

Math 254: Introduction to Linear Algebra

Notes #1.4 — Matrix–Vector Fundamentals Wrapup

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- 1 Student Learning Objectives
 - SLOs: 1.4 Fundamentals Wrapup
- 2 Basic Vector Operations
 - Definitions & Properties
- 3 The Dot / Inner Product — Competing Expressions
 - Two “Versions” of the Dot Product — Are They The Same?!?
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- 4 Supplemental Material
 - Metacognitive Reflection

SLOs 1.4

After this lecture you should:

- Know the functional definitions of the fundamental vector algebra operations
- Know the two ways to compute (inner) *dot products* of vectors
- Be able to compute the *norm* (length) of a vector
- Know what *unit vectors* are
- Understand *Orthogonality of Vectors*, and the relation to the dot product

Mostly a “formal review” of what we have done; with some new language added.



Vectors

Previously, we have defined vectors as matrices with only one column:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n \equiv \mathbb{R}^{n \times 1};$$

where the scalars v_k , $k = 1, \dots, n$ are the *components of the vector*.

Vector- and matrix-algebra is essentially the “same.” However, there is some “language” (properties) which only apply to vectors (matrices).

Here, we quickly go over some basic vector definitions and properties.



Vector Operations

Addition, Scaling

Definition (Vector Addition)

The sum of $\vec{v} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^n$ is defined component-wise, *i.e.* if

$$\vec{z} = \vec{v} + \vec{w},$$

then $\vec{z} \in \mathbb{R}^n$, and $z_k = v_k + w_k$, $k \in \{1, \dots, n\}$.

Definition (Scalar-Vector Multiplication)

Let $\vec{v} \in \mathbb{R}^n$ and $\rho \in \mathbb{R}$, then

$$\vec{z} = \rho \vec{v}$$

gives $\vec{z} \in \mathbb{R}^n$, and $z_k = \rho v_k$, $k \in \{1, \dots, n\}$.



Vector Operations

Negative, Difference

Definition (The *Negative (Opposite)* of a Vector (also “Additive Inverse”))

Using previous definitions:

$$-\vec{v} \equiv (-1)\vec{v}$$

Definition (Vector Difference)

For $\vec{v} \in \mathbb{R}^n$ and $\vec{w} \in \mathbb{R}^n$:

$$\vec{z} = \vec{v} - \vec{w} \equiv \vec{v} + (-\vec{w}),$$

so that $\vec{z} \in \mathbb{R}^n$, and $z_k = v_k - w_k$, $k \in \{1, \dots, n\}$

Definition (The Zero vector)

$\vec{0} \in \mathbb{R}^n$ is the vector with n components; all of which are 0.



Summary of Vector Algebra Rules

Fundamental Rules of Vector Algebra

(these make \mathbb{R}^n a “Vector Space”)

The following formulas hold $\forall \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $\forall c, k \in \mathbb{R}$:

1 Addition is *associative*[†]: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

2 Addition is *commutative*[‡]: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

3 $\vec{v} + \vec{0} = \vec{v}$

$\vec{0}$ is the “Additive Identity”

4 $\forall \vec{v} \in \mathbb{R}^n, \exists$ a unique $\vec{x} \in \mathbb{R}^n$: $\vec{v} + \vec{x} = \vec{0}$; $\vec{x} = -\vec{v}$.

$(-\vec{v})$ is the “Additive Inverse”

5 $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$.

distributive property

6 $(c + k)\vec{v} = c\vec{v} + k\vec{v}$

distributive property

7 $c(k\vec{v}) = (ck)\vec{v}$

8 $1\vec{v} = \vec{v}$

1 is the “Multiplicative Identity”

These rules “follow directly” from the properties of real numbers (scalars), and the component-by-component definition of addition (and subtraction) of vectors.

† Grouping does not matter.

‡ Order does not matter.

Geometric Interpretation: Parallel Vectors

Definition (Parallel Vectors)

We say that two vectors \vec{v} and \vec{w} are **parallel** if one of them is a scalar multiple of the other.

Recall: By definition, all our vectors go through the origin, so they cannot be parallel and not intersect.

By this definition the zero-vector is parallel to every vector, since

$$\vec{0} = 0\vec{v}.$$

Note: Linear-Algebra-Parallel is slightly different from (a special case of) Geometric-Parallel.

Rules for Dot Products

Rewind (Dot product of vectors)

Consider two vectors \vec{v} , and \vec{w} , both with n components (that is v_1, v_2, \dots, v_n and w_1, w_2, \dots, w_n). The **dot product** is defined as the sum of the element-wise products:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n = \sum_{k=1}^n v_k w_k$$

Rules for Dot Products

The following holds $\forall \vec{v}, \vec{u}, \vec{w}$ with n components; and $\forall k \in \mathbb{R}$:

- 1 $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ commutative
- 2 $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ distributive property
- 3 $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w})$
- 4 $\forall \vec{v} \neq \vec{0}: \vec{v} \cdot \vec{v} > 0$

Thinking in Geometrical Terms...

Definition (Length / Norm)

(2-norm, $\|\vec{x}\|_2$)

The *length* (or *norm*), of a vector $\vec{x} \in \mathbb{R}^n$ is denoted $\|\vec{x}\|$, and defined by

$$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Definition (Unit Vector)

A vector $\vec{u} \in \mathbb{R}^n$ is called a *unit vector* if $\|\vec{u}\| = 1$, i.e. the length of the vector is 1.

Definition (Unit Sphere / Circle)

The collection of all $\vec{u} \in \mathbb{R}^n$ with $\|\vec{u}\| = 1$, is called the *unit sphere* (in \mathbb{R}^n); when $n = 2$ we tend to call it the *unit circle*.

Orthogonality & An Alternative Expression for the Dot Product

Definition (Orthogonality of Vectors)

Two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are called *orthogonal* or *perpendicular* if $\vec{v} \cdot \vec{w} = 0$.

In vector spaces (here \mathbb{R}^n), orthogonality is *always* defined with respect to the inner product (here, the dot product).

Definition (Geometric Interpretation of the Dot Product)

If \vec{v} and $\vec{w} \in \mathbb{R}^n$ are two non-zero vectors, then

$$\vec{v} \cdot \vec{w} = \cos \theta \|\vec{v}\| \|\vec{w}\|$$

where θ is the angle between the vectors \vec{v} and \vec{w} .

We need some figures and examples...

Two “Versions” of the Dot Product — Are They The Same?!?

We have two “competing” expressions for the dot-product:

Definition (Dot Product of Vectors)

Consider two vectors \vec{v} , and \vec{w} , both with n components (that is v_1, v_2, \dots, v_n and w_1, w_2, \dots, w_n). The **dot product** is defined as the sum of the element-wise products:

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Definition (Geometric Interpretation of the Dot Product)

If \vec{v} and $\vec{w} \in \mathbb{R}^n$ are two non-zero vectors, then

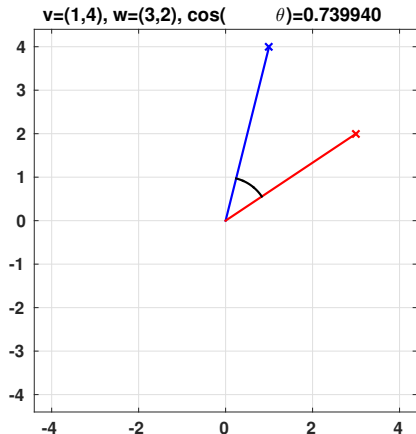
$$\vec{v} \cdot \vec{w} = \cos \theta \|\vec{v}\| \|\vec{w}\|,$$

where θ is the angle between the vectors \vec{v} and \vec{w} .

It is NOT obvious that these give the same values...

Are They The Same?!?

Example #1/4

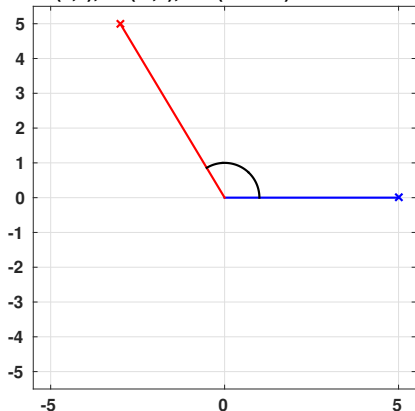


$$\begin{aligned} \|\vec{v}\| &= \sqrt{1^2 + 4^2} = \sqrt{17} \\ \|\vec{w}\| &= \sqrt{3^2 + 2^2} = \sqrt{13} \\ \cos \theta &= 0.739940\dots \\ \cos \theta \|\vec{v}\| \|\vec{w}\| &= 11 \end{aligned}$$

$$\vec{v} \cdot \vec{w} = 1 * 3 + 4 * 2 = 11$$

Are They The Same?!?

Example #2/4

 $v=(5,0), w=(-3,5), \cos(\theta)=-0.514496$


$$\|\vec{v}\| = \sqrt{5^2 + 0^2} = \sqrt{25}$$

$$\|\vec{w}\| = \sqrt{3^2 + 5^2} = \sqrt{34}$$

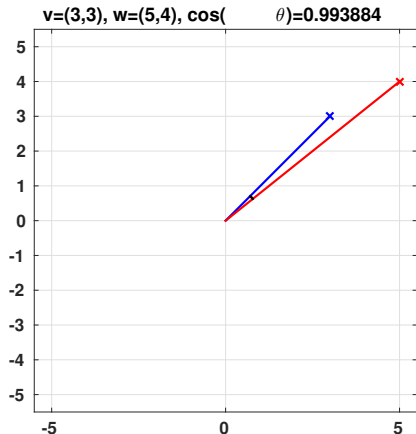
$$\cos \theta = -0.514496\dots$$

$$\cos \theta \|\vec{v}\| \|\vec{w}\| = -15$$

$$\vec{v} \cdot \vec{w} = 5 * (-3) + 0 * 5 = -15$$

Are They The Same?!?

Example #3/4

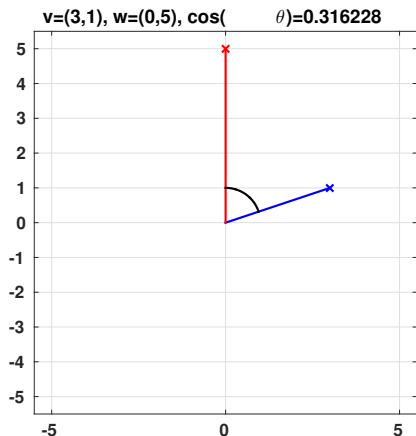


$$\begin{aligned} \|\vec{v}\| &= \sqrt{3^2 + 3^2} = \sqrt{18} \\ \|\vec{w}\| &= \sqrt{5^2 + 4^2} = \sqrt{41} \\ \cos \theta &= 0.993884\dots \\ \cos \theta \|\vec{v}\| \|\vec{w}\| &= 27 \end{aligned}$$

$$\vec{v} \cdot \vec{w} = 3 * 5 + 3 * 4 = 27$$

Are They The Same?!?

Example #4/4



$$\begin{aligned} \|\vec{v}\| &= \sqrt{3^2 + 1^2} = \sqrt{10} \\ \|\vec{w}\| &= \sqrt{0^2 + 5^2} = \sqrt{25} \\ \cos \theta &= 0.316228\dots \\ \cos \theta \|\vec{v}\| \|\vec{w}\| &= 5 \end{aligned}$$

$$\vec{v} \cdot \vec{w} = 3 * 0 + 1 * 5 = 5$$

OK, We Feel Better Now...

Whereas examples are NOT proof; it certainly seems like the two expressions agree.

Most of the time the first definition (using sums-of-products) is the most natural to work with.

However, we can use the equivalence of the two expressions

$$\sum_{k=1}^n v_k w_k = \vec{v} \cdot \vec{w} = \cos \theta \|\vec{v}\| \|\vec{w}\|$$

to...

Compute $\cos \theta$

$$\cos \theta = \frac{1}{\|\vec{v}\| \|\vec{w}\|} \sum_{k=1}^n v_k w_k$$

Lecture – Book Roadmap

Lecture	Book, [GS5–]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3
1.3	§1.1, §1.2, §1.3, §2.1, §2.3
1.4	§1.1–§1.3, §2.1–§2.3

Next major topic: “Linear Transformations” ([GS5–§8.1–§8.3])

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		