Math 254: Introduction to Linear Algebra

Notes #1.1 — Linear Equations

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Outline

SLOs 1.1 — Linear Equations

After this lecture you should:

- Be able to *Interpret* solutions of linear systems of 3 variables as intersections of planes.
- *Know* that linear systems can have
  - 0 no,
  - 1 one, or
  - ∞ infintely many

  solutions.
- Be able to *Perform* basic *Row-Operations* to determine all solutions of linear systems.

Why are the Notes Numbered the Way They Are???

The numbering of the topics originates from the the structure of Otto Bretcher's book (used Fall 2015 — Fall 2016).

Since adopting Gilbert Strang's book (Spring 2017 — Spring 2018) the main "chapter topic number" (the "1" in 1.n) is retained, but the "section number" the ("1" in n.1) is slowly being replaced by the the enumerated lecture number on each topic.

This is the first lecture on the First topic (Linear Equations); hence Notes #1.1.

References to the particular sections of Gilbert Strang's book will be added in the form [GS5–§1.1] (meaning section 1.1 in the 5th edition).

A First Example

We sweep the history lessons under our infinitely stretchable rug, and focus on the system of linear equations:

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ x & + & 3y & + & 2z & = & 34 \\ 3x & + & 2y & + & z & = & 26 \end{array}\right| \quad \begin{array}{ll} r_1 & \text{"row \#1"} \\ r_2 & \vdots \\ r_3 & \vdots \end{array}$$

and the question:

— **What values of x, y, and z satisfy this system?**

Manipulate the System to Find the Solution

We want to manipulate the system

$$\text{From} \left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ x & + & 3y & + & 2z & = & 34 \\ 3x & + & 2y & + & z & = & 26 \end{array}\right| \text{ To } \left|\begin{array}{ccccc} x & & & = & ??? \\ & y & & = & ??? \\ & & z & = & ??? \end{array}\right|$$

in order to reveal the solution.

But, before we do that, let's discuss a Graphical / Geometric Interpretation of the system...

→→→ **Quick Geometric "Detour!"** →→→

Geometric Interpretation

If we view each row/equation in the system

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ x & + & 3y & + & 2z & = & 34 \\ 3x & + & 2y & + & z & = & 26 \end{array}\right|$$

as a plane in 3D (x-y-z) space; the solution represents the point(s?) where the planes meet.

Solve for z in each one of the equations:

$$\begin{array}{rcl} z_1(x,y) & = & (39 - x - 2y)/3 \\ z_2(x,y) & = & (34 - x - 3y)/2 \\ z_3(x,y) & = & (26 - 3x - 2y) \end{array}$$

The Three Planes

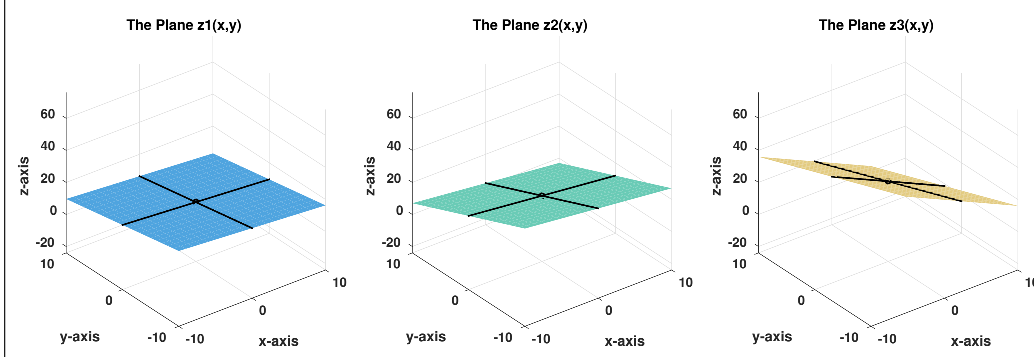


Figure: The planes $z_1(x,y)$, $z_2(x,y)$, and $z_3(x,y)$ visualized. As a reference, the black lines on the planes are aligned with the x- and y-axes.

The Three Planes — Intersecting

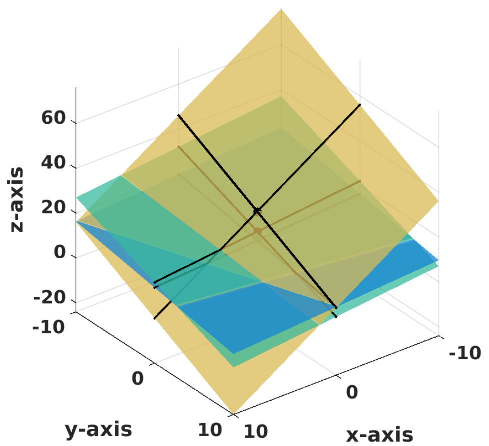


Figure: The planes $z_1(x,y)$, $z_2(x,y)$, and $z_3(x,y)$ visualized. We are looking for the (in this case) ONE POINT where they all meet. [∃ Movie]

OK, Back to Solving the System

We can

- ADD or SUBTRACT multiples (fractions) of any equation (row) to/from another equation; or
- MULTIPLY / DIVIDE / SCALE an equation,

without changing the solution.

We do this in order to successively *Eliminate* variables from the equations, so that in the end we reveal the solution.

[GS5–§2.2 — "The Idea of elimination"]

Solving ... "Forward elimination" ⇝ Upper Triangular System

"Forward elimination" stage:

$$\left|\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 39 \\ x &+& 3y &+& 2z &=& 34 \\ 3x &+& 2y &+& z &=& 26 \end{array}\right| \quad \begin{array}{l} \\ \text{subtract } r_1 \\ \text{subtract } 3r_1 \end{array}$$

we get:

$$\left|\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 39 \\ && y &-& z &=& -5 \\ &-& 4y &-& 8z &=& -91 \end{array}\right| \quad \begin{array}{l} \\ \\ \text{add } 4r_2 \end{array}$$

$$\left|\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 39 \\ && y &-& z &=& -5 \\ &&&-& 12z &=& -111 \end{array}\right| \quad \begin{array}{l} \\ \\ \text{divide by } (-12) \end{array}$$

... Solution "Back-substitution" ⇝ The Answer

"Backward elimination" or "Back-substitution" starts...

$$\left|\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 39 \\ && y &-& z &=& -5 \\ &&&& z &=& 9.25 \end{array}\right| \quad \begin{array}{r} \text{subtract } 3r_3 \\ \text{add } r_3 \\ \end{array}$$

$$\left|\begin{array}{rcrcr} x &+& 2y &=& 11.25 \\ && y &=& 4.25 \\ &&& z = & 9.25 \end{array}\right| \quad \begin{array}{l} \text{subtract } 2r_2 \\ \\ \\ \end{array}$$

$$\left|\begin{array}{rrrcr} x &&& = & 2.75 \\ & y && = & 4.25 \\ && z & = & 9.25 \end{array}\right|$$

Check the Solution!

Plug

$$\left|\begin{array}{ccccc} x & & & = & 2.75 \\ & y & & = & 4.25 \\ & & z & = & 9.25 \end{array}\right|$$

into

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 39 \\ x & + & 3y & + & 2z & = & 34 \\ 3x & + & 2y & + & z & = & 26 \end{array}\right|$$

and see that

$$\left|\begin{array}{ccccccc} 2.75 & + & 2\times 4.25 & + & 3\times 9.25 & = & 39 \\ 2.75 & + & 3\times 4.25 & + & 2\times 9.25 & = & 34 \\ 3\times 2.75 & + & 2\times 4.25 & + & 9.25 & = & 26 \end{array}\right|$$

Comments 1 of 2

- There are different paths to the solution:
  - for instance you can fully eliminate each column ("up" as well as "down") before proceeding to the next one;

  in the end you will arrive at the same solution.
  - This strategy does not separate into "forward" and "backward" stage
- When you do things by "hand" you will probably pick a path (order of operations) which is unique to the problem, and which makes the algebra as simple as possible.

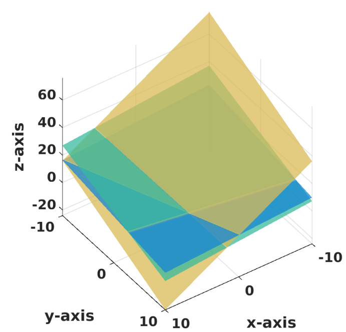
["Live Math" starting from bottom of slide #11]

Comments 2 of 2

- When you do things by "code" (in software), you develop an algorithm (recipe / step-by-step instructions) which always follows the same path, since no algebra is "hard" for the computer (except dividing by zero... which is very very bad.)
  - The forward/backward split has the added benefit that collecting the results gives some useful "side products" — see [GS5–§2.6 — "ELIMINATION = FACTORIZATION; $A = LU$"]

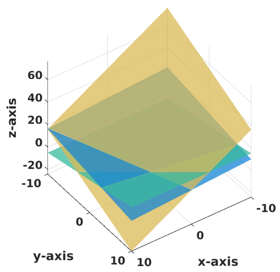
What About the Geometric Interpretation?!?

In a lot of books the Geometric Interpretation "lives" as its own separate "thing," but it is natural to wonder what the impact of the solution procedure (Elimination) has on the Geometry.

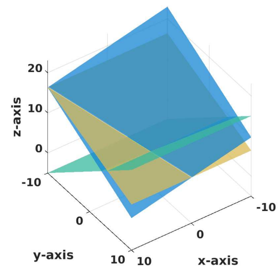


$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 39 \\ x & + & 3y & + & 2z & = & 34 \\ 3x & + & 2y & + & z & = & 26 \end{array}\right|$$

Eliminate x from Equations 2, and 3



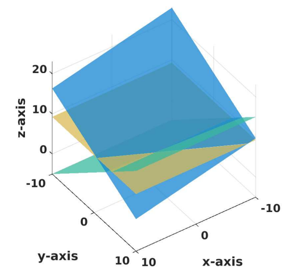
$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ 3x & + & 2y & + & z & = & 26 \end{array}\right|$$



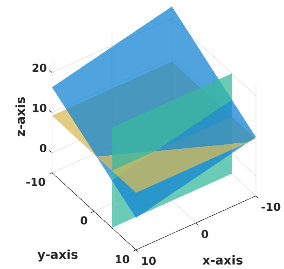
$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ & - & 4y & - & 8z & = & -91 \end{array}\right|$$

Note: the new $z_2(x,y) = y+5$, $z_3(x,y) = (91-4y)/8$ do not depend on x.

Eliminate y from Equation 3; and z from Equation 2

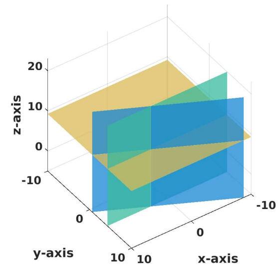


$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ & & & & z & = & 9.25 \end{array}\right|$$

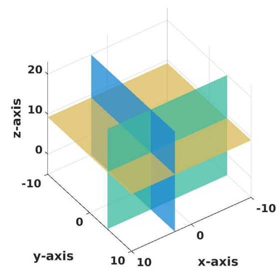


$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & & & = & 4.25 \\ & & & & z & = & 9.25 \end{array}\right|$$

Eliminate z from Equation 1; and y from Equation 1



$$\left|\begin{array}{rcrcrcr} x & + & 2y & & & = & 11.25 \\ & & y & & & = & 4.25 \\ & & & & z & = & 9.25 \end{array}\right|$$



$$\left|\begin{array}{rcrcrcr} x & & & & & = & 2.75 \\ & & y & & & = & 4.25 \\ & & & & z & = & 9.25 \end{array}\right|$$

Orthogonal Planes ⇝ Easy to "read" the Solution

Looking the sequence of intersection planes as we go through the elimination steps:

We realize that our **Geometric Goal** was to create **Orthogonal Planes** intersecting in the solution point.

The idea of *Orthogonalization* will show up in various contexts; but it is always a tool which makes it easy (well, *easier*) to identify the property we are after. — In this particular case the solution to a linear system.

What Can Happen?

A Linear System can have

- a unique solution (like our previous example)
- infinitely many solutions
- no solution

More solutions than you can shake an infinite stick at...

$$\left|\begin{array}{ccccccc} 2x & + & 4y & + & 6z & = & 0 \\ 4x & + & 5y & + & 6z & = & 3 \\ 7x & + & 8y & + & 9z & = & 6 \end{array}\right|$$

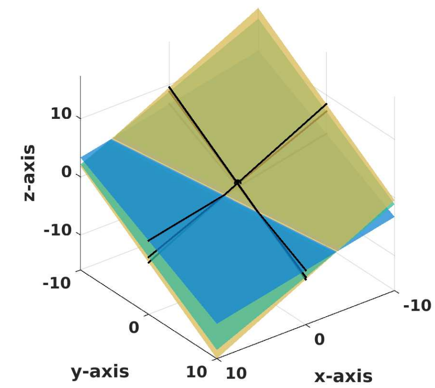


Figure: We sense trouble when the three planes meet not in one point, but along a common line in space. Let us see how that manifests itself in the elimination process.

Looking for infinitely many solutions...

$$\left|\begin{array}{ccccccc} 2x & + & 4y & + & 6z & = & 0 \\ 4x & + & 5y & + & 6z & = & 3 \\ 7x & + & 8y & + & 9z & = & 6 \end{array}\right| \quad \text{divide by 2}$$

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 0 \\ 4x & + & 5y & + & 6z & = & 3 \\ 7x & + & 8y & + & 9z & = & 6 \end{array}\right| \quad \begin{array}{l} \\ -4r_1 \\ -7r_1 \end{array}$$

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 0 \\ & - & 3y & - & 6z & = & 3 \\ & - & 6y & - & 12z & = & 6 \end{array}\right| \quad \begin{array}{l} \\ \text{divide by } -3 \\ \text{divide by } -6 \end{array}$$

Looking for infinitely many solutions...

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 0 \\ & & y & + & 2z & = & -1 \\ & & y & + & 2z & = & -1 \end{array}\right| \quad \begin{array}{l} \\ \\ -r_2 \end{array}$$

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 0 \\ & & y & + & 2z & = & -1 \\ & & & & 0 & = & 0 \end{array}\right| \quad \begin{array}{l} -2r_2 \\ \\ \text{d'oh!} \end{array}$$

$$\left|\begin{array}{ccccccc} x & & & - & z & = & 2 \\ & & y & + & 2z & = & -1 \\ & & & & 0 & = & 0 \end{array}\right|$$

... Oh, there they are!

We have:

$$\left|\begin{array}{rcrcrcr} x & & & - & z & = & 2 \\ & & y & + & 2z & = & -1 \\ & & & & 0 & = & 0 \end{array}\right|$$

This means that

$$\left|\begin{array}{rcr} x & = & 2+z \\ y & = & -1-2z \end{array}\right|$$

describes the line in space where the planes intersect; x and y are given as functions of z; the line is $(2+z, -1-2z, z)$, $z \in [-\infty, \infty]$.

We introduce a **parameter** (or "alias") $z = t$, which allows us to write

$$\left|\begin{array}{rcr} x & = & 2+t \\ y & = & -1-2t \\ z & = & t \end{array}\right|$$

We can write this in "line format" as:

$$(x, y, z) = (2, -1, 0) + t(1, -2, 1), \; t \in [-\infty, \infty].$$

Somebody ran away with the solution!

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ 4x & + & 5y & + & 6z & = & 3 \\ 7x & + & 8y & + & 9z & = & 0 \end{array}\right|$$

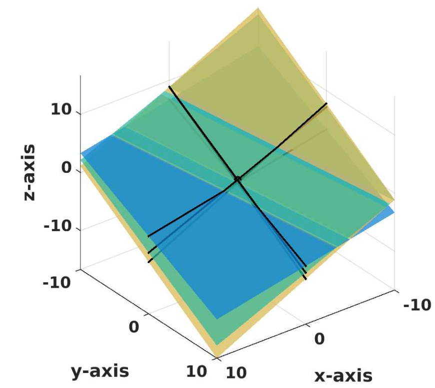


Figure: Here, the planes intersect (pair-wise), but not in any common point, or line. Again, we go through the elimination to see how this manifests itself in the computation.

Looking for Nothing...

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ 4x & + & 5y & + & 6z & = & 3 \\ 7x & + & 8y & + & 9z & = & 0 \end{array}\right| \quad \begin{array}{l} \\ -4r_1 \\ -7r_1 \end{array}$$

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ & - & 3y & - & 6z & = & 3 \\ & - & 6y & - & 12z & = & 0 \end{array}\right| \quad \begin{array}{l} \\ \text{divide by } -3 \\ \text{divide by } -6 \end{array}$$

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ & & y & + & 2z & = & -1 \\ & & y & + & 2z & = & 0 \end{array}\right| \quad \begin{array}{l} \\ \\ -r_2 \end{array}$$

Looking for Nothing... and Finding Nonsense!

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ & & y & + & 2z & = & -1 \\ & & & & 0 & = & 1 \end{array}\right| \quad \text{Say "What?!?"}$$

This system of equations is said to be *inconsistent*, and no solutions exist.

End-of-Problem

However:

It is still possible to find the pair-wise intersections of the planes. Since the first 2 equations are the same as in the previous ∞-many solutions example; one such line intersection is $(x, y, z) = (2, -1, 0) + t(1, -2, 1)$, $t \in [-\infty, \infty]$.

Finding More Pair-Wise Intersections (#2)

Equations #1 and #3

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ 7x & + & 8y & + & 9z & = & 0 \end{array}\right| \quad \begin{array}{c} \\ -7r_1 \end{array}$$

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ & - & 6y & - & 12z & = & 0 \end{array}\right| \quad \text{divide by } -6$$

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 0 \\ & & y & + & 2z & = & 0 \end{array}\right| \quad \begin{array}{c} -2r_2 \\ \\ \end{array}$$

$$\left|\begin{array}{rcrcrcr} x & & & - & z & = & 0 \\ & & y & + & 2z & = & 0 \end{array}\right|$$

Giving the line $(x, y, z) = (0, 0, 0) + t(1, -2, 1)$, $t \in [-\infty, \infty]$.

Finding More Pair-Wise Intersections (#3)

Equations #2 and #3

$$\left|\begin{array}{rcrcrcr} 4x & + & 5y & + & 6z & = & 3 \\ 7x & + & 8y & + & 9z & = & 0 \end{array}\right| \quad \begin{array}{c} \frac{1}{4} \\ \frac{1}{7} \end{array}$$

$$\left|\begin{array}{rcrcrcr} x & + & \frac{5}{4}y & + & \frac{6}{4}z & = & \frac{3}{4} \\ x & + & \frac{8}{7}y & + & \frac{9}{7}z & = & 0 \end{array}\right| \quad \begin{array}{c} \\ -r_1 \end{array}$$

$$\left|\begin{array}{rcrcrcr} x & + & \frac{5}{4}y & + & \frac{6}{4}z & = & \frac{3}{4} \\ & - & \frac{3}{28}y & - & \frac{3}{14}z & = & -\frac{3}{4} \end{array}\right| \quad \begin{array}{c} +\frac{35}{3}r_2 \\ \\ \end{array}$$

Finding More Pair-Wise Intersections (#3)

$$\left|\begin{array}{rcrcrcr} x & & & - & z & = & -8 \\ & - & \frac{3}{28}y & - & \frac{3}{14}z & = & -\frac{3}{4} \end{array}\right| \quad \begin{array}{c} \\ -\frac{28}{3} \end{array}$$

$$\left|\begin{array}{rcrcrcr} x & & & - & z & = & -8 \\ & & y & + & 2z & = & 7 \end{array}\right|$$

Giving the line $(x, y, z) = (-8, 7, 0) + t\,(1, -2, 1)$, $t \in [-\infty, \infty]$.

Leaving us with 3 lines in space

$$\begin{array}{rcl} (x, y, z)_1 & = & (2, -1, 0) + t(1, -2, 1) \\ (x, y, z)_2 & = & (0, 0, 0) + t(1, -2, 1) \\ (x, y, z)_3 & = & (-8, 7, 0) + t\,(1, -2, 1) \end{array}$$

Pair-Wise Intersections — Visualized

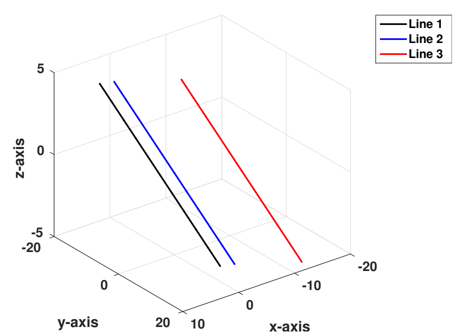


Figure: The lines are parallel, and never intersect.

$$\begin{array}{rcl} (x, y, z)_1 & = & (2, -1, 0) + t(1, -2, 1) \\ (x, y, z)_2 & = & (0, 0, 0) + t(1, -2, 1) \\ (x, y, z)_3 & = & (-8, 7, 0) + t\,(1, -2, 1) \end{array}$$

OK, we have squeezed the last “juice” out of this problem...

Suggested Problems 1.1

Available on "Learning Glass" videos:

- **(1.1.1)** Find all solutions to a 2-by-2 linear system using elimination.
- **(1.1.3)** Find all solutions to a 2-by-2 linear system using elimination.
- **(1.1.7)** Find all solutions to a 3-by-3 linear system using elimination.
- **(1.1.14)** Find all solutions to a 3-by-3 linear system using elimination.
- **(1.1.19)** Find all solutions to a 3-by-3 linear system *with a parameter k* using elimination. For what value(s) of k do we have one / no / infinitely many solutions?
- **(1.1.21)** Find three numbers, given then sums of all the pairs,
- **(1.1.42)** Solve upper and lower triangular linear systems.

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Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

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(1.1.1), (1.1.3)

(1.1.1) Find all solutions of the linear system using elimination; check your solution.

$$\left|\begin{array}{rcrcr} x & + & 2y & = & 1 \\ 2x & + & 3y & = & 1 \end{array}\right|$$

(1.1.3) Find all solutions of the linear system using elimination; check your solution.

$$\left|\begin{array}{rcrcr} 2x & + & 4y & = & 3 \\ 3x & + & 6y & = & 2 \end{array}\right|$$

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(1.1.7), (1.1.14)

(1.1.7) Find all solutions of the linear system using elimination; check your solution.

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 1 \\ x & + & 3y & + & 4z & = & 3 \\ x & + & 4y & + & 5z & = & 4 \end{array}\right|$$

(1.1.14) Find all solutions of the linear system.

$$\left|\begin{array}{rcrcrcr} x & + & 4y & + & z & = & 0 \\ 4x & + & 13y & + & 7z & = & 0 \\ 7x & + & 22y & + & 13z & = & 1 \end{array}\right|$$

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(1.1.19), (1.1.21)

(1.1.19) Consider the linear system

$$\left|\begin{array}{rcrcrcr} x & + & y & - & z & = & -2 \\ 3x & - & 5y & + & 13z & = & 18 \\ x & - & 2y & + & 5z & = & k \end{array}\right|$$

where k is an arbitrary number.

a. For which values of k does this system have one. or infinitely many solutions?

b. For each value of k you found in part a, how many solutions does the system have.

c. Find all solutions for each value of k.

(1.1.21) The sums of any two of three real numbers are 24, 28, and 30. Find these numbers.

Supplemental Material
Online Labs — Python / Jupyter Notebooks
Metacognitive Reflection
Problem Statements 1.1
Row-Reduction Redux

(1.1.42)

(1.1.42) Linear systems are particularly easy to solve when they are in triangular form (*i.e.* all entries above or below the diagonal are zero).

a. Solve the lower triangular system

$$\left|\begin{array}{rcrcrcrcr} x_1 & & & & & & & = & -3 \\ -3x_1 & + & x_2 & & & & & = & 14 \\ x_1 & + & 2x_2 & + & x_3 & & & = & 9 \\ -x_1 & + & 8x_2 & - & 5x_3 & + & x_4 & = & 33 \end{array}\right|$$

b. Solve the upper triangular system

$$\left|\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & 4x_4 & = & -3 \\ & & x_2 & + & 3x_3 & + & 7x_4 & = & 5 \\ & & & & x_3 & + & 2x_4 & = & 2 \\ & & & & & & x_4 & = & 0 \end{array}\right|$$

Supplemental Material
Online Labs — Python / Jupyter Notebooks
Metacognitive Reflection
Problem Statements 1.1
Row-Reduction Redux

Row-Reductions Revisited

"Forward elimination" ⇝ Upper Triangular System

"Forward elimination" stage:

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ x & + & 3y & + & 2z & = & 34 \\ 3x & + & 2y & + & z & = & 26 \end{array}\right| \quad \begin{array}{l} \\ \text{subtract } r_1 \\ \text{subtract } 3r_1 \end{array}$$

Details (subtracting r_1 from r_2):

$$\begin{array}{c|rcrcrcrl} (-1) & x & + & 2y & + & 3z & = & 39 & \\ (+1) & x & + & 3y & + & 2z & = & 34 & \\ \hline = & & & y & - & z & = & -5 & \text{new } r_2 \end{array}$$

Details (subtracting $3r_1$ from r_3):

$$\begin{array}{c|rcrcrcrl} (-3) & x & + & 2y & + & 3z & = & 39 & \\ (+1) & 3x & + & 2y & + & z & = & 26 & \\ \hline = & & - & 4y & - & 8z & = & -91 & \text{new } r_3 \end{array}$$

Supplemental Material
Online Labs — Python / Jupyter Notebooks
Metacognitive Reflection
Problem Statements 1.1
Row-Reduction Redux

Row-Reductions Revisited

"Forward elimination" ⇝ Upper Triangular System

After elimination of x from r_2 and r_3, we have:

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ & - & 4y & - & 8z & = & -91 \end{array}\right| \quad \begin{array}{l} \\ \\ \text{add } 4r_2 \end{array}$$

Details (adding $4r_2$ to r_3):

$$\begin{array}{c|rcrcrcrl} (+4) & & & y & - & z & = & -5 & \\ (+1) & & - & 4y & - & 8z & = & -91 & \\ \hline = & & & & - & 12z & = & -111 & \text{new } r_3 \end{array}$$

Row-Reductions Revisited — "Forward elimination" ⇝ Upper Triangular System

The system is now in "upper triangular" form; next we make the coefficient of z (in r_3) one (1):

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ & & & - & 12z & = & -111 \end{array}\right| \quad \text{divide by } (-12)$$

Details (dividing r_3 by (-12)):

$$\begin{array}{c|rcrcl} (\times \frac{1}{12}) & - & 12z & = & -111 & \\ \hline = & & z & = & \frac{111}{12} & \text{new } r_3 \end{array}$$

... and $\frac{111}{12} = 9.25$.

Row-Reductions Revisited — "Back-substitution" ⇝ The Answer

"Backward elimination" or "Back-substitution" starts...

$$\left|\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 39 \\ & & y & - & z & = & -5 \\ & & & & z & = & 9.25 \end{array}\right| \quad \begin{array}{l} \text{subtract } 3r_3 \\ \text{add } r_3 \end{array}$$

Details (adding r_3 to r_2):

$$\begin{array}{c|rcrcrl} (+1) & y & - & z & = & -5 & \\ (+1) & & & z & = & 9.25 & \\ \hline = & y & & & = & 4.25 & \text{new } r_2 \end{array}$$

Details (subtracting $3r_3$ from r_1):

$$\begin{array}{c|rcrcrcrl} (+1) & x & + & 2y & + & 3z & = & 39 & \\ (-3) & & & & & z & = & 9.25 & \\ \hline = & x & + & 2y & & & = & 11.25 & \text{new } r_1 \end{array}$$

Row-Reductions Revisited — "Back-substitution" ⇝ The Answer

Lastly, we eliminate y from r_1:

$$\left|\begin{array}{rcrcrcr} x & + & 2y & & & = & 11.25 \\ & & y & & & = & 4.25 \\ & & & & z & = & 9.25 \end{array}\right| \quad \text{subtract } 2r_2$$

Details (subtracting $2r_2$ from r_1):

$$\begin{array}{c|rcrcrl} (+1) & x & + & 2y & = & 11.25 & \\ (-2) & & & y & = & 4.25 & \\ \hline = & x & & & = & 2.75 & \text{new } r_1 \end{array}$$

We arrive at:

$$\left|\begin{array}{rrrcr} x & & & = & 2.75 \\ & y & & = & 4.25 \\ & & z & = & 9.25 \end{array}\right|$$

Online Labs — ⟨whine⟩Why???⟨/whine⟩ [New Fall 2019]

In an effort to give you another point of view of the material, the development of online "labs" and/or "demonstrations" has started. Since `python` is the computer language *du jour*, that is the choice. Everything is presented in terms of **jupyter notebooks**.

How Much is THIS going to cost??? — the Python language itself, and the Jupyter notebook environment can be downloaded and installed for the staggering cost of **$0** (with a full money-back guarantee).

All the necessary components: **Python**, **Pip**, **Anaconda**, and **Jupyter** are available for {Linux, Windows, MacOSX}

Wait. What? I have to download and install stuff?!? *C'mon, maaaaaan, this is a MATH class!!!* — R.E.L.A.X., there are multiple options for viewing and running the notebooks online, aka "in the cloud" (aka "on somebody elses hardware") without installing anything.

Lab 1.1, Getting Started [c'mon. give it a go!]

All lab materials can be found at http://terminus.SDSU.EDU/SDSU/Math254/?r=labs

At the start of Fall 2019, there are only two labs — 1.1, and 1.2.

There are two "modes" to view the labs:

- **Non-Interactive**
  - Just click the lab you want to view under the heading **"Static Views"**; the lab will open up using nbviewer.jupyter.org — no need to install anything.
- **Interactive**
  0. If you want to "play" with the lab (change the numbers, test other things)...
  1. First download (right-click + save) the lab under the heading **"Source Link"**
  2. Then you can upload/view/interact with the lab on *e.g.:*
     - https://colab.research.google.com/
     - https://mybinder.org/
     - Your local Python / Jupyter notebook installation

Ask "Uncle Google"!

There are (probably) other free services that will allow you to run Jupyter notebooks "in the cloud."

As these services change/are updated often, it is not worth trying to document HOW to use them here. The best thing to do it is just give it a try; read the help/instructions available on *e.g.* mybinder.org or colab.research.google.com.

You favorite uncle — "Uncle Google" — is always ready to help:

- "how do I run jupyter notebook on mybinder"
- "how do I run jupyter notebook on colab"
- "how do I install X on Y",
  - where X∈{python, jupyter, ...}, and
  - Y∈{Windows, MacOS, Fedora, Ubuntu, Debian, ...}