

Math 254: Introduction to Linear Algebra

Notes #2.1 — Linear Transformations (Intro)

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After this lecture you should:

- Understand how a Matrix[-Vector product] defines a *Linear Transformation*
- Understand the concept of *The Inverse of a Linear Transformation*
- Be able to compute *The Inverse of a (small-ish) Matrix*
- Understand the concept of, and identify *The Matrix of a Linear Transformation*



Outline

- 1 Student Learning Objectives
 - SLOs: Linear Transformations (Intro)
- 2 Linear Transformations
 - Silly Introductory Example
 - Formal Statements
- 3 Suggested Problems
 - Suggested Problems 2.1
 - Lecture–Book Roadmap
- 4 Supplemental Material
 - Metacognitive Reflection
 - Problem Statements 2.1



Imagine you are shopping online at ReallyWeakSecurity.com which “protects” your 16-digit credit card number (1234 5678 9098 7654) in the following way:

$$\begin{bmatrix} -1 & -4 & -1 & -3 \\ 2 & -4 & 0 & 4 \\ -5 & -3 & -1 & -5 \\ -2 & -2 & 2 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1234 \\ 5678 \\ 9098 \\ 7654 \end{bmatrix}}_{\text{Credit Card Number, } \vec{x}} = \underbrace{\begin{bmatrix} -56006 \\ 10372 \\ -70572 \\ 19680 \end{bmatrix}}_{\text{"Coded" transmission, } \vec{y}}$$

You enter your CC#, the “coded” version gets transmitted to the website...

The operation $\vec{y} = A\vec{x}$ is called a **linear transformation**. The matrix A is the **coefficient matrix** of the transformation.



Online Shopping

Coding-Decoding

Your "coded" payment details get processed at
QuestionableOffshorePaymentProcessor.com:

$$\left[\begin{array}{cccc|c} -1 & -4 & -1 & -3 & -56006 \\ 2 & -4 & 0 & 4 & 10372 \\ -5 & -3 & -1 & -5 & -70572 \\ -2 & -2 & 2 & 2 & 19680 \end{array} \right] \quad \text{Divide by } -1$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 2 & -4 & 0 & 4 & 10372 \\ -5 & -3 & -1 & -5 & -70572 \\ -2 & -2 & 2 & 2 & 19680 \end{array} \right] \quad \begin{array}{l} -2 \times \text{row \#1} \\ +5 \times \text{row \#1} \\ +2 \times \text{row \#1} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & -12 & -2 & -2 & -101640 \\ 0 & 17 & 4 & 10 & 209458 \\ 0 & 6 & 4 & 8 & 131692 \end{array} \right] \quad \text{Divide by } -12$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 17 & 4 & 10 & 209458 \\ 0 & 6 & 4 & 8 & 131692 \end{array} \right] \quad \begin{array}{l} -17 \times \text{row \#2} \\ -6 \times \text{row \#2} \end{array}$$



Online Shopping

Coding-Decoding

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & \frac{7}{6} & \frac{43}{6} & 65468 \\ 0 & 0 & 3 & 7 & 80872 \end{array} \right] \quad \text{Divide by } \frac{7}{6}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{392808}{7} \\ 0 & 0 & 3 & 7 & 80872 \end{array} \right] \quad -3 \times \text{row \#3}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{392808}{7} \\ 0 & 0 & 0 & -\frac{80}{7} & -\frac{612320}{7} \end{array} \right] \quad \text{Divide by } -\frac{80}{7}$$



Online Shopping

Coding-Decoding

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{392808}{7} \\ 0 & 0 & 0 & 1 & 7654 \end{array} \right] \quad \begin{array}{l} -3 \times \text{row \#4} \\ -\frac{1}{6} \times \text{row \#4} \\ -\frac{43}{7} \times \text{row \#4} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 0 & 33044 \\ 0 & 1 & \frac{1}{6} & 0 & \frac{21583}{3} \\ 0 & 0 & 1 & 0 & 9098 \\ 0 & 0 & 0 & 1 & 7654 \end{array} \right] \quad \begin{array}{l} -1 \times \text{row \#3} \\ -\frac{1}{6} \times \text{row \#3} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & 23946 \\ 0 & 1 & 0 & 0 & 5678 \\ 0 & 0 & 1 & 0 & 9098 \\ 0 & 0 & 0 & 1 & 7654 \end{array} \right] \quad -4 \times \text{row \#2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1234 \\ 0 & 1 & 0 & 0 & 5678 \\ 0 & 0 & 1 & 0 & 9098 \\ 0 & 0 & 0 & 1 & 7654 \end{array} \right]$$



Online Shopping

Smarter Decoding?

Now, if you are "in charge" of decoding by hand, this process will get a little bit old somewhat fast... and you want to find the decoding (inverse) transformation $\vec{y} \mapsto \vec{x}$ once and for all.

Find the Inverse Transformation

Given

$$A\vec{x} = \vec{y}.$$

Find a matrix B so that

$$B\vec{y} = \vec{x}.$$


Online Shopping

Looking for the Decoding Matrix

$$\begin{bmatrix} -1 & -4 & -1 & -3 & | & y_1 \\ 2 & -4 & 0 & 4 & | & y_2 \\ -5 & -3 & -1 & -5 & | & y_3 \\ -2 & -2 & 2 & 2 & | & y_4 \end{bmatrix} \quad \text{Divide by } -1$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 2 & -4 & 0 & 4 & | & y_2 \\ -5 & -3 & -1 & -5 & | & y_3 \\ -2 & -2 & 2 & 2 & | & y_4 \end{bmatrix} \quad \begin{array}{l} -2 \times \text{row \#1} \\ +5 \times \text{row \#1} \\ +2 \times \text{row \#1} \end{array}$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 0 & -12 & -2 & -2 & | & 2y_1 + y_2 \\ 0 & 17 & 4 & 10 & | & y_3 - 5y_1 \\ 0 & 6 & 4 & 8 & | & y_4 - 2y_1 \end{bmatrix} \quad \text{Divide by } -12$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & | & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 17 & 4 & 10 & | & y_3 - 5y_1 \\ 0 & 6 & 4 & 8 & | & y_4 - 2y_1 \end{bmatrix} \quad \begin{array}{l} -17 \times \text{row \#2} \\ -6 \times \text{row \#2} \end{array}$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & | & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & \frac{7}{6} & \frac{43}{6} & | & \frac{17y_2}{12} - \frac{13y_1}{6} + y_3 \\ 0 & 0 & 3 & 7 & | & \frac{y_2}{2} - y_1 + y_4 \end{bmatrix} \quad \text{Divide by } \frac{7}{6}$$



Online Shopping

Looking for the Decoding Matrix

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & | & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & 1 & \frac{43}{7} & | & \frac{17y_2}{14} - \frac{13y_1}{7} + \frac{6y_3}{7} \\ 0 & 0 & 3 & 7 & | & \frac{y_2}{2} - y_1 + y_4 \end{bmatrix} \quad \begin{array}{l} -3 \times \text{row \#3} \\ -3 \times \text{row \#4} \\ -\frac{1}{6} \times \text{row \#4} \\ -\frac{43}{7} \times \text{row \#4} \end{array}$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & | & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & 1 & \frac{43}{7} & | & \frac{17y_2}{14} - \frac{13y_1}{7} + \frac{6y_3}{7} \\ 0 & 0 & 0 & -\frac{80}{7} & | & \frac{32y_1}{7} - \frac{22y_2}{7} - \frac{18y_3}{7} + y_4 \end{bmatrix} \quad \text{Divide by } -\frac{80}{7}$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & | & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & 1 & \frac{43}{7} & | & \frac{17y_2}{14} - \frac{13y_1}{7} + \frac{6y_3}{7} \\ 0 & 0 & 0 & 1 & | & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} \end{bmatrix} \quad \begin{array}{l} -1 \times \text{row \#3} \\ -\frac{1}{6} \times \text{row \#3} \end{array}$$



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Looking for the Decoding Matrix

$$\begin{bmatrix} 1 & 4 & 0 & 0 & | & -\frac{2y_1}{5} - \frac{7y_2}{20} - \frac{3y_3}{20} - \frac{11y_4}{40} \\ 0 & 1 & 0 & 0 & | & \frac{y_3}{20} - \frac{y_2}{20} - \frac{y_1}{5} - \frac{3y_4}{40} \\ 0 & 0 & 1 & 0 & | & \frac{3y_1}{5} - \frac{19y_2}{40} - \frac{21y_3}{40} + \frac{43y_4}{80} \\ 0 & 0 & 0 & 1 & | & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} \end{bmatrix} \quad -4 \times \text{row \#2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & \frac{2y_1}{5} - \frac{3y_2}{20} - \frac{7y_3}{20} + \frac{y_4}{40} \\ 0 & 1 & 0 & 0 & | & \frac{y_3}{20} - \frac{y_2}{20} - \frac{y_1}{5} - \frac{3y_4}{40} \\ 0 & 0 & 1 & 0 & | & \frac{3y_1}{5} - \frac{19y_2}{40} - \frac{21y_3}{40} + \frac{43y_4}{80} \\ 0 & 0 & 0 & 1 & | & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} \end{bmatrix}$$

Which tells us

$$\underbrace{\begin{bmatrix} \frac{2}{5} & -\frac{3}{20} & -\frac{7}{20} & \frac{1}{40} \\ -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{bmatrix}}_B \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$



Online Shopping

Looking for the Decoding Matrix (Again)

$$\begin{bmatrix} -1 & -4 & -1 & -3 & | & y_1 \\ 2 & -4 & 0 & 4 & | & y_2 \\ -5 & -3 & -1 & -5 & | & y_3 \\ -2 & -2 & 2 & 2 & | & y_4 \end{bmatrix} \quad \begin{array}{l} 1 \ 0 \ 0 \ 0 \\ 0 \ 1 \ 0 \ 0 \\ 0 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 0 \ 1 \end{array} \quad \text{Divide by } -1$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 2 & -4 & 0 & 4 & | & y_2 \\ -5 & -3 & -1 & -5 & | & y_3 \\ -2 & -2 & 2 & 2 & | & y_4 \end{bmatrix} \quad \begin{array}{l} -2 \times \text{row \#1} \\ +5 \times \text{row \#1} \\ +2 \times \text{row \#1} \end{array}$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 0 & -12 & -2 & -2 & | & 2y_1 + y_2 \\ 0 & 17 & 4 & 10 & | & y_3 - 5y_1 \\ 0 & 6 & 4 & 8 & | & y_4 - 2y_1 \end{bmatrix} \quad \text{Divide by } -12$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & | & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 17 & 4 & 10 & | & y_3 - 5y_1 \\ 0 & 6 & 4 & 8 & | & y_4 - 2y_1 \end{bmatrix} \quad \begin{array}{l} -17 \times \text{row \#2} \\ -6 \times \text{row \#2} \end{array}$$

$$\begin{bmatrix} 1 & 4 & 1 & 3 & | & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & | & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & \frac{7}{6} & \frac{43}{6} & | & \frac{17y_2}{12} - \frac{13y_1}{6} + y_3 \\ 0 & 0 & 3 & 7 & | & \frac{y_2}{2} - y_1 + y_4 \end{bmatrix} \quad \begin{array}{l} -1 \ 0 \ 0 \ 0 \\ -\frac{1}{6} \ -\frac{1}{12} \ 0 \ 0 \\ -\frac{13}{6} \ \frac{17}{12} \ 1 \ 0 \end{array} \quad \text{Divide by } \frac{7}{6}$$



Online Shopping

Looking for the Decoding Matrix (Again)

$$\left[\begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17y_2}{14} - \frac{13y_1}{7} + \frac{6y_3}{7} & -\frac{13}{7} & \frac{17}{14} & \frac{6}{7} & 0 \\ 0 & 0 & 3 & 7 & \frac{y_2}{2} - y_1 + y_4 & -1 & \frac{1}{2} & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ -3 \times \text{row} \#3 \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17y_2}{14} - \frac{13y_1}{7} + \frac{6y_3}{7} & -\frac{13}{7} & \frac{17}{14} & \frac{6}{7} & 0 \\ 0 & 0 & 0 & -\frac{80}{7} & \frac{32y_1}{7} - \frac{22y_2}{7} - \frac{18y_3}{7} + y_4 & \frac{32}{7} & -\frac{22}{7} & -\frac{18}{7} & 1 \end{array} \right] \begin{array}{l} \\ \\ \\ \text{Divide by } -\frac{80}{7} \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17y_2}{14} - \frac{13y_1}{7} + \frac{6y_3}{7} & -\frac{13}{7} & \frac{17}{14} & \frac{6}{7} & 0 \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array} \right] \begin{array}{l} -3 \times \text{row} \#4 \\ -\frac{1}{6} \times \text{row} \#4 \\ -\frac{43}{7} \times \text{row} \#4 \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 4 & 1 & 0 & \frac{y_1}{5} - \frac{33y_2}{40} - \frac{27y_3}{40} + \frac{21y_4}{80} & \frac{1}{5} & -\frac{33}{40} & -\frac{27}{40} & \frac{21}{80} \\ 0 & 1 & \frac{1}{6} & 0 & \frac{7y_4}{480} - \frac{31y_2}{240} - \frac{3y_3}{80} - \frac{y_1}{10} & -\frac{1}{10} & -\frac{31}{240} & -\frac{3}{80} & \frac{7}{480} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} - \frac{19y_2}{40} - \frac{21y_3}{40} + \frac{43y_4}{80} & \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array} \right] \begin{array}{l} -1 \times \text{row} \#3 \\ -\frac{1}{6} \times \text{row} \#3 \\ \end{array}$$



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Looking for the Decoding Matrix (Again)

$$\left[\begin{array}{cccc|cccc} 1 & 4 & 0 & 0 & -\frac{2y_1}{5} - \frac{7y_2}{20} - \frac{3y_3}{20} - \frac{11y_4}{40} & -\frac{2}{5} & -\frac{7}{20} & -\frac{3}{20} & -\frac{11}{40} \\ 0 & 1 & 0 & 0 & \frac{y_3}{20} - \frac{y_2}{20} - \frac{y_1}{5} - \frac{3y_4}{40} & -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} - \frac{19y_2}{40} - \frac{21y_3}{40} + \frac{43y_4}{80} & \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array} \right] \begin{array}{l} \\ -4 \times \text{row} \#2 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{2y_1}{5} - \frac{3y_2}{20} - \frac{7y_3}{20} + \frac{y_4}{40} & \frac{2}{5} & -\frac{3}{20} & -\frac{7}{20} & \frac{1}{40} \\ 0 & 1 & 0 & 0 & \frac{y_3}{20} - \frac{y_2}{20} - \frac{y_1}{5} - \frac{3y_4}{40} & -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} - \frac{19y_2}{40} - \frac{21y_3}{40} + \frac{43y_4}{80} & \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array} \right]$$

Which tells us (again!)

$$\underbrace{\begin{bmatrix} \frac{2}{5} & -\frac{3}{20} & -\frac{7}{20} & \frac{1}{40} \\ -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{bmatrix}}_B \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$



Summarizing Language and Notation : Linear Transformations

Definition (Linear Transformations)

A function $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is called a **linear transformation** if there exists an $(n \times m)$ matrix A such that (MATHSPEAK: "if $\exists A \in \mathbb{R}^{n \times m}$:")

$$T(\vec{x}) = A\vec{x}$$

for all $\vec{x} \in \mathbb{R}^m$. (MATHSPEAK: " $\forall \vec{x} \in \mathbb{R}^m$ ")

A linear transformation is a special type of *function*. The input/argument is a vector (in \mathbb{R}^m), and the output/result is a vector (in \mathbb{R}^n); usually we write " $\vec{y} = A\vec{x}$."



Summarizing Language and Notation : Inverse

Inverse Linear Transformation / Inverse of a Matrix; A^{-1}

If $\vec{y} = A\vec{x}$ is a linear transformation, and $\vec{x} = B\vec{y}$ is the inverse linear transformation; then we say that B is the **inverse** of A , and we write

$$B = A^{-1}.$$

Note: Not all linear transformations have inverses; a linear transformation of the form $A\vec{x}$ has an inverse **if and only if** $\text{rref}(A) = I_n$ (the $(n \times n)$ identity matrix).



Qualifying "Linear" vs. "Affine"

In the realm of Linear Algebra, we have:

Definition (Linear Function)

A **Linear Function** takes the form

$$\vec{y} = A\vec{x},$$

so that $A\vec{0} = \vec{0}$.

Definition (Affine Function)

An **Affine Function** takes the form

$$\vec{y} = A\vec{x} + \vec{b},$$

so that $A\vec{0} + \vec{b} = \vec{b}$.



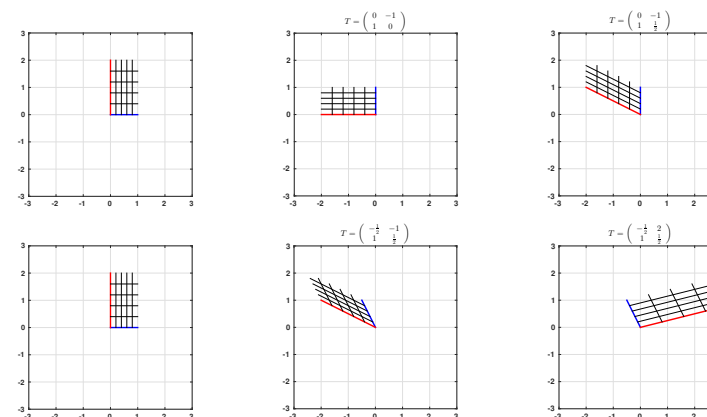
Linear Transformation Examples

2D

See Movies

We consider 4 linear transformations of the vectors

$$\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Linear Transformation Examples

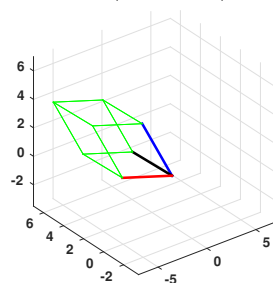
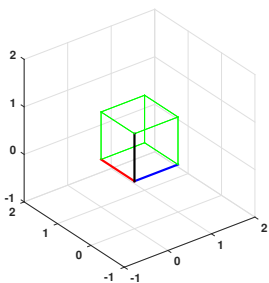
3D

See Movies

We consider a linear transformation of the vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T = \begin{pmatrix} -1 & -2 & -3 \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{pmatrix}$$



Columns of the Matrix of a Linear Transformation

Theorem (The Columns of the Matrix of a Linear Transformation)

Consider a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$. Then, the matrix of T is

$$A = \begin{bmatrix} | & | & \dots & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \\ | & | & \dots & | \end{bmatrix},$$

where $\vec{e}_i \in \mathbb{R}^m$ is the vector of all zeros, except entry $\#i$ which is 1.

Consider

$$A\vec{e}_i = \begin{bmatrix} | & | & \dots & | & \dots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_i & \dots & \vec{v}_n \\ | & | & \dots & | & \dots & | \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{v}_i.$$



Linearity of a Transform

Theorem (Linear Transforms)

A transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is linear *if and only if*

- **Vector Addition** — [Additivity Condition]
 $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}), \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^m, \text{ and}$
- **Scalar Multiplication** — [Homogeneity-of-Degree-1 Condition]
 $T(k\vec{v}) = kT(\vec{v}), \quad \forall \vec{v} \in \mathbb{R}^m, \text{ and } \forall k \in \mathbb{R}.$

This follows directly from the **Algebraic Rules for $A\vec{x}$** :

Theorem (Algebraic Rules for $A\vec{x}$ [NOTES#1.3])

If $A \in \mathbb{R}^{n \times m}$, $\vec{x} \in \mathbb{R}^m$, $\vec{y} \in \mathbb{R}^m$, and $k \in \mathbb{R}$, then

- $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
- $A(k\vec{x}) = k(A\vec{x})$



Lecture – Book Roadmap

Lecture	Book, [GS5-]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3
1.3	§1.1, §1.2, §1.3, §2.1, §2.3
1.4	§1.1–§1.3, §2.1–§2.3
2.1	§8.1, §8.2*, §2.5*

§2.5* (p.86–88) “Calculating A^{-1} by Gauss-Jordan Elimination”

§8.2* We will talk about “Basis” / “Bases” soon... don’t worry about those concepts... yet.



Suggested Problems 2.1

Available on Learning Glass videos:

2.1 — 1, 2, 3, 4, 5, 7, 8, 13, 14, 16, 17, 41, 42



Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		



(2.1.1), (2.1.2)

(2.1.1) Consider the transformation from \mathbb{R}^3 to \mathbb{R}^3 defined by

$$\begin{aligned}y_1 &= 2x_2 \\y_2 &= x_2 + 2, \\y_3 &= 2x_2\end{aligned}$$

is this a **linear** transformation?**(2.1.2)** Consider the transformation from \mathbb{R}^3 to \mathbb{R}^3 defined by

$$\begin{aligned}y_1 &= 2x_2 \\y_2 &= 3x_3, \\y_3 &= x_1\end{aligned}$$

is this a **linear** transformation?

(2.1.3), (2.1.4)

(2.1.3) Consider the transformation from \mathbb{R}^3 to \mathbb{R}^3 defined by

$$\begin{aligned}y_1 &= x_2 - x_3 \\y_2 &= x_1 x_3, \\y_3 &= x_1 - x_2\end{aligned}$$

is this a **linear** transformation?**(2.1.4)** Find the matrix of the linear transformation

$$\begin{aligned}y_1 &= 9x_1 + 3x_2 - 3x_3 \\y_2 &= 2x_1 - 9x_2 + x_3 \\y_3 &= 4x_1 - 9x_2 - 2x_3 \\y_4 &= 5x_1 + x_2 + 5x_3\end{aligned}$$



(2.1.5), (2.1.7)

(2.1.5) Consider the transformation from \mathbb{R}^3 to \mathbb{R}^2 with

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ 11 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 9 \end{bmatrix}, \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -13 \\ 17 \end{bmatrix}.$$

Find the matrix A of $T: \mathbb{R}^3 \mapsto \mathbb{R}^2$.**(2.1.7)** Suppose $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are arbitrary vectors in \mathbb{R}^n . Consider the transformation from \mathbb{R}^m to \mathbb{R}^n given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}\right) = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_m \vec{v}_m.$$

Is this transformation linear? If so, find its matrix A in terms of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$.

(2.1.8), (2.1.13)

(2.1.8) Find the inverse of the linear transformation

$$\begin{aligned}y_1 &= x_1 + 7x_2 \\y_2 &= 3x_1 + 20x_2\end{aligned}$$

(2.1.13) Prove the following facts:

- The (2×2) matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible **if and only if** $(ad - bc) \neq 0$. (Consider the cases $a = 0$, and $a \neq 0$ separately.)
- If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$



(2.1.14)

(2.1.14)

a. For which values of the constant k is the matrix

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$$

invertible?

b. For which values of the constant k are all the entries of

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$$

integers?



(2.1.16)

(2.1.16) Give a geometric interpretation of the linear transformation defined by the matrix

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Show the effect of the transformation on the letter L , described by the two vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Is the transformation invertible? Find the inverse if it exists, and interpret it geometrically.



(2.1.17), (2.1.41)

(2.1.17) Give a geometric interpretation of the linear transformation defined by the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show the effect of the transformation on the letter L , described by the two vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Is the transformation invertible? Find the inverse if it exists, and interpret it geometrically.

(2.1.41) Describe all linear transformations from \mathbb{R}^2 to \mathbb{R} ($= \mathbb{R}^1$). What do their graphs look like?



(2.1.42)

(2.1.42) When you represent a 3-dimensional object in the plane (on paper, whiteboard, learning glass, screen...), you have to transform spatial coordinates $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, into plane coordinates $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. The simplest choice is a linear transformation, for example, the one given by the matrix $\begin{bmatrix} -0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix}$.

a. Use this transformation to represent the unit cube with corner points

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

b. Represent the image of the point $\begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix}$, in your figure in part (a).

c. Find all the points $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, in \mathbb{R}^3 that are transformed into $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Explain.



Two Separate Websites



<https://piazza.com/>

- **Piazza Account**

- Mini-Quizzes
- Access using app, or web interface



<https://Canvas.SDSU.edu/>

- **GRADEBOOK**



Goal: Your Points in the GRADEBOOK!

How do we get there?

- 1 Set up a **Piazza Account**
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 - In Piazza go to Account/Email Settings



Goal: Your Points in the GRADEBOOK!

How do we get there?

- 3 Scores will be transferred after EACH LECTURE
 - The transfer is manual, so there may be a bit of a delay.
- 4 *If something is not right, please let me know!* — Email me!

If you cannot participate in the use of Piazza (due to lack of a suitable device) — let me know, and we'll figure something out.

