Math 254: Introduction to Linear Algebra

Notes #1.4 — Matrix–Vector Fundamentals Wrapup

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Outline

SLOs 1.4

After this lecture you should:

- Know the functional definitions of the fundamental vector algebra operations
- Know the two ways to compute (inner) *dot products* of vectors
- Be able to compute the *norm* (length) of a vector
- Know what *unit vectors* are
- Understand *Ortogonality of Vectors*, and the relation to the dot product

Mostly a "formal review" of what we have done; with some new language added.

Vectors

Previously, we have defined vectors as matrices with only one column:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \equiv \mathbb{R}^{n\times 1};$$

where the scalars v_k, $k = 1, \dots, n$ are the *components of the vector*.

Vector- and matrix-algebra is essentially the "same." However, there is some "language" (properties) which only apply to vectors (matrices).

Here, we quickly go over some basic vector definitions and properties.

Vector Operations — Addition, Scaling

Definition (Vector Addition)

The sum of $\vec{v} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^n$ is defined component-wise, *i.e.* if

$$\vec{z} = \vec{v} + \vec{w},$$

then $\vec{z} \in \mathbb{R}^n$, and $z_k = v_k + w_k$, $k \in \{1, \dots, n\}$.

Definition (Scalar-Vector Multiplication)

Let $\vec{v} \in \mathbb{R}^n$ and $\rho \in \mathbb{R}$, then

$$\vec{z} = \rho\vec{v}$$

gives $\vec{z} \in \mathbb{R}^n$, and $z_k = \rho\, v_k$, $k \in \{1, \dots, n\}$.

Vector Operations — Negative, Difference

Definition (The *Negative (Opposite)* of a Vector (also "Additive Inverse"))

Using previous definitions:

$$-\vec{v} \equiv (-1)\vec{v}$$

Definition (Vector Difference)

For $\vec{v} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^n$:

$$\vec{z} = \vec{v} - \vec{w} \equiv \vec{v} + (-\vec{w}),$$

so that $\vec{z} \in \mathbb{R}^n$, and $z_k = v_k - w_k$, $k \in \{1, \dots, n\}$

Definition (The Zero vector)

$\vec{0} \in \mathbb{R}^n$ is the vector with n components; all of which are 0.

Summary of Vector Algebra Rules

Fundamental Rules of Vector Algebra — (these make $\mathbb{R}^n$ a "Vector Space")

The following formulas hold $\forall\ \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $\forall\ c, k \in \mathbb{R}$:

1. Addition is *associative*†: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
2. Addition is *commutative*‡: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $\vec{v} + \vec{0} = \vec{v}$ — $\vec{0}$ is the "Additive Identity"
4. $\forall \vec{v} \in \mathbb{R}^n$, $\exists$ a unique $\vec{x} \in \mathbb{R}^n$: $\vec{v} + \vec{x} = \vec{0}$; $\vec{x} = -\vec{v}$. — $(-v)$ is the "Additive Inverse"
5. $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$. — distributive property
6. $(c + k)\vec{v} = c\vec{v} + k\vec{v}$ — distributive property
7. $c(k\vec{v}) = (ck)\vec{v}$
8. $1\vec{v} = \vec{v}$ — 1 is the "Multiplicative Identity"

These rules "follow directly" from the propertial of real numbers (scalars), and the component-by-component definition of addition (and subtraction) of vectors.

† Grouping does not matter.
‡ Order does not matter.

Geometric Interpretation: Parallel Vectors

Definition (Parallel Vectors)

We say that two vectors $\vec{v}$ and $\vec{w}$ are **parallel** if one of them is a scalar multiple of the other.

Recall: By definition, all our vectors go through the origin, so they cannot be parallel and not intersect.

By this definition the zero-vector is parallel to every vector, since

$$\vec{0} = 0\vec{v}.$$

Note: Linear-Algebra-Parallel is slightly different from (a special case of) Geometric-Parallel.

Rules for Dot Products

Rewind (Dot product of vectors)

Consider two vectors $\vec{v}$, and $\vec{w}$, both with n components (that is $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$). The **dot product** is defined as the sum of the element-wise products:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k$$

Rules for Dot Products

The following holds $\forall$ $\vec{v}$, $\vec{u}$, $\vec{w}$ with n components; and $\forall k \in \mathbb{R}$:

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ — commutative
2. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ — distributive property
3. $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w})$
4. $\forall v \neq \vec{0}$: $\vec{v} \cdot \vec{v} > 0$

Thinking in Geometrical Terms...

Definition (Length / Norm) (2-norm, $\|\vec{x}\|_2$)

The *length* (or *norm*), of a vector $\vec{x} \in \mathbb{R}^n$ is denoted $\|\vec{x}\|$, and defined by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Definition (Unit Vector)

A vector $\vec{u} \in \mathbb{R}^n$ is called a *unit vector* if $\|\vec{u}\| = 1$, *i.e.* the length of the vector is 1.

Definition (Unit Sphere / Circle)

The collection of all $\vec{u} \in \mathbb{R}^n$ with $\|\vec{u}\| = 1$, is called the *unit sphere* (in $\mathbb{R}^n$); when $n = 2$ the we tend to call it the *unit circle*.

Orthogonality & An Alternative Expression for the Dot Product

Definition (Orthogonality of Vectors)

Two vectors $\vec{v}$, $\vec{w} \in \mathbb{R}^n$ are called *orthogonal* or *perpendicular* if $\vec{v} \cdot \vec{w} = 0$.

In vector spaces (here $\mathbb{R}^n$), orthogonality is *always* defined with respect to the inner product (here, the dot product).

Definition (Geometric Interpretation of the Dot Product)

If $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$ are two non-zero vectors, then

$$\vec{v} \cdot \vec{w} = \cos\theta \, \|\vec{v}\| \, \|\vec{w}\|$$

where θ is the angle between the vectors $\vec{v}$ and $\vec{w}$.

We need some figures and examples...

Two "Versions" of the Dot Product — Are They The Same?!?

We have two "competing" expressions for the dot-product:

Definition (Dot Product of Vectors)

Consider two vectors $\vec{v}$, and $\vec{w}$, both with n components (that is $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_n$). The **dot product** is defined as the sum of the element-wise products:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k$$

Definition (Geometric Interpretation of the Dot Product)

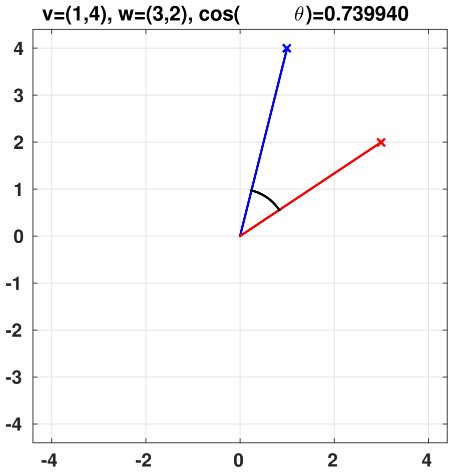
If $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$ are two non-zero vectors, then

$$\vec{v} \cdot \vec{w} = \cos\theta \, \|\vec{v}\| \, \|\vec{w}\|,$$

where θ is the angle between the vectors $\vec{v}$ and $\vec{w}$.

It is NOT obvious that these give the same values...

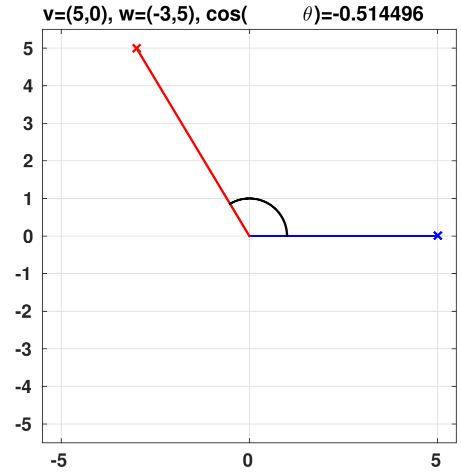
Are They The Same?!? Example #1/4



$$
\begin{aligned}
\|\vec{v}\| &= \sqrt{1^2+4^2} = \sqrt{17} \\
\|\vec{w}\| &= \sqrt{3^2+2^2} = \sqrt{13} \\
\cos\theta &= 0.739940\ldots \\
\cos\theta\, \|v\|\, \|w\| &= 11 \\
\hline
\vec{v}\cdot\vec{w} &= 1*3+4*2 = 11
\end{aligned}
$$

Basic Vector Operations
The Dot / Inner Product — Competing Expressions
Two "Versions" of the Dot Product — Are They The Same?!?
Examples...
Lecture – Book Roadmap

Are They The Same?!? Example #2/4



$$
\begin{aligned}
\|\vec{v}\| &= \sqrt{5^2+0^2} = \sqrt{25} \\
\|\vec{w}\| &= \sqrt{3^2+5^2} = \sqrt{34} \\
\cos\theta &= -0.514496\ldots \\
\cos\theta\, \|v\|\, \|w\| &= -15 \\
\hline
\vec{v}\cdot\vec{w} &= 5*(-3)+0*5 = -15
\end{aligned}
$$

Basic Vector Operations
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Examples...
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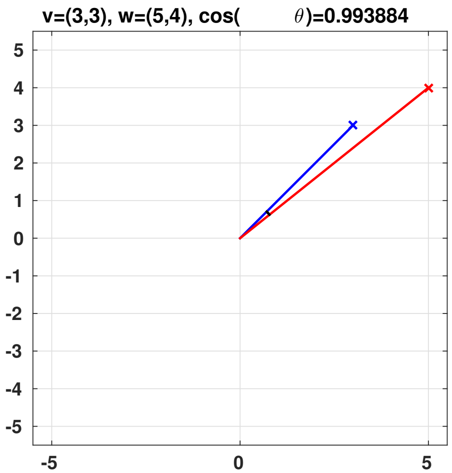
Are They The Same?!? Example #3/4



$$
\begin{aligned}
\|\vec{v}\| &= \sqrt{3^2+3^2} = \sqrt{18} \\
\|\vec{w}\| &= \sqrt{5^2+4^2} = \sqrt{41} \\
\cos\theta &= 0.993884\ldots \\
\cos\theta\, \|v\|\, \|w\| &= 27 \\
\hline
\vec{v}\cdot\vec{w} &= 3*5+3*4 = 27
\end{aligned}
$$

Basic Vector Operations
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Two "Versions" of the Dot Product — Are They The Same?!?
Examples...
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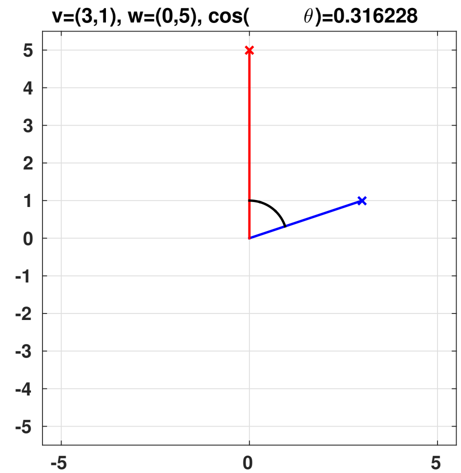
Are They The Same?!? Example #4/4



$$
\begin{aligned}
\|\vec{v}\| &= \sqrt{3^2+1^2} = \sqrt{10} \\
\|\vec{w}\| &= \sqrt{0^2+5^2} = \sqrt{25} \\
\cos\theta &= 0.316228\ldots \\
\cos\theta\, \|v\|\, \|w\| &= 5 \\
\hline
\vec{v}\cdot\vec{w} &= 3*0+1*5 = 5
\end{aligned}
$$

OK, We Feel Better Now...

Whereas examples are NOT proof; it certainly seems like the two expressions agree.

Most of the time the first definition (using sums-of-products) is the most natural to work with.

However, we can use the equivalence of the two expressions

$$\sum_{k=1}^{n} v_k w_k = \vec{v} \cdot \vec{w} = \cos\theta \, \|\vec{v}\| \, \|\vec{w}\|$$

to...

Compute $\cos\theta$

$$\cos\theta = \frac{1}{\|\vec{v}\| \, \|\vec{w}\|} \sum_{k=1}^{n} v_k w_k$$

Lecture – Book Roadmap

Lecture	Book, [GS5–]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3
1.3	§1.1, §1.2, §1.3, §2.1, §2.3
1.4	§1.1–§1.3, §2.1–§2.3

Next major topic: "Linear Transformations" ([GS5–§8.1-§8.3])

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		