Math 254: Introduction to Linear Algebra

Notes #2.1 — Linear Transformations (Intro)

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Outline

SLOs 2.1 — Linear Transformations (Intro)

After this lecture you should:

- Understand how a Matrix[-Vector product] defines a *Linear Transformation*
- Understand the concept of *The Inverse of a Linear Transformation*
- Be able to compute *The Inverse of a (small-ish) Matrix*
- Understand the concept of, and identify *The Matrix of a Linear Transformation*

Online Shopping **Coding**-Decoding

Imagine you are shopping online at `ReallyWeakSecurity.com` which "protects" your 16-digit credit card number (1234 5678 9098 7654) in the following way:

$$\begin{bmatrix} -1 & -4 & -1 & -3 \\ 2 & -4 & 0 & 4 \\ -5 & -3 & -1 & -5 \\ -2 & -2 & 2 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1234 \\ 5678 \\ 9098 \\ 7654 \end{bmatrix}}_{\text{Credit Card Number, } \vec{x}} = \underbrace{\begin{bmatrix} -56006 \\ 10372 \\ -70572 \\ 19680 \end{bmatrix}}_{\text{"Coded" transmission, } \vec{y}}$$

You enter your CC#, the "coded" version gets transmitted to the website...

The operation $\vec{y} = A\vec{x}$ is called a **linear transformation**. The matrix A is the **coefficient matrix** of the transformation.

Online Shopping — Coding-**Decoding**

Your "coded" payment details get processed at `QuestionableOffshorePaymentProcessor.com`:

$$\left[\begin{array}{cccc|c} -1 & -4 & -1 & -3 & -56006 \\ 2 & -4 & 0 & 4 & 10372 \\ -5 & -3 & -1 & -5 & -70572 \\ -2 & -2 & 2 & 2 & 19680 \end{array}\right] \quad \text{Divide by } -1$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 2 & -4 & 0 & 4 & 10372 \\ -5 & -3 & -1 & -5 & -70572 \\ -2 & -2 & 2 & 2 & 19680 \end{array}\right] \quad \begin{array}{l} \\ -2\times\text{row}\#1 \\ +5\times\text{row}\#1 \\ +2\times\text{row}\#1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & -12 & -2 & -2 & -101640 \\ 0 & 17 & 4 & 10 & 209458 \\ 0 & 6 & 4 & 8 & 131692 \end{array}\right] \quad \text{Divide by } -12$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 17 & 4 & 10 & 209458 \\ 0 & 6 & 4 & 8 & 131692 \end{array}\right] \quad \begin{array}{l} \\ \\ -17\times\text{row}\#2 \\ -6\times\text{row}\#2 \end{array}$$

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Online Shopping — Coding-**Decoding**

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & \frac{7}{6} & \frac{43}{6} & 65468 \\ 0 & 0 & 3 & 7 & 80872 \end{array}\right] \quad \text{Divide by } \frac{7}{6}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{392808}{7} \\ 0 & 0 & 3 & 7 & 80872 \end{array}\right] \quad -3\times\text{row}\#3$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{392808}{7} \\ 0 & 0 & 0 & -\frac{80}{7} & -\frac{612320}{7} \end{array}\right] \quad \text{Divide by } -\frac{80}{7}$$

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Online Shopping — Coding-**Decoding**

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{392808}{7} \\ 0 & 0 & 0 & 1 & 7654 \end{array}\right] \quad \begin{array}{l} -3\times\text{row}\#4 \\ -\frac{1}{6}\times\text{row}\#4 \\ -\frac{43}{7}\times\text{row}\#4 \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 0 & 33044 \\ 0 & 1 & \frac{1}{6} & 0 & \frac{21583}{3} \\ 0 & 0 & 1 & 0 & 9098 \\ 0 & 0 & 0 & 1 & 7654 \end{array}\right] \quad \begin{array}{l} -1\times\text{row}\#3 \\ -\frac{1}{6}\times\text{row}\#3 \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & 23946 \\ 0 & 1 & 0 & 0 & 5678 \\ 0 & 0 & 1 & 0 & 9098 \\ 0 & 0 & 0 & 1 & 7654 \end{array}\right] \quad \begin{array}{l} -4\times\text{row}\#2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1234 \\ 0 & 1 & 0 & 0 & 5678 \\ 0 & 0 & 1 & 0 & 9098 \\ 0 & 0 & 0 & 1 & 7654 \end{array}\right]$$

Online Shopping — Smarter Decoding?

Now, if you are "in charge" of decoding by hand, this process will get a little bit old somewhat fast... and you want to find the decoding (inverse) transformation $\vec{y} \mapsto \vec{x}$ once and for all.

Find the Inverse Transformation

Given

$$A\vec{x} = \vec{y}.$$

Find a matrix B so that

$$B\vec{y} = \vec{x}.$$

Online Shopping — Looking for the Decoding Matrix

$$\left[\begin{array}{rrrr|c} -1 & -4 & -1 & -3 & y_1 \\ 2 & -4 & 0 & 4 & y_2 \\ -5 & -3 & -1 & -5 & y_3 \\ -2 & -2 & 2 & 2 & y_4 \end{array}\right] \quad \text{Divide by } -1$$

$$\left[\begin{array}{rrrr|c} 1 & 4 & 1 & 3 & -y_1 \\ 2 & -4 & 0 & 4 & y_2 \\ -5 & -3 & -1 & -5 & y_3 \\ -2 & -2 & 2 & 2 & y_4 \end{array}\right] \quad \begin{array}{l} \\ -2\times\text{row}\#1 \\ +5\times\text{row}\#1 \\ +2\times\text{row}\#1 \end{array}$$

$$\left[\begin{array}{rrrr|r} 1 & 4 & 1 & 3 & -y_1 \\ 0 & -12 & -2 & -2 & 2\,y_1 + y_2 \\ 0 & 17 & 4 & 10 & y_3 - 5\,y_1 \\ 0 & 6 & 4 & 8 & y_4 - 2\,y_1 \end{array}\right] \quad \text{Divide by } -12$$

$$\left[\begin{array}{rrrr|r} 1 & 4 & 1 & 3 & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 17 & 4 & 10 & y_3 - 5\,y_1 \\ 0 & 6 & 4 & 8 & y_4 - 2\,y_1 \end{array}\right] \quad \begin{array}{l} \\ \\ -17\times\text{row}\#2 \\ -6\times\text{row}\#2 \end{array}$$

$$\left[\begin{array}{rrrr|r} 1 & 4 & 1 & 3 & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & \frac{7}{6} & \frac{43}{6} & \frac{17\,y_2}{12} - \frac{13\,y_1}{6} + y_3 \\ 0 & 0 & 3 & 7 & \frac{y_2}{2} - y_1 + y_4 \end{array}\right] \quad \text{Divide by } \frac{7}{6}$$

Online Shopping — Looking for the Decoding Matrix

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17\,y_2}{14} - \frac{13\,y_1}{7} + \frac{6\,y_3}{7} \\ 0 & 0 & 3 & 7 & \frac{y_2}{2} - y_1 + y_4 \end{array}\right] \quad -3\times \text{row}\#3$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17\,y_2}{14} - \frac{13\,y_1}{7} + \frac{6\,y_3}{7} \\ 0 & 0 & 0 & -\frac{80}{7} & \frac{32\,y_1}{7} - \frac{22\,y_2}{7} - \frac{18\,y_3}{7} + y_4 \end{array}\right] \quad \text{Divide by } -\frac{80}{7}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 3 & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17\,y_2}{14} - \frac{13\,y_1}{7} + \frac{6\,y_3}{7} \\ 0 & 0 & 0 & 1 & \frac{11\,y_2}{40} - \frac{2\,y_1}{5} + \frac{9\,y_3}{40} - \frac{7\,y_4}{80} \end{array}\right] \quad \begin{array}{l} -3\times \text{row}\#4 \\ -\frac{1}{6}\times \text{row}\#4 \\ -\frac{43}{7}\times \text{row}\#4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 1 & 0 & \frac{y_1}{5} - \frac{33\,y_2}{40} - \frac{27\,y_3}{40} + \frac{21\,y_4}{80} \\ 0 & 1 & \frac{1}{6} & 0 & \frac{7\,y_4}{480} - \frac{31\,y_2}{240} - \frac{3\,y_3}{80} - \frac{y_1}{10} \\ 0 & 0 & 1 & 0 & \frac{3\,y_1}{5} - \frac{19\,y_2}{40} - \frac{21\,y_3}{40} + \frac{43\,y_4}{80} \\ 0 & 0 & 0 & 1 & \frac{11\,y_2}{40} - \frac{2\,y_1}{5} + \frac{9\,y_3}{40} - \frac{7\,y_4}{80} \end{array}\right] \quad \begin{array}{l} -1\times \text{row}\#3 \\ -\frac{1}{6}\times \text{row}\#3 \end{array}$$

Online Shopping — Looking for the Decoding Matrix

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & 0 & -\frac{2y_1}{5}-\frac{7y_2}{20}-\frac{3y_3}{20}-\frac{11y_4}{40} \\ 0 & 1 & 0 & 0 & \frac{y_3}{20}-\frac{y_2}{20}-\frac{y_1}{5}-\frac{3y_4}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5}-\frac{19y_2}{40}-\frac{21y_3}{40}+\frac{43y_4}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40}-\frac{2y_1}{5}+\frac{9y_3}{40}-\frac{7y_4}{80} \end{array}\right] \quad -4\times\text{row}\#2$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{2y_1}{5}-\frac{3y_2}{20}-\frac{7y_3}{20}+\frac{y_4}{40} \\ 0 & 1 & 0 & 0 & \frac{y_3}{20}-\frac{y_2}{20}-\frac{y_1}{5}-\frac{3y_4}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5}-\frac{19y_2}{40}-\frac{21y_3}{40}+\frac{43y_4}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40}-\frac{2y_1}{5}+\frac{9y_3}{40}-\frac{7y_4}{80} \end{array}\right]$$

Which tells us

$$\underbrace{\begin{bmatrix} \frac{2}{5} & -\frac{3}{20} & -\frac{7}{20} & \frac{1}{40} \\ -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{bmatrix}}_{B} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Online Shopping — Looking for the Decoding Matrix (Again)

$$\left[\begin{array}{cccc|c|cccc} -1 & -4 & -1 & -3 & y_1 & 1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 4 & y_2 & 0 & 1 & 0 & 0 \\ -5 & -3 & -1 & -5 & y_3 & 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 2 & y_4 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{Divide by } -1$$

$$\left[\begin{array}{cccc|c|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 4 & y_2 & 0 & 1 & 0 & 0 \\ -5 & -3 & -1 & -5 & y_3 & 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 2 & y_4 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2\times \mathrm{row}\#1 \\ +5\times \mathrm{row}\#1 \\ +2\times \mathrm{row}\#1 \end{array}$$

$$\left[\begin{array}{cccc|c|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & -12 & -2 & -2 & 2\,y_1 + y_2 & 2 & 1 & 0 & 0 \\ 0 & 17 & 4 & 10 & y_3 - 5\,y_1 & -5 & 0 & 1 & 0 \\ 0 & 6 & 4 & 8 & y_4 - 2\,y_1 & -2 & 0 & 0 & 1 \end{array}\right] \quad \text{Divide by } -12$$

$$\left[\begin{array}{cccc|c|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 17 & 4 & 10 & y_3 - 5\,y_1 & -5 & 0 & 1 & 0 \\ 0 & 6 & 4 & 8 & y_4 - 2\,y_1 & -2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ -17\times \mathrm{row}\#2 \\ -6\times \mathrm{row}\#2 \end{array}$$

$$\left[\begin{array}{cccc|c|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{7}{6} & \frac{43}{6} & \frac{17\,y_2}{12} - \frac{13\,y_1}{6} + y_3 & -\frac{13}{6} & \frac{17}{12} & 1 & 0 \\ 0 & 0 & 3 & 7 & \frac{y_2}{2} - y_1 + y_4 & -1 & \frac{1}{2} & 0 & 1 \end{array}\right] \quad \text{Divide by } \frac{7}{6}$$

Online Shopping — Looking for the Decoding Matrix (Again)

$$\left[\begin{array}{cccc|c|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17\,y_2}{14} - \frac{13\,y_1}{7} + \frac{6y_3}{7} & -\frac{13}{7} & \frac{17}{14} & \frac{6}{7} & 0 \\ 0 & 0 & 3 & 7 & \frac{y_2}{2} - y_1 + y_4 & -1 & \frac{1}{2} & 0 & 1 \end{array}\right] \quad -3\times \text{row}\#3$$

$$\left[\begin{array}{cccc|c|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17\,y_2}{14} - \frac{13\,y_1}{7} + \frac{6y_3}{7} & -\frac{13}{7} & \frac{17}{14} & \frac{6}{7} & 0 \\ 0 & 0 & 0 & -\frac{80}{7} & \frac{32\,y_1}{7} - \frac{22\,y_2}{7} - \frac{18\,y_3}{7} + y_4 & \frac{32}{7} & -\frac{22}{7} & -\frac{18}{7} & 1 \end{array}\right] \quad \text{Divide by } -\frac{80}{7}$$

$$\left[\begin{array}{cccc|c|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17\,y_2}{14} - \frac{13\,y_1}{7} + \frac{6y_3}{7} & -\frac{13}{7} & \frac{17}{14} & \frac{6}{7} & 0 \\ 0 & 0 & 0 & 1 & \frac{11\,y_2}{40} - \frac{2\,y_1}{5} + \frac{9\,y_3}{40} - \frac{7\,y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array}\right] \quad \begin{array}{l} -3\times\text{row}\#4 \\ -\frac{1}{6}\times\text{row}\#4 \\ -\frac{43}{7}\times\text{row}\#4 \\ \end{array}$$

$$\left[\begin{array}{cccc|c|cccc} 1 & 4 & 1 & 0 & \frac{y_1}{5} - \frac{33\,y_2}{40} - \frac{27\,y_3}{40} + \frac{21\,y_4}{80} & \frac{1}{5} & -\frac{33}{40} & -\frac{27}{40} & \frac{21}{80} \\ 0 & 1 & \frac{1}{6} & 0 & \frac{7\,y_4}{480} - \frac{31\,y_2}{240} - \frac{3\,y_3}{80} - \frac{y_1}{10} & -\frac{1}{10} & -\frac{31}{240} & -\frac{3}{80} & \frac{7}{480} \\ 0 & 0 & 1 & 0 & \frac{3\,y_1}{5} - \frac{19\,y_2}{40} - \frac{21\,y_3}{40} + \frac{43\,y_4}{80} & \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ 0 & 0 & 0 & 1 & \frac{11\,y_2}{40} - \frac{2\,y_1}{5} + \frac{9\,y_3}{40} - \frac{7\,y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array}\right] \quad \begin{array}{l} -1\times\text{row}\#3 \\ -\frac{1}{6}\times\text{row}\#3 \\ \\ \end{array}$$

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Online Shopping

Looking for the Decoding Matrix (Again)

$$\left[\begin{array}{cccc|c|cccc} 1 & 4 & 0 & 0 & -\frac{2y_1}{5} - \frac{7y_2}{20} - \frac{3y_3}{20} - \frac{11y_4}{40} & -\frac{2}{5} & -\frac{7}{20} & -\frac{3}{20} & -\frac{11}{40} \\ 0 & 1 & 0 & 0 & \frac{y_3}{20} - \frac{y_2}{20} - \frac{y_1}{5} - \frac{3y_4}{40} & -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} - \frac{19y_2}{40} - \frac{21y_3}{40} + \frac{43y_4}{80} & \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array}\right] \quad -4\times\text{row}\#2$$

$$\left[\begin{array}{cccc|c|cccc} 1 & 0 & 0 & 0 & \frac{2y_1}{5} - \frac{3y_2}{20} - \frac{7y_3}{20} + \frac{y_4}{40} & \frac{2}{5} & -\frac{3}{20} & -\frac{7}{20} & \frac{1}{40} \\ 0 & 1 & 0 & 0 & \frac{y_3}{20} - \frac{y_2}{20} - \frac{y_1}{5} - \frac{3y_4}{40} & -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} - \frac{19y_2}{40} - \frac{21y_3}{40} + \frac{43y_4}{80} & \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array}\right]$$

Which tells us (again!)

$$\underbrace{\begin{bmatrix} \frac{2}{5} & -\frac{3}{20} & -\frac{7}{20} & \frac{1}{40} \\ -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{bmatrix}}_{B} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Summarizing Language and Notation : Linear Transformations

Definition (Linear Transformations)

A function $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is called a **linear transformation** if there exists an $(n \times m)$ matrix A such that (MathSpeak: "if $\exists A \in \mathbb{R}^{n\times m}$:")

$$T(\vec{x}) = A\vec{x}$$

for all $\vec{x} \in \mathbb{R}^m$. (MathSpeak: "$\forall \vec{x} \in \mathbb{R}^m$")

A linear transformation is a special type of *function*. The input/argument is a vector (in $\mathbb{R}^m$), and the output/result is a vector (in $\mathbb{R}^n$); usually we write "$\vec{y} = A\vec{x}$."

Summarizing Language and Notation : Inverse

Inverse Linear Transformation / Inverse of a Matrix; A^{-1}

If $\vec{y} = A\vec{x}$ is a linear transformation, and $\vec{x} = B\vec{y}$ is the inverse linear transformation; then we say that B is the **inverse** of A, and we write

$$B = A^{-1}.$$

Note: Not all linear transformations have inverses; a linear transformation of the form $A\vec{x}$ has an inverse if and only if $\mathrm{rref}(A) = I_n$ (the $(n \times n)$ identity matrix).

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Qualifying "Linear" vs. "Affine"

In the realm of Linear Algebra, we have:

Definition (Linear Function)

A Linear Function takes the form

$$\vec{y} = A\vec{x},$$

so that $A\vec{0} = \vec{0}$.

Definition (Affine Function)

An Affine Function takes the form

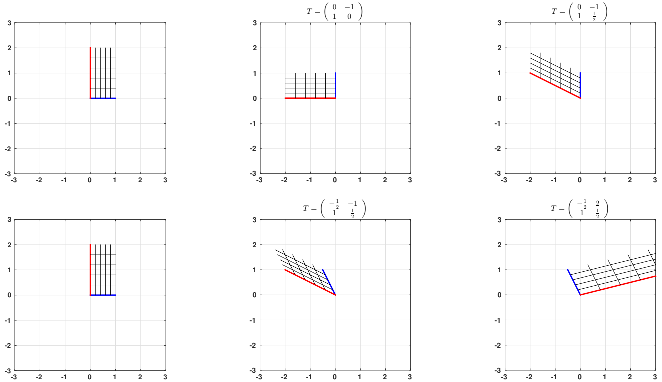
$$\vec{y} = A\vec{x} + \vec{b},$$

so that $A\vec{0} + \vec{b} = \vec{b}$.

Linear Transformation Examples 2D See Movies

We consider 4 linear transformations of the vectors

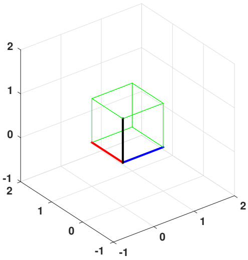
$$\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



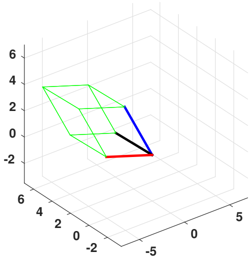
Linear Transformation Examples 3D See Movies

We consider a linear transformation of the vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$$T = \begin{pmatrix} -1 & -2 & -3 \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{pmatrix}$$



Columns of the Matrix of a Linear Transformation

Theorem (The Columns of the Matrix of a Linear Transformation)

Consider a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$. Then, the matrix of T is

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \\ | & | & & | \end{bmatrix},$$

where $\vec{e}_i \in \mathbb{R}^m$ is the vector of all zeros, except entry $\#i$ which is 1.

Consider

$$A\vec{e}_i = \begin{bmatrix} | & | & & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_i & \dots & \vec{v}_n \\ | & | & & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{v}_i.$$

Linearity of a Transform

Theorem (Linear Transforms)

A transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ *is linear if and only if*

- **Vector Addition —** *[Additivity Condition]*
  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}), \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^m$, *and*
- **Scalar Multiplication —** *[Homogeneity-of-Degree-1 Condition]*
  $T(k\vec{v}) = kT(\vec{v}), \quad \forall \vec{v} \in \mathbb{R}^m$, *and* $\forall k \in \mathbb{R}$.

This follows directly from the **Algebraic Rules for** $A\vec{x}$:

Theorem (Algebraic Rules for $A\vec{x}$ [Notes#1.3])

If $A \in \mathbb{R}^{n \times m}$, $\vec{x} \in \mathbb{R}^m$, $\vec{y} \in \mathbb{R}^m$, *and* $k \in \mathbb{R}$, *then*

- $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
- $A(k\vec{x}) = k(A\vec{x})$

Suggested Problems 2.1

Available on Learning Glass videos:
2.1 — 1, 2, 3, 4, 5, 7, 8, 13, 14, 16, 17, 41, 42

San Diego State University

Lecture – Book Roadmap

Lecture	Book, [GS5–]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3
1.3	§1.1, §1.2, §1.3, §2.1, §2.3
1.4	§1.1–§1.3, §2.1–§2.3
2.1	§8.1, §8.2*, §2.5*

§2.5* (p.86–88) "Calculating A^{-1} by Gauss-Jordan Elimination"

§8.2* We will talk about "Basis" / "Bases" soon... don't worry about those concepts... yet.

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(2.1.1), (2.1.2)

(2.1.1) Consider the transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$\begin{aligned} y_1 &= 2x_2 \\ y_2 &= x_2 + 2 \\ y_3 &= 2x_2 \end{aligned},$$

is this a **linear** transformation?

(2.1.2) Consider the transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$\begin{aligned} y_1 &= 2x_2 \\ y_2 &= 3x_3 \\ y_3 &= x_1 \end{aligned},$$

is this a **linear** transformation?

(2.1.3), (2.1.4)

(2.1.3) Consider the transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$\begin{array}{rcl} y_1 &=& x_2 - x_3 \\ y_2 &=& x_1 x_3 \\ y_3 &=& x_1 - x_2 \end{array},$$

is this a **linear** transformation?

(2.1.4) Find the matrix of the linear transformation

$$\begin{array}{rcl} y_1 &=& 9x_1 + 3x_2 - 3x_3 \\ y_2 &=& 2x_1 - 9x_2 + x_3 \\ y_3 &=& 4x_1 - 9x_2 - 2x_3 \\ y_4 &=& 5x_1 + x_2 + 5x_3 \end{array}.$$

(2.1.5), (2.1.7)

(2.1.5) Consider the transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ with

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}7\\11\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}6\\9\end{bmatrix}, \quad \text{and } T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-13\\17\end{bmatrix}.$$

Find the matrix A of $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$.

(2.1.7) Suppose $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_m$ are arbitrary vectors in $\mathbb{R}^n$. Consider the transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ given by

$$T\left(\begin{bmatrix}x_1\\x_2\\\vdots\\x_m\end{bmatrix}\right) = x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_m\vec{v}_m.$$

Is this transformation linear? If so, find its matrix A in terms of the vectors $\vec{v}_1$, $\vec{v}_2$, $\dots$, $\vec{v}_m$.

(2.1.8), (2.1.13)

(2.1.8) Find the inverse of the linear transformation

$$\begin{aligned} y_1 &= x_1 + 7x_2 \\ y_2 &= 3x_1 + 20x_2 \end{aligned}$$

(2.1.13) Prove the following facts:

a. The (2×2) matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $(ad - bc) \neq 0$. (Consider the cases $a = 0$, and $a \neq 0$ separately.)

b. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(2.1.14)

(2.1.14)

a. For which values of the constant k is the matrix

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$$

invertible?

b. For which values of the constant k are all the entries of

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$$

integers?

(2.1.16)

(2.1.16) Give a geometric interpretation of the linear transformation defined by the matrix

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Show the effect of the transformation on the letter L, described by the two vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Is the transformation invertible? Find the inverse if it exists, and interpret it geometrically.

(2.1.17), (2.1.41)

(2.1.17) Give a geometric interpretation of the linear transformation defined by the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show the effect of the transformation on the letter L, described by the two vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Is the transformation invertible? Find the inverse if it exists, and interpret it geometrically.

(2.1.41) Describe all linear transformations from $\mathbb{R}^2$ to $\mathbb{R}$ $(= \mathbb{R}^1)$. What do their graphs look like?

(2.1.42)

(2.1.42) When you represent a 3-dimensional object in the plane (on paper, whiteboard, learning glass, screen...), you have to transform spatial coordinates $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, into plane coordinates $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. The simplest choice is a linear transformation, for example, the one given by the matrix $\begin{bmatrix} -0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix}$.

a. Use this transformation to represent the unit cube with corner points

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

b. Represent the image of the point $\begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix}$, in your figure in part (a).

c. Find all the points $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, in $\mathbb{R}^3$ that are transformed into $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Explain.

Piazza Setup and Use, 1 of 3 — Does not apply Fall 2021 — [TECH::SETUP]

Two Separate Websites

https://piazza.com/

- **Piazza Account**
  - Mini-Quizzes
  - Access using app, or web interface

https://Canvas.SDSU.edu/

- **GRADEBOOK**

Piazza Setup and Use, 2 of 3 Does not apply Fall 2021 [TECH::SETUP]

Goal: Your Points in the GRADEBOOK!

How do we get there?

1. Set up a **Piazza Account**
   - You should have gotten an inviatation sent to your `[...]@sdsu.edu` email address; or go to
   - `https://piazza.com/sdsu/fall2021/math254blomgren`
   - It is FREE
   - Download the app from the iOS App Store, or Google Play
2. **If you want credit:** You can register multiple email addresses to a piazza account — one of them must match your email address in Canvas.
   - In Piazza go to `Account/Email Settings`

Piazza Setup and Use, 3 of 3 Does not apply Fall 2021 [TECH::SETUP]

Goal: Your Points in the GRADEBOOK!

How do we get there?

3. Scores will be transferred after EACH LECTURE
   - The transfer is manual, so there may be a bit of a delay.
4. *If something is not right, please let me know!* — Email me!

If you cannot participate in the use of Piazza (due to lack of a suitable device) — let me know, and we'll figure something out.