

Math 254: Introduction to Linear Algebra

Notes #1.2 — Matrices, Vectors, ...

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Outline

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 - SLOs: Matrices, Vectors, ...
- 2 Matrices, Vectors; Gauss-Jordan Elimination
 - Matrix – Vector Notation
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 - Suggested Problems
 - Lecture – Book Roadmap
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SLOs 1.2

Matrices, Vectors, ...

After this lecture you should:

- Know basic language and concepts:
 - Matrices, vectors, and their components
 - Matrix types: square, diagonal, triangular, zero, identity
 - The collection of all n -vectors, denoted \mathbb{R}^n is a vector space
- Vector addition
- Know the difference between the Coefficient matrix, and the Augmented matrix; their uses in the solution of linear systems
- Know how to use elimination to identify leading and non-leading (a.k.a. *free variables*), and when necessary introduce parameters to express *all* solutions of linear systems.
- Know what **Reduced-Row-Echelon-Form** (RREF) of a Matrix is, and how to achieve it using elementary row operations.



Equation Manipulation to Isolate Variables

Our current “business” is manipulating linear systems, of the form

$$\begin{cases} 3x + 21y - 3z = 0 \\ -6x - 2y - z = 62 \\ 2x - 3y + 8z = 32 \end{cases}$$

into a form which reveals the values of x , y , and z :

$$\begin{cases} x & & = -3574/281 \\ & y & = 844/281 \\ & & z = 2334/281 \end{cases}.$$

We achieve this by cleverly adding/subtracting rows (equations) from each other.

Matrix Notation — “Encoding” the Information

We realize that all the important information is in the coefficients (numbers), and that the variables (x , y , z) just get carried around. We can “encode” all the information about the linear system

$$\begin{cases} 3x + 21y - 3z = 0 \\ -6x - 2y - z = 62 \\ 2x - 3y + 8z = 32 \end{cases}$$

in a **matrix**

$$\underbrace{\begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix}}_{\text{Augmented Matrix}}, \text{ or, sometimes: } \underbrace{\begin{bmatrix} 3 & 21 & -3 & | & 0 \\ -6 & -2 & -1 & | & 62 \\ 2 & -3 & 8 & | & 32 \end{bmatrix}}_{\text{Augmented Matrix with Coefficient Matrix and right-hand-side "separated."}}$$

Augmented Matrix

Augmented Matrix with
Coefficient Matrix and
right-hand-side “separated.”

Row – Column Indexing

Ponder the matrix “A” with 3 rows, and 4 columns:

$$A = \begin{bmatrix} 3 & 21 & -3 & 0 \\ -6 & -2 & -1 & 62 \\ 2 & -3 & 8 & 32 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

that is, usually we refer to the entries of a matrix A (upper-case), using double subscripts a_{ij} (lower-case); the subscripts i , and j are “standard” but r (row) and c (column) would be more intuitive.

Sometimes you see the notation $A \in \mathbb{R}^{3 \times 4}$ to denote a 3-by-4 (always [ROWS-by-COLUMNS]) matrix where the entries are real (\mathbb{R}) numbers.

Note: The entries can be other mathematical objects, e.g. complex numbers, \mathbb{C} , polynomials, etc... but we will work with \mathbb{R} for quite while.



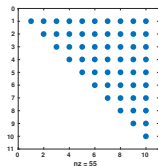
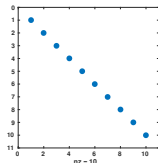
[Focus :: CS] What is a Matrix?

From a computer science point-of-view a matrix can be viewed as *data structure*, and depending on your mood (and/or preference of programming paradigm) you can think of it as e.g. a

- C—C++ style *2-dimensional array*,
 - :: `double A[3][3]; /* A is a 3-by-3 matrix */`
 - :: `A[0][0] = 1; /* Assigning 1 to a_{11} */`
 - :: `A[2][2] = 14; /* Assigning 14 to a_{33} */`
 - :: Yes, some languages count from 0 to (n-1); others from 1 to n.
- or an abstract *container class*.
- Python
 - :: uses (...) for immutable “tuples” and [...] for “lists”...
 - :: a matrix is a lists-of-lists: [[...], ..., [...]]

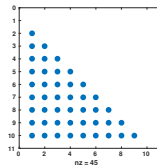
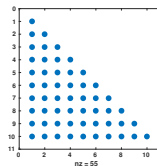
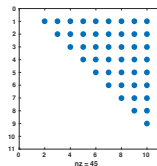
Types of Matrices

- When $A \in \mathbb{R}^{n \times n}$, i.e. the matrix has the same number of rows and columns, it is a **square matrix**
- A matrix is **diagonal** if all entries $a_{ij} = 0$ for all $i \neq j$. (Only entries of the type a_{ii} are non-zero.)
- A square matrix is **upper triangular** if all entries $a_{ij} = 0$ for all $i > j$.



Types of Matrices

- A square matrix is **strictly upper triangular** if all entries $a_{ij} = 0$ for all $i \geq j$.
- A square matrix is **lower triangular** if all entries $a_{ij} = 0$ for all $i < j$.
- A square matrix is **strictly lower triangular** if all entries $a_{ij} = 0$ for all $i \leq j$.



Types of Matrices

- A matrix where all entries are zero is (surprisingly?) called a **zero matrix**.
- A square matrix where all diagonal entries are *ones*, and the off-diagonal entries are *zeros*

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

is called an **identity matrix**. “ \cdots ” and “ \vdots ” denote padding with 0-entries, and “ \ddots ” diagonal 1-entries; filling the matrix out to its full size (whatever that may be).

Matrices of size $n \times 1$ and $1 \times n \Rightarrow$ “Vectors”

- A “matrix” with only one column is called a **column vector**:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A “matrix” with only one row is called a **row vector**:

$$\vec{w}^T = [w_1 \quad w_2 \quad \cdots \quad w_n]$$

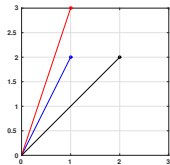
By mathematical convention a **vector** is a **column vector**; so \vec{v} and \vec{w} are (column) vectors. The notation \vec{w}^T is the *transpose* of the vector \vec{w} , which is a *row vector*.

Vector “Language”

- The entries of a vector are called its **components**; v_k is the k^{th} component of the vector \vec{v} .
- The set (collection) of *all* (column) vectors with n components is denoted by \mathbb{R}^n ; we refer to \mathbb{R}^n as a **vector space**.

It is easy to visualize vectors in \mathbb{R}^2 ; we can think of the vector \vec{v} as an arrow from the origin $(0, 0)$ to the point $(x, y) = (v_1, v_2)$:

In the figure we have



$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

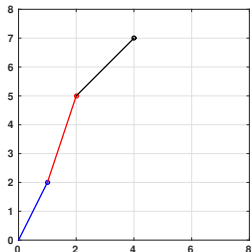
Without confusion, we can just let the terminal points $(x, y) = (v_1, v_2)$ represent the vectors.

Adding Vectors

With:

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

We can graphically show how to add vectors:



That is

$$\vec{u} + \vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \vec{u} + \vec{v} + \vec{w} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

Making Use of our Matrix – Vector Notation

Now, given a linear system:

$$\begin{cases} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{cases}$$

We can extract the **Coefficient Matrix** (containing the coefficients of the unknown variables in the system)

$$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix},$$

or the **augmented matrix**

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix},$$

which captures all the information in the linear system.

The Augmented Matrix

Often, we separate the coefficients from the right-hand-side information in the Augmented Matrix:

$$\begin{bmatrix} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 8 & 4 & | & 2 \\ 2 & 5 & 1 & | & 5 \\ 4 & 10 & -1 & | & 1 \end{bmatrix}$$

Solving Linear Systems Using the Augmented Matrix

We can solve the linear system by manipulating the Augmented Matrix:

$$\begin{array}{l} \begin{array}{c} \bullet \\ \downarrow \\ \left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array} \right] \end{array} \quad /2 \\ \begin{array}{c} \bullet \\ \downarrow \\ \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array} \right] \end{array} \quad \begin{array}{l} -2r_1 \\ -4r_1 \end{array} \\ \begin{array}{c} \bullet \\ \downarrow \\ \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & -3 & -3 & 3 \\ 0 & -6 & -9 & -3 \end{array} \right] \end{array} \quad \begin{array}{l} /(-3) \\ /(-3) \end{array} \end{array}$$

Solving Linear Systems Using the Augmented Matrix

$$\begin{array}{l}
 \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 3 & 1 \end{array} \right] \quad -2r_2 \\
 \left[\begin{array}{ccc|c} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 3 \end{array} \right] \quad \begin{array}{l} -2r_3 \\ -r_3 \end{array} \\
 \left[\begin{array}{ccc|c} 1 & 4 & 0 & -5 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array} \right] \quad -4r_2 \\
 \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array} \right], \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}
 \end{array}$$

The (Math) World is not limited to 2, or 3 Variables (Dimensions)

We can (easily?) imagine a system of 4 linear equations with 7 unknowns:

$$\left| \begin{array}{ccccccc} x_1 & - & x_2 & & + & 4x_5 & & & 7777x_7 & = & 1 \\ & & & x_3 & & + & 2x_5 & & - & 777x_7 & = & 2 \\ & & & & x_4 & & & & & 77x_7 & = & 3 \\ & & & & & & & x_6 & - & 7x_7 & = & 4 \end{array} \right| ,$$

so that solving for the leading variables* gives:

$$\left| \begin{array}{l} x_1 = 1 + x_2 - 4x_5 - 7777x_7 \\ x_3 = 2 - 2x_5 + 777x_7 \\ x_4 = 3 - 77x_7 \\ x_6 = 4 + 7x_7 \end{array} \right|$$

* **Leading variables** are the first ones to appear in each equation (after elimination); here x_1 , x_3 , x_4 , and x_6 .

Infinitely Many Solutions

If we parameterize (the non-leading, or “free” variables): $x_2 = s$, $x_5 = t$, and $x_7 = u$, we can write the infinitely many solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 & + & s & - & 4t & - & 7777u \\ & & s & & & & \\ 2 & & & - & 2t & + & 777u \\ 3 & & & & & - & 77u \\ & & & & t & & \\ 4 & & & & & + & 7u \\ & & & & & & u \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} -7777 \\ 0 \\ 777 \\ -77 \\ 0 \\ 7 \\ 1 \end{bmatrix}$$

If we plug that into the original system of linear equations we see that it indeed is the (collection of) solution(s)!

Note that s , t , and u are allowed to take any values in \mathbb{R} (independent of each other)...

What Makes a System “Easy” to Solve?

Three properties make a system “easy” to solve:

- P1 The leading coefficient in each equation is 1.
- P2 The leading variable in each equation does not appear in any other equation.
- P3 The leading variables appear in “natural order,” with increasing indices as we go down the system: x_1 , x_3 , x_4 , and x_6 as opposed to any other ordering.

If/when the system does not satisfy these properties, we use elimination to get there...



Another Example...

Keeping P1, P2, P3 in mind...

We go straight to the Augmented Matrix:

$$\left[\begin{array}{ccccc|c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array} \right] \quad (/2)$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array} \right] \quad \begin{array}{l} -r_1 \\ -3r_1 \\ -5r_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array} \right] \quad \begin{array}{l} -r_2 \\ +2r_2 \end{array}$$

Another Example...

Keeping P1, P2, P3 in mind...

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array} \right] \begin{array}{l} +r_3 \\ \\ -r_3 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \rightsquigarrow r_3 \\ \rightsquigarrow r_2 \end{array}$$

$$\left[\begin{array}{ccccc|c} \textcircled{1} & 2 & 0 & 0 & 3 & 2 \\ & & \textcircled{1} & 0 & -1 & 4 \\ & & & \textcircled{1} & -2 & 3 \end{array} \right] \begin{array}{l} \text{leading zeros} \\ \text{suppressed} \\ \text{for clarity} \end{array}$$

... Identifying the Solutions

$$\left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \text{value} \\ \hline 1 & 2 & & & 3 & 2 \\ & & 1 & & -1 & 4 \\ & & & 1 & -2 & 3 \end{array} \right]$$

So that:

$$\begin{cases} x_1 = 2 - 2x_2 - 3x_5 \\ x_3 = 4 + x_5 \\ x_4 = 3 + 2x_5 \end{cases}, \text{ or}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - 2s - 3t \\ s \\ 4 + t \\ 3 + 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Parameters: $x_2 = s$, $x_5 = t$

Solving a System of Linear Equations

“Elimination Strategy”

We go equation-by-equation from top to bottom (i runs from 1 to n):

- For the i^{th} equation: if the leading variable is x_j with non-zero coefficient c ; divide the equation by c to make the leading coefficient 1.
- Eliminate x_j from all other equations.
- Go to the next equation.

“Exit Strategy”

- If we get $zero = nonzero$ at any point; then there are no solutions. [STOP]
- If we complete without inconsistencies:
 - **rearrange** the equations so that the leading variables are in “*natural order*”
 - Solve each equation for the leading variable
 - Choose parameters for the **non-leading variables** [if there are any] (appropriate “alphabet soup”)
 - Express leading variables using parameters.

Reduced Row Echelon Form

After elimination according to this strategy, the matrix is in:

Reduced Row Echelon Form

A Matrix is said to be in *Reduced Row Echelon Form* if it satisfies the following conditions:

- 1 If a row has non-zero entries, then the first non-zero entry is a 1, called *the leading 1* (or *pivot*) of this row.
- 2 If a column contains a leading 1, then all other entries in that column are 0. [ELIMINATION IS COMPLETE]
- 3 If a row contains a leading 1, then each row above it contains a leading 1 further to the left. [SORTING OF ROWS]

The last condition implies that rows of 0's, if any, must appear at the bottom of the matrix.

Getting to Reduced Row Echelon Form

Elementary Row Operations

We get to *Reduced Row Echelon Form* by performing

Elementary Row Operations

- Divide a row by a non-zero scalar
- Subtract a multiple of a row from another row
- Swap two rows

This strategy of solving linear systems by reduction to Reduced Row Echelon Form is referred to as **Gaussian Elimination**, or **Gauss-Jordan Elimination**.

Gauss (1777–1855), Jordan (1842–1899); but the Chinese used it loooooong before that.

“Gauss-Jordan Elimination” \rightsquigarrow RREF

“Gaussian Elimination” \rightsquigarrow REF (leading variables NOT 1's; LU -factorization)



Suggested Problems 1.2

Available on “Learning Glass” videos:

- (1.2.1) Find all solutions to a 2-by-3 linear system using elimination.
- (1.2.3) Find all solutions to a 1-by-3 linear system using elimination.
- (1.2.9) Find all solutions to a 3-by-6 linear system using elimination.
- (1.2.11) Find all solutions to a 4-by-4 linear system using elimination.
- (1.2.18) Determine which matrices are in RREF.
- (1.2.21) Find values of matrix entries so that the resulting matrix is in RREF.

Lecture – Book Roadmap

Lecture	Book, [GS5-]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(1.2.1), (1.2.3)

(1.2.1) Find all solutions to the linear system using elimination:

$$\left| \begin{array}{cccc} x & + & y & - 2z & = & 5 \\ 2x & + & 3y & + 4z & = & 2 \end{array} \right|$$

(1.2.3) Find all solutions to the linear system using elimination:

$$\left| x + 2y + 3z = 4 \right|$$

(1.2.9), (1.2.11)

(1.2.9) Find all solutions to the linear system using elimination:

$$\left| \begin{array}{cccccc} & & & x_4 & + & 2x_5 & - & x_6 & = & 2 \\ x_1 & + & 2x_2 & & & & & & & & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & & & - & x_5 & + & x_6 & = & 2 \end{array} \right|$$

(1.2.11) Find all solutions to the linear system using elimination:

$$\left| \begin{array}{cccccc} x_1 & & & + & 2x_3 & + & 4x_4 & = & -8 \\ & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - & x_2 & + & 3x_3 & + & 4x_4 & = & -12 \end{array} \right|$$

(1.2.18)

(1.2.18) Determine which of the matrices are in Reduced Row Echelon Form:

a.
$$\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

b.
$$\begin{bmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c.
$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

d.
$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

(1.2.21)

(1.2.21) For which values of a , b , c , d , and e is the following matrix in reduced-row-echelon-form?

$$\begin{bmatrix} 1 & a & b & 3 & 0 & -2 \\ 0 & 0 & c & 1 & d & 3 \\ 0 & e & 0 & 0 & 1 & 1 \end{bmatrix}$$

Two Separate Websites



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- **GRADEBOOK**

Goal: Your Points in the GRADEBOOK!

How do we get there?

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- You should have gotten an invitation sent to your [...]@sdsu.edu email address; or go to
- <https://piazza.com/sdsu/fall2021/math254blomgren>
- It is FREE
- Download the app from the iOS App Store, or Google Play

2 **If you want credit:** You can register multiple email addresses to a piazza account — one of them must match your email address in Canvas.

- In Piazza go to Account/Email Settings

Goal: Your Points in the GRADEBOOK!

How do we get there?

- 3 Scores will be transferred after EACH LECTURE
 - The transfer is manual, so there may be a bit of a delay.
- 4 *If something is not right, please let me know!* — Email me!

If you cannot participate in the use of Piazza (due to lack of a suitable device) — let me know, and we'll figure something out.