Math 254: Introduction to Linear Algebra

Notes #1.3 —
Solutions of Linear systems; Matrix Algebra

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Outline

1. Student Learning Objectives
   - SLOs: Solutions of Linear systems; Matrix Algebra
2. The Number of Solutions to a System of Linear Equations
   - Collecting the Results... and Adding More Language / Notation
   - Mathematical Language: Logic
   - Using Logic to Derive More Results Re: Variables and Rank
3. Definitions and Rules of Matrix Algebra
   - Fundamentals of Matrix and Vector Algebra
4. Suggested Problems
   - Suggested Problems 1.3
   - Lecture – Book Roadmap
5. Supplemental Material
   - Metacognitive Reflection
   - Problem Statements 1.3

SLOs 1.3 — Solutions of Linear systems; Matrix Algebra

After this lecture you should:

- Know what the **Rank** of a Matrix is; and its connection to total/leading/free variables and the number of solutions of a linear system
- Know the Fundamentals of:
  - Matrix-Vector algebra
  - Vector-Vector Dot Product / Inner Product
  - Matrix-Vector Product: Linear combinations

How Many Solutions Are There?!?

That's a good question; and it ties in with last lecture...

Let's ponder the three (augmented, eliminated) systems:

$$\text{a.}\ \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \quad \text{b.}\ \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 0 & 1 \\ 0 & 0 & \textcircled{1} & 2 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \quad \text{c.}\ \left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 1 \\ 0 & \textcircled{1} & 0 & 2 \\ 0 & 0 & \textcircled{1} & 3 \end{array}\right]$$

Where the leading coefficients have been circled in red. Notice that in a, we did not circle the **1** in the third row, since it belongs to the right-hand-side (and NOT the coefficient matrix).

System a. — No Solutions

"Move" non-leading variables to right-hand-side

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \quad \text{interpretation} \quad \left|\begin{array}{rcl} x_1 & = & 0 - \mathbf{2x_2} \\ x_3 & = & 0 \\ \mathbf{0} & = & \mathbf{1} \\ 0 & = & 0 \end{array}\right|$$

right-hand-side constants

Here, the third row shows that there are **no solutions** to this system. We say that **the system is inconsistent**.

System b. — Infinitely Many Solutions

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right] \quad \text{interpretation} \quad \left|\begin{array}{rcl} x_1 & = & 1 - \mathbf{2x_2} \\ x_3 & = & 2 \\ 0 & = & 0 \end{array}\right|$$

We are left with one un-determined (free) variable; and introduce a **parameter for x_2** (let's pick the Greek letter η for fun), and write the infinitely many solutions as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-2\eta \\ \eta \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \eta \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{where } \eta \in \mathbb{R} \Leftrightarrow \eta \in [-\infty, +\infty]$$

System c. — One (Unique) Solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \text{interpretation} \quad \left|\begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 2 \\ x_3 & = & 3 \end{array}\right|$$

Here, there are no un-determined (free) variables; so there's only one solution.

Consistent vs. Inconsistent Linear Systems

Theorem (Number of Solutions of a Linear System)

A system of equations is said to be **consistent** *if there is at least one solution; it is* **inconsistent** *if there are no solutions.*

A linear system is inconsistent if and only if the reduced row-echelon form of its augmented matrix contains the row

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

representing the equation "$0 = 1$."

If a linear system is consistent, then it has either

- *infinitely many solutions, if there is at least one free variable,* or
- *exactly one solution, if all the variables are leading.*

Definitions — The Rank of a Matrix

Definition (The RANK of a Matrix)

The **rank** of a matrix is the number of leading 1s in $\mathrm{rref}(A)$ — the Reduced Row Echelon Form of A — and is denoted

$$\mathrm{rank}(A).$$

Definition (Full RANK)

If $A \in \mathbb{R}^{n \times n}$ (a square matrix of size n), and $\mathrm{rank}(A) = n$, then the matrix is said to have **full rank**.

Heads-up! In terms of linear systems; the important rank is that of the *coefficient matrix*...

Properties of the $\mathrm{rank}(A)$

Consider a matrix $A \in \mathbb{R}^{n \times m}$, corresponding to a linear system of n equations with m unknowns:

Property #1a, and #1b

The inequalities

$$\mathrm{rank}(A) \le n, \quad \text{and} \quad \mathrm{rank}(A) \le m$$

hold.

"Proof:" If we transform A into $\mathrm{rref}(A)$, there is *at most* one leading 1 in each of the n rows (showing #1a); and there is at most one leading 1 in each of the m columns (showing #1b).

Properties of the $\mathrm{rank}(A)$ — $A \in \mathbb{R}^{n \times m}$

Property #2

If the system is inconsistent, then

$$\mathrm{rank}(A) < n.$$

"Proof:" For an inconsistent matrix A, $\mathrm{rref}(A)$ will contain (at least) a row of the form $[\,0 \;\; 0 \;\; 0 \;\; 0 \mid 1\,]$ — which does not have a leading one — so the rank can be at most $(n-1)$.

Properties of the $\mathrm{rank}(A)$ — $A \in \mathbb{R}^{n \times m}$

Property #3

If the system has exactly one solution, then

$$\mathrm{rank}(A) = m.$$

"Proof:" A leading 1 for each variable leaves no free (un-determined) variables.

Property #4

If the system has infinitely many solutions, then

$$\mathrm{rank}(A) < m.$$

"Proof:" In this case, there's at least one free (un-determined) variable, which does not have a corresponding leading 1.

Properties of the $\text{rank}(A)$

It is true that (for $A \in \mathbb{R}^{n \times m}$)

$$\begin{aligned} \#\text{Free\_Variables} &= \#\text{Total\_Variables} - \#\text{Leading\_Variables} \\ &= m - \text{rank}(A). \end{aligned}$$

More Mathematical Language: The Contrapositive

Definition (The Contrapositive of a Statement)

The contrapositive of a logic statement *"if p then q"*, in math notation: $p \to q$; is: *"if not-q then not-p"*, notation: $(\sim q) \to (\sim p)$.

The contrapositive of

$$\text{if } \underbrace{\textit{you are in this room}}_{p} \text{ then } \underbrace{\textit{you are in this building}}_{q}$$

is

$$\text{if } \underbrace{\textit{you are not in this building}}_{(\sim q)} \text{ then } \underbrace{\textit{you are not in this room}}_{(\sim p)}$$

A statement and its contrapositive are logically equivalent; that is if the statement is true, then the contrapositive is true.

Using the Contrapositive

We have some true statements (for $A \in \mathbb{R}^{n \times m}$):

1. if **the system is inconsistent**, then $\text{rank}(A) < n$.
2. if **the system has exactly one solution**, then $\text{rank}(A) = m$.
3. if **the system has infinitely many solutions**, then $\text{rank}(A) < m$.

Using the contrapositive, we immediately can say that

1. if $\text{rank}(A) = n$, then **the system is consistent**.
2. if $\text{rank}(A) < m$, then **the system has either no solutions, or infinitely many solutions**.
3. if $\text{rank}(A) = m$, then **the system has no solutions, or exactly one solution**.

Additional Discussion I

In all cases below, $A \in \mathbb{R}^{n \times m}$, $\text{rank}(A) \le \min(n, m)$.

1. For an **inconsistent** system, there must be (as least) one row with zeros one the coefficient-side, and a non-zero on the right-hand-side:

$$\begin{matrix} \text{at most} \\ n-1 \\ \text{leading ones} \\ \text{No leading one in this row} \end{matrix} \left[\begin{array}{ccc|c} \times & \cdots & \times & \times \\ \vdots & & \vdots & \vdots \\ \times & \cdots & \times & \times \\ 0 & \cdots & 0 & 1 \end{array}\right]$$

therefore, $\text{rank}(A) < n$.

Additional Discussion II

- When a system has **exactly one solution**, then $\text{rref}(A)$ must have a *leading one* in each column (no free variables can remain). The number of columns (m) equals the number of variables; so we must have $\text{rank}(A) = m$. Note that therefore $n \geq m$ — there can only be a single *leading one* in each row. We get two cases:
  - $(n = m) \Rightarrow \text{rref}(A) = I_n$
  - $(n > m) \Rightarrow$ Rows $(m + 1)$ to (n) must be all zeros, with zero right-hand-side.
- When a system has infinitely many solutions, there is at least one *free variable*. Therefore $\text{rref}(A)$ must have at least one column *without* a leading one, which means that $\text{rank}(A) \leq (m - 1)$. $\Rightarrow \text{rank}(A) < m$.

Additional Discussion III

Thinking about the contrapositive statements...

- When $\text{rank}(A) = n$, there are leading ones in each row of the reduced system. Therefore, there cannot be any row of the form

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right]$$

  which would indicate inconsistency. Hence, the system must be consistent. Again, we have two cases:
  - $(m = n) \Rightarrow \text{rref}(A) = I_n \Rightarrow$ the solution is unique.
  - $(m > n) \Rightarrow$ there are $(m - n)$ free variables $\Rightarrow$ there are infinitely many solutions.

Additional Discussion IV

- When $\text{rank}(A) < m$, there is at least one column without a leading one $\Rightarrow$ there is at least one free variable. Note that this does not rule out rows of the form

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right].$$

  if such a row exists, the system is inconsistent and has no solutions, otherwise the system is consistent with (at least) one free variable, and has infinitely many solutions.
- When $\text{rank}(A) = m$, there is a leading one in each column $\Rightarrow$ there are no free variables. If there is a row of the form

$$\left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \end{array}\right].$$

  the system is inconsistent and has no solutions, otherwise the system is consistent with a unique solution.

The number of equations vs. the number of unknowns

Theorem (#Equations vs. #Unknowns)

- **statement**: *If a linear system has exactly one solution, then there must be at least as many equations as there are variables; $(m \leq n)$ using previous notation.* [The coefficient matrix is either square, or "tall and skinny."]
- **contrapositive**: *If a linear system has fewer equations than unknowns $(n < m)$, then it either has no solutions or infinitely many solutions.* [The coefficient matrix is "short and wide."]

Proof (of statement).

A system with exactly one solution has $m = \text{rank}(A)$ [PROPERTY #3]; further $\text{rank}(A) \leq n$ [PROPERTY #1A], therefore

$$m = \text{rank}(A) \leq n$$

which shows $(m \leq n)$. □

Square Matrices, and Their Reduced-Row-Echelon-Form

"Square" systems play a huge role in linear algebra:

Theorem (Systems of n Equations in n Variables)

A linear system of n equations (rows in the coefficient matrix) in n variables (columns in the coefficient matrix) has a unique solution if and only if the rank of the coefficient matrix A satisfies $\mathrm{rank}(A) = n$. *When that is true the Reduced Row Echelon Form of A satisfies*

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

that is $\mathrm{rref}(A)$ *is the* $(n \times n)$ *identity matrix, usually denoted* I_n.

Fundamentals of Matrix and Vector Algebra

We now define ways that our Matrix and Vector objects can "interact"; we are adding some "verbs" to our Mathematical language!

Definition (Matrix Sums)

The sum of two matrices of the same size $A, B \in \mathbb{R}^{n\times m}$ is determined by the entry-by-entry sums, that is if

$$C = A + B$$

then $C \in \mathbb{R}^{n\times m}$, and $c_{ij} = a_{ij} + b_{ij}$ for $i \in [1, \dots, n]$, $j \in [1, \dots, m]$.

Definition (Scalar Multiple of a Matrix)

If $A \in \mathbb{R}^{n\times m}$ is a matrix, and $\rho \in \mathbb{R}$ is a real scalar, then the scalar-matrix-product

$$C = \rho\, A$$

gives $C \in \mathbb{R}^{n\times m}$, and $c_{ij} = \rho\, a_{ij}$.

Fundamentals of Matrix and Vector Algebra — Dot Product

Definition (Dot Product of Vectors)

Consider two vectors $\vec{v}$, and $\vec{w}$, both with n components (that is $v_1, v_2, \dots, v_n$ and $w_1, w_2, \dots, w_n$). The **dot product** is defined as the sum of the element-wise products:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k$$

Note: The way we have defined the *dot product* it is not row/column sensitive. However if you stick with the standard notation that "vectors" are column-vectors, it is common to see the equivalent notation:

$$\vec{v}^T \vec{w} \equiv \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k.$$

A common alternative name for the dot product, is the **inner product.**

Fundamentals of Matrix and Vector Algebra — Matrix-Vector Product

Definition (Matrix-Vector Product)

If $A \in \mathbb{R}^{n\times m}$ matrix with row-vectors $\vec{r}_1^T, \dots, \vec{r}_n^T \in \mathbb{R}^m$, and $\vec{x} \in \mathbb{R}^m$ is a (column) vector, then

$$A\vec{x} = \begin{bmatrix} — & \vec{r}_1^T & — \\ & \vdots & \\ — & \vec{r}_n^T & — \end{bmatrix} \vec{x} = \underbrace{\begin{bmatrix} \vec{r}_1^T \vec{x} \\ \vdots \\ \vec{r}_n^T \vec{x} \end{bmatrix}}_{\text{Using Inner Product Notation}} \equiv \underbrace{\begin{bmatrix} \vec{r}_1^T \cdot \vec{x} \\ \vdots \\ \vec{r}_n^T \cdot \vec{x} \end{bmatrix}}_{\text{Using Dot Product Notation}}$$

The i^{th} component of the resulting vector $\vec{y} = A\vec{x}$ is given by the dot (inner) product of the i^{th} row of A and the vector $\vec{x}$. Note that if $m \neq n$ then $\vec{y} \in \mathbb{R}^n$ is not the same size as $\vec{x} \in \mathbb{R}^m$.

Size and Shape Do Matter in Matrix-Vector Multiplication

For the matrix-vector product to make sense, the matrix $A \in \mathbb{R}^{n\times m}$ and the vector $\vec{x} \in \mathbb{R}^{\mathbf{m}} \equiv \mathbb{R}^{\mathbf{m}\times 1}$ must have compatible sizes:

$$\underbrace{A}_{[n\times m]}\ \underbrace{\vec{x}}_{[m\times 1]} = \underbrace{\vec{y}}_{[n\times 1]}$$

Looking Ahead (Matrix Multiplication): thinking about size, it's probably OK to multiply $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times p}$; a solid "guess" for the size of the result? —

$$\underbrace{A}_{[n\times m]}\ \underbrace{B}_{[m\times p]} = \underbrace{C}_{[n\times p]},$$

however the product BA does not make sense (unless $n = p$).

We will formally define Matrix-Matrix products in [NOTES#3.3].

Thinking About $A\vec{x}$ in a Different Way

So far, we have thought of the components of $A\vec{x}$ as the result of dot-products of the rows of A and the vector $\vec{x}$; to inspire a different view:

Consider $A \in \mathbb{R}^{2\times 3}$ and $\vec{x} \in \mathbb{R}^3$, then

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

We realize that

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

Which means that we can think of $\vec{y} = A\vec{x}$ as a sum of vectors (where the vectors are the columns of A, scaled by the components of $\vec{x}$)

Thinking about $A\vec{x}$ as the Linear Combination of the Columns

Theorem (The Product $A\vec{x}$ in Terms of the Columns of A)

If the column vectors of an $n \times m$ matrix A are $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{x} \in \mathbb{R}^m$ with components $x_1, \ldots, x_m$, then

$$A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

Definition (Linear Combinations)

A vector $\vec{b}$ in $\mathbb{R}^n$ is called a **linear combination** of the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ if there exists scalars $x_1, \ldots, x_m$ such that

$$\vec{b} = x_1\vec{v}_1 + \ldots x_m\vec{v}_m.$$

Challenge Question

Think, again, about the linear systems:

$$\text{a.} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{b.} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{c.} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Let $A_a \in \mathbb{R}^{4\times 3}$, $A_b \in \mathbb{R}^{3\times 3}$, $A_c \in \mathbb{R}^{3\times 3}$ be the coefficient matrices; and $\vec{b}_a \in \mathbb{R}^4$, $\vec{b}_b, \vec{b}_c \in \mathbb{R}^3$ be the right-hand-sides. We are seeking solutions $\vec{x}_a, \vec{x}_b, \vec{x}_c \in \mathbb{R}^3$, so that $A_a\vec{x}_a = \vec{b}_a$, $A_b\vec{x}_b = \vec{b}_b$, $A_c\vec{x}_c = \vec{b}_c$.

If we think of the matrix-vector products as linear combinations of the columns; how can we characterize the 3 possible scenarios (no, ∞, 1) solutions?

Does the rank have anything to do with it?

This will be answered very soon, but do *think* about it...

Two More Theorems...

Theorem (Algebraic Rules for $A\vec{x}$)

If $A \in \mathbb{R}^{n\times m}$, $\vec{x} \in \mathbb{R}^m$, $\vec{y} \in \mathbb{R}^m$, and $k \in \mathbb{R}$, then

- $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$
- $A(k\vec{x}) = k(A\vec{x})$

Theorem (Matrix Form of Linear System)

We can write the linear system with Augmented Matrix $\left[A \mid \vec{b}\right]$ in matrix-vector form as

$$A\vec{x} = \vec{b}.$$

Suggested Problems 1.3

Available on "Learning Glass" videos:

- **1.3.1** Given rref, how many solutions does each system have?
- **1.3.2** Find the rank of a matrix.
- **1.3.3** Find the rank of a matrix.
- **1.3.7** How many solutions? (Geometrical argument).
- **1.3.13** Compute matrix-vector product.
- **1.3.22** Given a system + properties of the solution; what is the form of rref(A)?
- **1.3.23** Given a system + properties of the solution; what is the form of rref(A)?
- **1.3.37** Find all solutions of $A\vec{x} = \vec{b}$.
- **1.3.46** Find rank(A).
- **1.3.55** Is a given vector a linear combination of two other vectors?

Lecture – Book Roadmap

Lecture	Book, [GS5–]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3
1.3	§1.1, §1.2, §1.3, §2.1, §2.3

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(1.3.1)

(1.3.1) The reduced-row-echelon-forms (RREF) of the augmented matrices of three systems are given. How many solutions does each system have?

$$\textbf{(a)} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad \textbf{(b)} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \end{array}\right], \quad \textbf{(c)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

(1.3.2), (1.3.3)

(1.3.2) Find the rank of

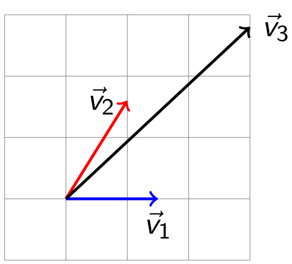
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(1.3.3) Find the rank of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(1.3.7)

(1.3.7) Consider the vectors $\bar{\mathbf{v}}_1$, $\bar{\mathbf{v}}_2$, $\bar{\mathbf{v}}_3 \in \mathbb{R}^2$:



How many solutions x, y does the system

$$x\bar{\mathbf{v}}_1 + y\bar{\mathbf{v}}_2 = \bar{\mathbf{v}}_3$$

have? Argue geometrically.

(1.3.13), (1.3.22), (1.3.23)

(1.3.13) Compute the matrix-vector product $A\bar{\mathbf{x}}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}.$$

(1.3.22) Consider a linear system of 3 equations with 3 unknowns, $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$. GIVEN: This system has a unique solution. What does the reduced-row-echelon-form of the coefficient matrix, $\mathrm{rref}(A)$ of this system look like?

(1.3.23) Consider a linear system of 4 equations with 3 unknowns, $A\bar{\mathbf{x}} = \bar{\mathbf{b}}$. GIVEN: This system has a unique solution. What does the reduced-row-echelon-form of the coefficient matrix, $\mathrm{rref}(A)$ of this system look like?

(1.3.37), (1.3.46)

(1.3.37) Find all vectors $\vec{x}$ such that $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

(1.3.46) Find the rank of the matrix

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix},$$

where $a, d, f \neq 0$; and $b, c, e \in \mathbb{R}^n$ are arbitrary numbers.

(1.3.55)

(1.3.55) Is the vector

$$\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$