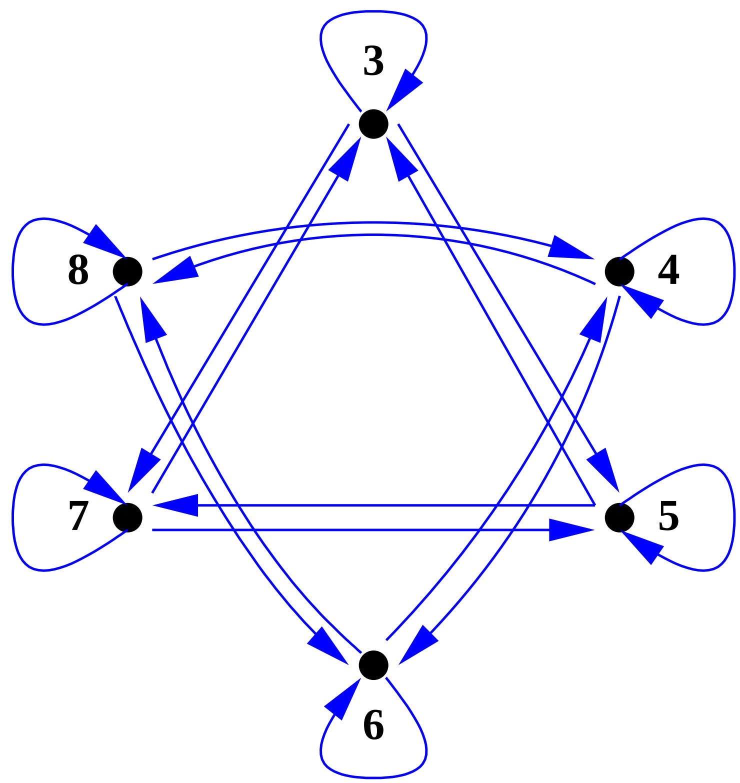


Figure: We notice that the graph must be symmetric, since if $2|n$, then $2|(-n)$. Since $2|0$, there is a loop at every node in the graph.

Properties of a Binary Relation on One Set A

Recall:

> ***Definition:*** A **binary relation on a set $\mathbf{A}$** is a binary relation from A to A.

In the context of a binary relation on a set, we can name 3 properties:

> ***Definition:*** Let R be a binary relation on a set A
>
> 1. R is **Reflexive** if and only if $\forall x \in A$, $x\,R\,x$.
> 2. R is **Symmetric** if and only if $\forall x, y \in A$, if $x\,R\,y$ then $y\,R\,x$.
> 3. R is **Transitive** if and only if $\forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

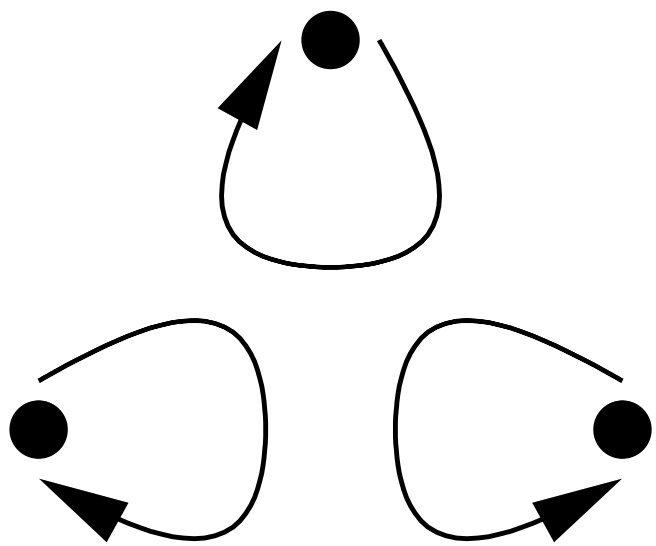
Reflexivity

Formal: R is **Reflexive** if and only if $\forall x \in A,\ x\,R\,x$.

Functional: R is **Reflexive** $\Leftrightarrow$ for all $x \in A$, $(x, x) \in R$.

Informal: Each element is related to itself.

Graph: Each point of the graph has an arrow looping around back to itself.



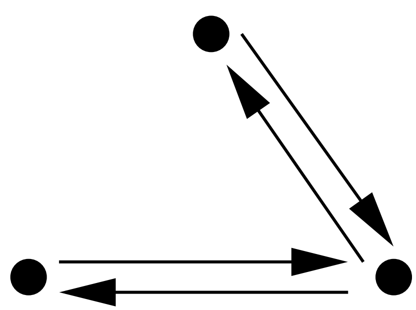
Symmetry

Formal: R is **Symmetric** if and only if $\forall x, y \in A$, if $x\,R\,y$ then $y\,R\,x$.

Functional: R is **Symmetric** $\Leftrightarrow$ for all $x, y \in A$, if $(x, y) \in R$ then $(y, x) \in R$.

Informal: If one element is related to a second element, then the second element is related to the first.

Graph: In all cases where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.



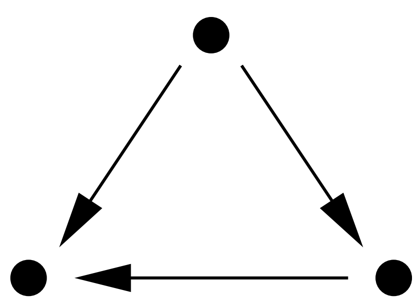
Transitivity

Formal: R is **Transitive** if and only if $\forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$ then $x\,R\,z$.

Functional: R is **Transitive** $\Leftrightarrow$ for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Informal: If one element is related to a second element, and that second element is related to a third element, then the first element is related to the third element.

Graph: In all cases where there is an arrow going from one point to a second, and from the second point to a third, there is an arrow going from the first point to the third.



Non-Reflexivity, Non-Symmetry, and Non-Transitivity

If R is a binary relation defined on a set A, then

1. R **is not reflexive** $\Leftrightarrow$ there is an element $x \in A$ such that $x \not R\, x$, *i.e.* $(x, x) \notin R$.

2. R **is not symmetric** $\Leftrightarrow$ there are elements $x, y \in A$ such that $x\, R\, y$ but $y \not R\, x$, *i.e.* $(x, y) \in R$, but $(y, x) \notin R$.

3. R **is not transitive** $\Leftrightarrow$ there are elements $x, y, z \in A$ such that $x\, R\, y$ and $y\, R\, z$ but $x \not R\, z$, *i.e.* $(x, y), (y, z) \in R$, but $(x, z) \notin R$.

To show that a binary relation does ***not*** have one of the properties, it is sufficient to find a counterexample.

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T:

$$R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$$

$$S = \{(0,0), (0,2), (0,3), (2,3)\}$$

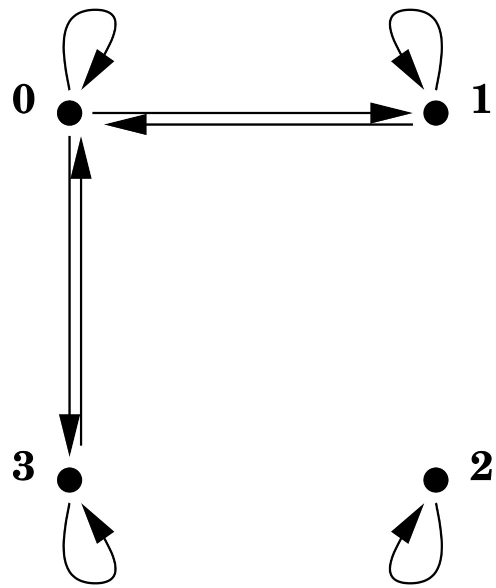
$$T = \{(0,1), (2,3)\}$$

Fill in the table:

	Reflexive	Symmetric	Transitive
R			
S			
T			

We have $A = \{0, 1, 2, 3\}$ and

$$R = \{(0,0), (0,1), (0,3), (1,0), (1,1), (2,2), (3,0), (3,3)\}$$



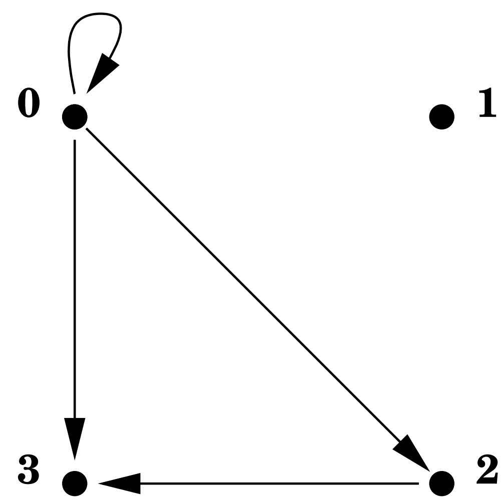
R is reflexive since there is a loop at each point in the directed graph.

R is symmetric since in for every arrow going from one point to another, there is another arrow going back.

R is not transitive since *e.g.* $1\,R\,0$ and $0\,R\,3$ but $1\,\not R\,3$ *i.e.* there is no “short-cut” arrow connecting 1 and 3.

We have $A = \{0, 1, 2, 3\}$ and

$$S = \{(0,0), (0,2), (0,3), (2,3)\}$$



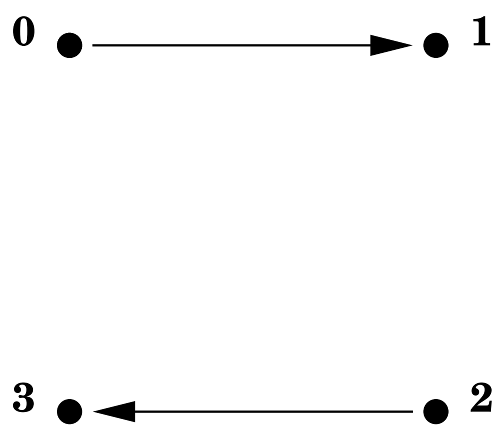
S is not reflexive since there are missing loops at 1, 2, and 3.

S is not symmetric, the arrows from 2-to-0, 3-to-0, and 3-to-2 are missing.

S is transitive since there is always a “short-cut” arrow so that if $(x, y) \in S$ and $(y, z) \in S$ then $(x, z) \in S$.

We have $A = \{0, 1, 2, 3\}$ and

$$T = \{(0, 1), (2, 3)\}$$



T is not reflexive since there are missing loops at 0, 1, 2, and 3.

T is not symmetric, the arrows from 1-to-0, and 3-to-2 are missing.

T is transitive since it is ***not*** *not transitive*.

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T:

$$R = \{(0,0),(0,1),(0,3),(1,0),(1,1),(2,2),(3,0),(3,3)\}$$

$$S = \{(0,0),(0,2),(0,3),(2,3)\}$$

$$T = \{(0,1),(2,3)\}$$

Fill in the table:

	Reflexive	Symmetric	Transitive
R	Yes	Yes	No
S	No	No	Yes
T	No	No	Yes

Irreflexivity, Anti-Symmetry, and Intransitivity

Definition: Let R be a binary relation on a set A

1. R is **Irreflexive** if and only if $\forall x \in A$, $x \not R\, x$.
2. R is **Anti-symmetric** if and only if $\forall x, y \in A$, if $x\, R\, y$ then $y \not R\, x$.
3. R is **Intransitive** if and only if $\forall x, y, z \in A$, if $x\, R\, y$ and $y\, R\, z$ then $x \not R\, z$.

- R can be reflexive, non-reflexive, or irreflexive,
- R can be symmetric. non-symmetric, or anti-symmetric
- R can be transitive, non-transitive, or intransitive.

Think about these definitions!!!

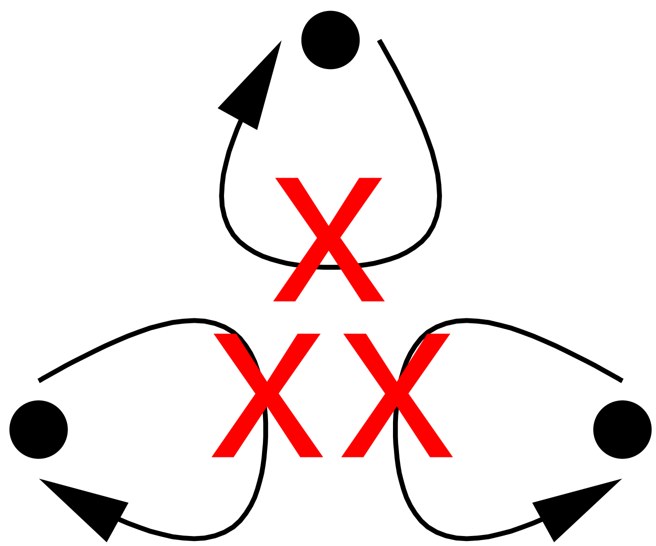
Irreflexivity

Formal: R is **Irreflexive** if and only if $\forall x \in A$, $\mathbf{x} \not\mathbf{R}\, \mathbf{x}$.

Functional: R is **Irreflexive** $\Leftrightarrow$ for all $x \in A$, $(\mathbf{x}, \mathbf{x}) \notin \mathbf{R}$.

Informal: No element is related to itself.

Graph: No point of the graph has an arrow looping around back to itself.



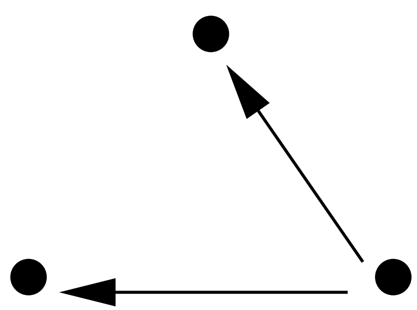
Anti-Symmetry

Formal: R is **Anti-Symmetric** if and only if $\forall x, y \in A$, if $x \, R \, y$ then $\mathbf{y} \not\mathbf{R} \, \mathbf{x}$.

Functional: R is **Anti-Symmetric** $\Leftrightarrow$ for all $x, y \in A$, if $(x, y) \in R$ then $(\mathbf{y}, \mathbf{x}) \notin \mathbf{R}$.

Informal: If one element is related to a second element, then the second element is **NOT** related to the first.

Graph: In all cases where there is an arrow going from one point to a second, there is **no** arrow going from the second point back to the first.



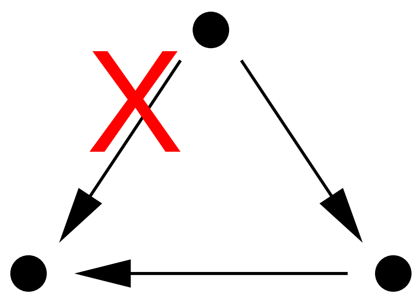
Intransitivity

Formal: R is **Intransitive** if and only if $\forall x, y, z \in A$, if $x\,R\,y$ and $y\,R\,z$ then $\mathbf{x} \not{\mathbf{R}}\ \mathbf{z}$.

Functional: R is **Intransitive** $\Leftrightarrow$ for all $x, y, z \in A$, if $(x, y) \in R$ and $(y, z) \in R$ then $(\mathbf{x}, \mathbf{z}) \notin \mathbf{R}$.

Informal: If one element is related to a second element, and that second element is related to a third element, then the first element is **not** related to the third element.

Graph: In all cases where there is an arrow going from one point to a second, and from the second point to a third, there is **never** an arrow going from the first point to the third (no shortcut exist, anywhere.).



Example: Equality ($=$) on $\mathbb{R}$

Let $A = \mathbb{R}$ (the set of real numbers), and define the relation R

$$x \, R \, y \quad \Leftrightarrow \quad x = y$$

Properties:

R is reflexive: R is reflexive if and only if $\forall x \in \mathbb{R}$, $x \, R \, x$. Here, this means $x = x$, *i.e.* $\forall x \in \mathbb{R}$ $x = x$. This statement is certainly true; every real number equals itself.

R is symmetric: This is true since if $x = y$ then $y = x$, hence $(x, y) \in R$ and $(y, x) \in R$.

R is transitive: This is true since if $x = y$ and $y = z$, then $x = z$.

Example: Less Than ($<$) on $\mathbb{R}$

Let $A = \mathbb{R}$ (the set of real numbers), and define the relation R

$$x \, R \, y \quad \Leftrightarrow \quad x < y$$

Properties:

R is irreflexive: If $x \, R \, x$ then $x < x$, but that is never true, hence $x \not R \, x$ $\forall x \in \mathbb{R}$.

R is anti-symmetric: If $x \, R \, y$ then $x < y$, which means $y \not< x$ *i.e.* $y \not R \, x$.

R is transitive: This is true since if $x < y$ and $y < z$, then $x < z$.

Example: Congruence Modulo 3 on $\mathbb{Z}$

We define a relation R on $\mathbb{Z}$ as follows

$$\forall m, n \in \mathbb{Z} : \quad m\,R\,n \quad \Leftrightarrow \quad 3|(m-n)$$

R is reflexive: Suppose m is an integer. Now, $m - m = 0$ and $3|0$ since $0 = 3 \cdot 0$, so by definition of R we have $m\,R\,m$. $\square$

R is symmetric: Suppose $m, n \in \mathbb{Z}$ such that $m\,R\,n$. By definition of R we have $3|(m-n) \Leftrightarrow m - n = 3 \cdot k$, for some $k \in \mathbb{Z}$. Multiplying both sides by (-1) gives $n - m = 3 \cdot (-k)$, which shows $3|(n-m)$, hence $n\,R\,m$. $\square$

R is transitive: Suppose $m, n, p \in \mathbb{Z}$ such that $m\,R\,n$ and $n\,R\,p$. We have $3|(m-n)$ and $3|(n-p)$, and we can write $(m-n) = 3r$ and $(n-p) = 3s$ for some $r, s \in \mathbb{Z}$. Adding the two gives $(m-n)+(n-p) = (m-p) = 3(r+s)$ which shows that $3|(m-p)$. Hence $m\,R\,p$, and it follows that R is transitive. $\square$

Equivalence Relations: Different, but the Same...

Idea: We are going to group elements that look different, but really are the same...

Example: Think about the rational numbers, there are several ways of writing the same fraction, *e.g.*

$$\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{4711}{9422}$$

We can define a relation on $\mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers

$$R = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x = y\}$$

now $\left(\frac{1}{2}, \frac{2}{4}\right) \in R$, $\left(\frac{4711}{9422}, \frac{2}{4}\right) \in R$, $\left(\frac{1}{3}, \frac{2}{6}\right) \in R$, etc...

A Relation Induced by a Partition

Recall:

Definition: A collection of non-empty sets $\{A_1, A_2, \dots, A_n\}$ is a **partition** of a set A if and only if

1. $A = A_1 \cup A_2 \cup \dots \cup A_n$.

2. $A_1, A_2, \dots, A_n$ are mutually disjoint.

Definition: Given a partition of a set A the **binary relation induced by the partition**, R, is defined on A as follows

$\forall x, y \in A, \quad x\,R\,y \quad \Leftrightarrow$ there is a set A_i of the partition such that both $x \in A_i$ and $y \in A_i$.

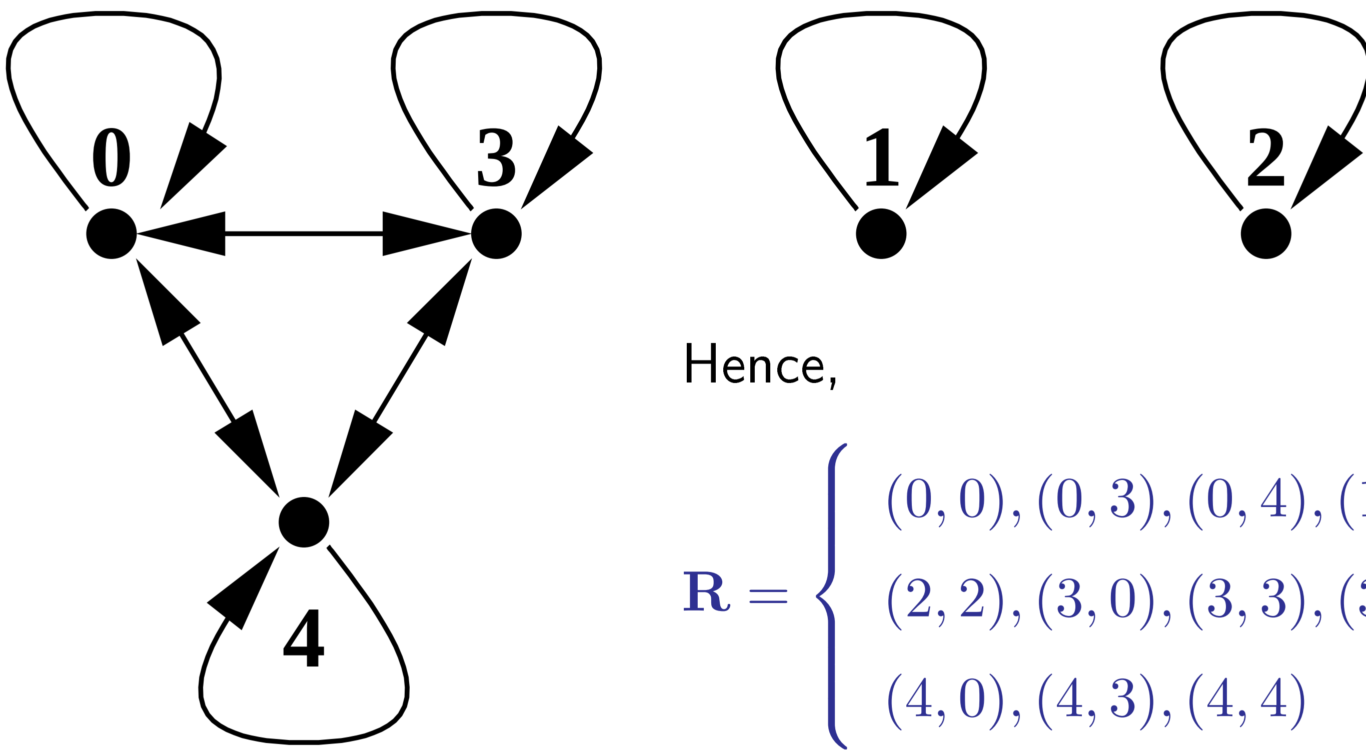
We need an example to make sense out of this definition...

Example: Relation Induced by a Partition

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$A_1 = \{0, 3, 4\}, \quad A_2 = \{1\}, \quad A_3 = \{2\}$$

Now, two elements $x, y \in A$ are related if and only if they belong to the same subset of the partition...



Hence,

$$\mathbf{R} = \left\{ \begin{array}{l} (0,0), (0,3), (0,4), (1,1), \\ (2,2), (3,0), (3,3), (3,4), \\ (4,0), (4,3), (4,4) \end{array} \right\}$$

Equivalence Relations

Theorem: Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Definition: ***Equivalence Relation*** —
Let A be a non-empty set and R a binary relation on A. R is an **equivalence relation** if and only if R is reflexive, symmetric, and transitive.

Example: By the theorem the relation induced by a partition is an equivalence relation.

Notation: Congruence Modulo n

Notation: Let $m, n, d \in \mathbb{Z}$ with $d > 0$. The notation

$$m \equiv n \pmod{d}$$

is read "m ***is congruent to*** n ***modulo*** d" and means that

$$d|(m-n)$$

Symbolically,

$$m \equiv n \pmod{d} \qquad \Leftrightarrow \qquad d|(m-n)$$

Recall the ***Quotient-Remainder Theorem:***

Theorem: Given any integer n and a positive integer d, there exist unique integers q (the quotient) and r (the remainder) such that

$$n = d \cdot q + r, \quad \text{and} \quad 0 \leq r < d$$

Equivalence Relation: Congruence Modulo 3

Let R be the relation $R = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : \; m = n(\text{mod } 3)\}$. We show that this is an equivalence relation.

[Reflexivity] Let $m \in Z$, then $3|(m - m)$ since $0 = 3 \cdot 0$, and it follows that $m\,R\,m$.

[Symmetry] Let $m, n \in Z$, so that $m\,R\,n$. We have $3|(m - n) \Leftrightarrow (m - n) = 3 \cdot k$ for some $k \in \mathbb{Z} \Leftrightarrow (n - m) = 3 \cdot (-k)$ $\Leftrightarrow 3|(n - m) \Leftrightarrow n\,R\,m$.

[Transitivity] Let $m, n, p \in Z$, so that $m\,R\,n$ and $n\,R\,p$. We have

$$\begin{array}{ccl} 3|(m-n) & \Leftrightarrow & (m-n) = 3 \cdot r, \; r \in \mathbb{Z} \\ 3|(n-p) & \Leftrightarrow & (n-p) = 3 \cdot s, \; s \in \mathbb{Z} \\ \hline & \textbf{add} & (m-p) = 3 \cdot (r+s) \end{array}$$

Hence $3|(m - p)$ and we have $m\,R\,p$. $\square$

Equivalence Classes

Suppose we have a set A and an equivalence relation R on A. Given a particular element $x \in A$ it is natural to ask the question ***"what elements are related to x?"***

All the elements that are related to x form a subset of A and this subset is called ***the equivalence class of*** x:

> **Definition:** Suppose A is a set and R is an equivalence relation on A. For each element $x \in A$, the **equivalence class of** x, denoted $[\mathbf{x}]$ and called the **class of** x for short, is the set of all elements $y \in A$ such that $y\,R\,x$.

Symbolically,

$$[x] = \{y \in A |\ y\,R\,x\}$$

Let $A = \{0, 1, 2, 3, 4\}$ and define a binary relation R on A

$$R = \{(0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4)\}$$

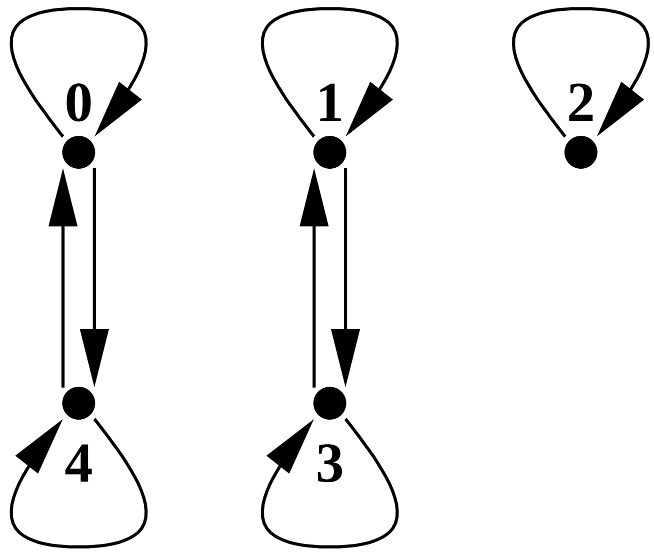
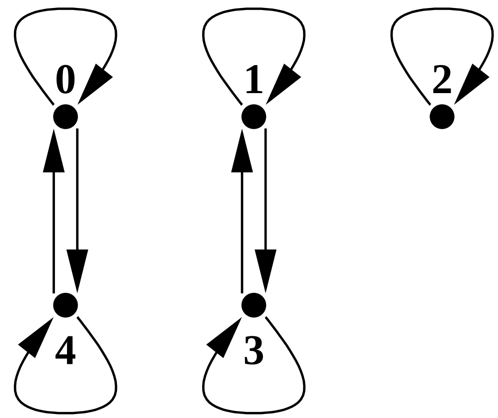


Figure: The array diagram (directed graph) corresponding to the relation.

By quick inspection we see that R is reflexive, symmetric, and transitive, hence an equivalence relation.



The equivalence classes are:

$$\begin{array}{lclcl} [0] & = & \{x \in A \,|\, x\,R\,0\} & = & \{0,4\} \\ [1] & = & \{x \in A \,|\, x\,R\,1\} & = & \{1,3\} \\ [2] & = & \{x \in A \,|\, x\,R\,2\} & = & \{2\} \\ [3] & = & \{x \in A \,|\, x\,R\,3\} & = & \{1,3\} \\ [4] & = & \{x \in A \,|\, x\,R\,4\} & = & \{0,4\} \end{array}$$

Note that $[0] = [4]$ and $[1] = [3]$, hence the ***distinct*** equivalence classes are: $\{0,4\}$, $\{1,3\}$, $\{2\}$.

Equivalence Classes: A Theorem

The following theorem tells us that an equivalence relation induces a partition:

Theorem: If A is a non-empty set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; *i.e.* the union of the equivalence classes is all of A and the intersection of any two distinct classes is empty.

Let R be the relation of congruence modulo 3 on the set $\mathbb{Z}$, *i.e.* $\forall m, n \in \mathbb{Z}$

$$m \, R \, n \;\Leftrightarrow\; 3|(m-n) \;\Leftrightarrow\; m \equiv n (\bmod 3)$$

We describe the equivalence classes: For each integer a, the class of a is

$$\begin{aligned} [a] &= \{x \in \mathbb{Z} \,|\, x \, R \, a\} \\ &= \{x \in \mathbb{Z} \,|\, 3|(x-a)\} \\ &= \{x \in \mathbb{Z} \,|\, x - a = 3 \cdot k, \; k \in \mathbb{Z}\} \\ &= \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + a, \; k \in \mathbb{Z}\} \end{aligned}$$

In particular

$$\begin{aligned} [0] = \{x \in \mathbb{Z} \,|\, x = 3 \cdot k, \; k \in \mathbb{Z}\} \quad &= \{0, 3, -3, 6, -6, 9, -9, \ldots\} \\ [1] = \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 1, \; k \in \mathbb{Z}\} \quad &= \{1, 4, -2, 7, -5, 10, -8, \ldots\} \\ [2] = \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 2, \; k \in \mathbb{Z}\} \quad &= \{2, 5, -1, 8, -4, 11, -7, \ldots\} \end{aligned}$$

We have

$$[0] = \{x \in \mathbb{Z} \,|\, x = 3 \cdot k,\ \ k \in \mathbb{Z}\} = \{0, 3, -3, 6, -6, 9, -9, \ldots\}$$
$$[1] = \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 1,\ \ k \in \mathbb{Z}\} = \{1, 4, -2, 7, -5, 10, -8, \ldots\}$$
$$[2] = \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 2,\ \ k \in \mathbb{Z}\} = \{2, 5, -1, 8, -4, 11, -7, \ldots\}$$

By lemma#1

$$[0] = [3] = [-3] = [6] = [-6] = [9] = [-9] = \ldots$$
$$[1] = [4] = [-2] = [7] = [-5] = [10] = [-8] = \ldots$$
$$[2] = [5] = [-1] = [8] = [-4] = [11] = [-7] = \ldots$$

Hence the distinct equivalence classes are

$$\{x \in \mathbb{Z} \,|\, x = 3 \cdot k,\ \ k \in \mathbb{Z}\}, \quad \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 1,\ \ k \in \mathbb{Z}\},$$

$$\{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 2,\ \ k \in \mathbb{Z}\}$$

The distinct equivalence classes are

$$\{x \in \mathbb{Z} \,|\, x = 3 \cdot k,\ k \in \mathbb{Z}\}, \quad \{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 1,\ k \in \mathbb{Z}\},$$

$$\{x \in \mathbb{Z} \,|\, x = 3 \cdot k + 2,\ k \in \mathbb{Z}\}$$

The class of $[0]$ can also be called the class of $[3]$ or the class of $[96]$, but the class ***is*** the set $\{x \in \mathbb{Z} \,|\, x = 3 \cdot k,\ k \in \mathbb{Z}\}$.

Definition: Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element $a \in A$ such that $[a] = S$.

Notes

- It is possible to define multiplication and addition of the equivalence classes corresponding to the rational numbers (previous example).
- The rational numbers can be defined as equivalence classes of ordered integers.
- It can be shown that all integers — negative, zero, and positive — can be defined as equivalence classes of ordered pairs of positive integers.
- Frege and Peano showed (late 19th century) that the positive integers can be defined entirely in terms of sets.
- Dedekind (1848–1916) showed that all real numbers can be defined as sets of rational numbers.
- All together, these results show that the real numbers can be defined using logic and set theory alone!

(Epp-v3.0)

10.1.1, 10.1.5, 10.1.7, 10.1.15, 10.1.23, 10.1.25, 10.2.3, 10.2.4, 10.2.12, 10.2.14, 10.2.37, 10.3.3, 10.3.17, 10.3.19, 10.3.40

(Epp-v2.0)

10.1.1, 10.1.5, 10.1.7, 10.1.15, 10.1.23, 10.1.25, 10.2.3, 10.2.4, 10.2.12, 10.2.14, 10.2.37, 10.3.2, 10.3.14, 10.3.16, 10.3.35