

# Math 254: Introduction to Linear Algebra

## Notes #2.1 — Linear Transformations (Intro)

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Spring 2022

(Revised: January 18, 2022)



# Outline

- 1 Student Learning Objectives
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  - Problem Statements 2.1

## SLOs 2.1

## Linear Transformations (Intro)

After this lecture you should:

- Understand how a Matrix[-Vector product] defines a *Linear Transformation*
- Understand the concept of *The Inverse of a Linear Transformation*
- Be able to compute *The Inverse of a (small-ish) Matrix*
- Understand the concept of, and identify *The Matrix of a Linear Transformation*



Imagine you are shopping online at ReallyWeakSecurity.com which “protects” your 16-digit credit card number (1234 5678 9098 7654) in the following way:

$$\begin{bmatrix} -1 & -4 & -1 & -3 \\ 2 & -4 & 0 & 4 \\ -5 & -3 & -1 & -5 \\ -2 & -2 & 2 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1234 \\ 5678 \\ 9098 \\ 7654 \end{bmatrix}}_{\text{Credit Card Number, } \vec{x}} = \underbrace{\begin{bmatrix} -56006 \\ 10372 \\ -70572 \\ 19680 \end{bmatrix}}_{\text{"Coded" transmission, } \vec{y}}$$

You enter your CC#, the “coded” version gets transmitted to the website...

The operation  $\vec{y} = A\vec{x}$  is called a **linear transformation**. The matrix  $A$  is the **coefficient matrix** of the transformation.



# Online Shopping

# Coding-Decoding

Your "coded" payment details get processed at  
QuestionableOffshorePaymentProcessor.com:

$$\left[ \begin{array}{cccc|c} -1 & -4 & -1 & -3 & -56006 \\ 2 & -4 & 0 & 4 & 10372 \\ -5 & -3 & -1 & -5 & -70572 \\ -2 & -2 & 2 & 2 & 19680 \end{array} \right] \quad \text{Divide by } -1$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 2 & -4 & 0 & 4 & 10372 \\ -5 & -3 & -1 & -5 & -70572 \\ -2 & -2 & 2 & 2 & 19680 \end{array} \right] \begin{array}{l} \leftarrow -2 \times \text{row\#1} \\ \leftarrow +5 \times \text{row\#1} \\ \leftarrow +2 \times \text{row\#1} \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & -12 & -2 & -2 & -101640 \\ 0 & 17 & 4 & 10 & 209458 \\ 0 & 6 & 4 & 8 & 131692 \end{array} \right] \quad \text{Divide by } -12$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 17 & 4 & 10 & 209458 \\ 0 & 6 & 4 & 8 & 131692 \end{array} \right] \begin{array}{l} \leftarrow -17 \times \text{row\#2} \\ \leftarrow -6 \times \text{row\#2} \end{array}$$



$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & \frac{7}{6} & \frac{43}{6} & 65468 \\ 0 & 0 & 3 & 7 & 80872 \end{array} \right] \quad \text{Divide by } \frac{7}{6}$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{392808}{7} \\ 0 & 0 & 3 & 7 & 80872 \end{array} \right] \quad \begin{array}{l} \leftarrow -3 \times \text{row\#3} \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{392808}{7} \\ 0 & 0 & 0 & -\frac{80}{7} & -\frac{612320}{7} \end{array} \right] \quad \text{Divide by } -\frac{80}{7}$$

# Online Shopping

# Coding-Decoding

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & 56006 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & 8470 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{392808}{7} \\ 0 & 0 & 0 & 1 & 7654 \end{array} \right] \begin{array}{l} \leftarrow -3 \times \text{row\#4} \\ \leftarrow -\frac{1}{6} \times \text{row\#4} \\ \leftarrow -\frac{43}{7} \times \text{row\#4} \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 0 & 33044 \\ 0 & 1 & \frac{1}{6} & 0 & \frac{21583}{3} \\ 0 & 0 & 1 & 0 & 9098 \\ 0 & 0 & 0 & 1 & 7654 \end{array} \right] \begin{array}{l} \leftarrow -1 \times \text{row\#3} \\ \leftarrow -\frac{1}{6} \times \text{row\#3} \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 0 & 0 & 23946 \\ 0 & 1 & 0 & 0 & 5678 \\ 0 & 0 & 1 & 0 & 9098 \\ 0 & 0 & 0 & 1 & 7654 \end{array} \right] \leftarrow -4 \times \text{row\#2}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1234 \\ 0 & 1 & 0 & 0 & 5678 \\ 0 & 0 & 1 & 0 & 9098 \\ 0 & 0 & 0 & 1 & 7654 \end{array} \right]$$

Now, if you are “in charge” of decoding by hand, this process will get a little bit old somewhat fast... and you want to find the decoding (inverse) transformation  $\vec{y} \mapsto \vec{x}$  once and for all.

### Find the Inverse Transformation

Given

$$A\vec{x} = \vec{y}.$$

Find a matrix  $B$  so that

$$B\vec{y} = \vec{x}.$$



# Online Shopping

# Looking for the Decoding Matrix

$$\left[ \begin{array}{cccc|c} -1 & -4 & -1 & -3 & y_1 \\ 2 & -4 & 0 & 4 & y_2 \\ -5 & -3 & -1 & -5 & y_3 \\ -2 & -2 & 2 & 2 & y_4 \end{array} \right] \quad \text{Divide by } -1$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & -y_1 \\ 2 & -4 & 0 & 4 & y_2 \\ -5 & -3 & -1 & -5 & y_3 \\ -2 & -2 & 2 & 2 & y_4 \end{array} \right] \begin{array}{l} \left. \begin{array}{l} \leftarrow -2 \times \text{row\#1} \\ \leftarrow +5 \times \text{row\#1} \\ \leftarrow +2 \times \text{row\#1} \end{array} \right\} \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & -y_1 \\ 0 & -12 & -2 & -2 & 2y_1 + y_2 \\ 0 & 17 & 4 & 10 & y_3 - 5y_1 \\ 0 & 6 & 4 & 8 & y_4 - 2y_1 \end{array} \right] \quad \text{Divide by } -12$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 17 & 4 & 10 & y_3 - 5y_1 \\ 0 & 6 & 4 & 8 & y_4 - 2y_1 \end{array} \right] \begin{array}{l} \left. \begin{array}{l} \leftarrow -17 \times \text{row\#2} \\ \leftarrow -6 \times \text{row\#2} \end{array} \right\} \end{array}$$

$$\left[ \begin{array}{cccc|c} 1 & 4 & 1 & 3 & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} \\ 0 & 0 & \frac{7}{6} & \frac{43}{6} & \frac{17y_2}{12} - \frac{13y_1}{6} + y_3 \\ 0 & 0 & 3 & 7 & \frac{y_2}{2} - y_1 + y_4 \end{array} \right] \quad \text{Divide by } \frac{7}{6}$$

# Online Shopping

# Looking for the Decoding Matrix

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & & & & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & & -\frac{y_1}{6} & -\frac{y_2}{12} & \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17y_2}{14} & -\frac{13y_1}{7} & +\frac{6y_3}{7} & \\ 0 & 0 & 3 & 7 & \frac{y_2}{2} & -y_1 & +y_4 & \end{array} \right] \begin{array}{l} \\ \\ \text{red bracket} \\ \end{array} \quad -3 \times \text{row\#3}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & & & & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & & -\frac{y_1}{6} & -\frac{y_2}{12} & \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17y_2}{14} & -\frac{13y_1}{7} & +\frac{6y_3}{7} & \\ 0 & 0 & 0 & -\frac{80}{7} & \frac{32y_1}{7} & -\frac{22y_2}{7} & -\frac{18y_3}{7} & +y_4 \end{array} \right] \quad \text{Divide by } -\frac{80}{7}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & & & & -y_1 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & & -\frac{y_1}{6} & -\frac{y_2}{12} & \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17y_2}{14} & -\frac{13y_1}{7} & +\frac{6y_3}{7} & \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} & -\frac{2y_1}{5} & +\frac{9y_3}{40} & -\frac{7y_4}{80} \end{array} \right] \begin{array}{l} \text{blue bracket} \\ \text{blue bracket} \\ \text{blue bracket} \\ \end{array} \quad \begin{array}{l} -3 \times \text{row\#4} \\ -\frac{1}{6} \times \text{row\#4} \\ -\frac{43}{7} \times \text{row\#4} \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 0 & \frac{y_1}{5} & -\frac{33y_2}{40} & -\frac{27y_3}{40} & +\frac{21y_4}{80} \\ 0 & 1 & \frac{1}{6} & 0 & \frac{7y_4}{480} & -\frac{31y_2}{240} & -\frac{3y_3}{80} & -\frac{y_1}{10} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} & -\frac{19y_2}{40} & -\frac{21y_3}{40} & +\frac{43y_4}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} & -\frac{2y_1}{5} & +\frac{9y_3}{40} & -\frac{7y_4}{80} \end{array} \right] \begin{array}{l} \text{blue bracket} \\ \text{blue bracket} \\ \end{array} \quad \begin{array}{l} -1 \times \text{row\#3} \\ -\frac{1}{6} \times \text{row\#3} \end{array}$$

## Online Shopping

## Looking for the Decoding Matrix

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 0 & 0 & -\frac{2y_1}{5} & -\frac{7y_2}{20} & -\frac{3y_3}{20} & -\frac{11y_4}{40} \\ 0 & 1 & 0 & 0 & & \frac{y_3}{20} & -\frac{y_2}{20} & -\frac{y_1}{5} & -\frac{3y_4}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} & -\frac{19y_2}{40} & -\frac{21y_3}{40} & +\frac{43y_4}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} & -\frac{2y_1}{5} & +\frac{9y_3}{40} & -\frac{7y_4}{80} \end{array} \right] \begin{array}{l} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \begin{array}{l} -4 \times \text{row} \#2 \\ \\ \\ \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{2y_1}{5} & -\frac{3y_2}{20} & -\frac{7y_3}{20} & +\frac{y_4}{40} \\ 0 & 1 & 0 & 0 & & \frac{y_3}{20} & -\frac{y_2}{20} & -\frac{y_1}{5} & -\frac{3y_4}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} & -\frac{19y_2}{40} & -\frac{21y_3}{40} & +\frac{43y_4}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} & -\frac{2y_1}{5} & +\frac{9y_3}{40} & -\frac{7y_4}{80} \end{array} \right]$$

Which tells us

$$\underbrace{\begin{bmatrix} \frac{2}{5} & -\frac{3}{20} & -\frac{7}{20} & \frac{1}{40} \\ -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{bmatrix}}_B \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

# Online Shopping

# Looking for the Decoding Matrix (Again)

$$\left[ \begin{array}{cccc|cccc} -1 & -4 & -1 & -3 & y_1 & 1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 4 & y_2 & 0 & 1 & 0 & 0 \\ -5 & -3 & -1 & -5 & y_3 & 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 2 & y_4 & 0 & 0 & 0 & 1 \end{array} \right] \quad \text{Divide by } -1$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 2 & -4 & 0 & 4 & y_2 & 0 & 1 & 0 & 0 \\ -5 & -3 & -1 & -5 & y_3 & 0 & 0 & 1 & 0 \\ -2 & -2 & 2 & 2 & y_4 & 0 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \leftarrow -2 \times \text{row\#1} \\ \leftarrow +5 \times \text{row\#1} \\ \leftarrow +2 \times \text{row\#1} \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & -12 & -2 & -2 & 2y_1 + y_2 & 2 & 1 & 0 & 0 \\ 0 & 17 & 4 & 10 & y_3 - 5y_1 & -5 & 0 & 1 & 0 \\ 0 & 6 & 4 & 8 & y_4 - 2y_1 & -2 & 0 & 0 & 1 \end{array} \right] \quad \text{Divide by } -12$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 17 & 4 & 10 & y_3 - 5y_1 & -5 & 0 & 1 & 0 \\ 0 & 6 & 4 & 8 & y_4 - 2y_1 & -2 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} \leftarrow -17 \times \text{row\#2} \\ \leftarrow -6 \times \text{row\#2} \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & \frac{7}{6} & \frac{43}{6} & \frac{17y_2}{12} - \frac{13y_1}{6} + y_3 & -\frac{1}{6} & \frac{17}{12} & 1 & 0 \\ 0 & 0 & 3 & 7 & \frac{y_2}{2} - y_1 + y_4 & -1 & \frac{1}{2} & 0 & 1 \end{array} \right] \quad \text{Divide by } \frac{7}{6}$$

# Online Shopping

# Looking for the Decoding Matrix (Again)

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17y_2}{14} - \frac{13y_1}{7} + \frac{6y_3}{7} & -\frac{13}{7} & \frac{17}{14} & \frac{6}{7} & 0 \\ 0 & 0 & 3 & 7 & \frac{y_2}{2} - y_1 + y_4 & -1 & \frac{1}{2} & 0 & 1 \end{array} \right] \begin{array}{l} \\ \\ \text{red arrow} \\ \text{red arrow} \end{array} \quad -3 \times \text{row} \# 3$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17y_2}{14} - \frac{13y_1}{7} + \frac{6y_3}{7} & -\frac{13}{7} & \frac{17}{14} & \frac{6}{7} & 0 \\ 0 & 0 & 0 & -\frac{80}{7} & \frac{32y_1}{7} - \frac{22y_2}{7} - \frac{18y_3}{7} + y_4 & \frac{32}{7} & -\frac{22}{7} & -\frac{18}{7} & 1 \end{array} \right] \quad \text{Divide by } -\frac{80}{7}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 3 & -y_1 & -1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} & -\frac{y_1}{6} - \frac{y_2}{12} & -\frac{1}{6} & -\frac{1}{12} & 0 & 0 \\ 0 & 0 & 1 & \frac{43}{7} & \frac{17y_2}{14} - \frac{13y_1}{7} + \frac{6y_3}{7} & -\frac{13}{7} & \frac{17}{14} & \frac{6}{7} & 0 \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array} \right] \begin{array}{l} \text{blue arrow} \\ \text{blue arrow} \\ \text{blue arrow} \\ \text{blue arrow} \end{array} \quad \begin{array}{l} -3 \times \text{row} \# 4 \\ -\frac{1}{6} \times \text{row} \# 4 \\ -\frac{43}{7} \times \text{row} \# 4 \\ \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 1 & 0 & \frac{y_1}{5} - \frac{33y_2}{40} - \frac{27y_3}{40} + \frac{21y_4}{80} & \frac{1}{5} & -\frac{33}{40} & -\frac{27}{40} & \frac{21}{80} \\ 0 & 1 & \frac{1}{6} & 0 & \frac{7y_4}{480} - \frac{31y_2}{240} - \frac{3y_3}{80} - \frac{y_1}{10} & -\frac{1}{10} & -\frac{31}{240} & -\frac{3}{80} & \frac{7}{480} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} - \frac{19y_2}{40} - \frac{21y_3}{40} + \frac{43y_4}{80} & \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} - \frac{2y_1}{5} + \frac{9y_3}{40} - \frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array} \right] \begin{array}{l} \text{blue arrow} \\ \text{blue arrow} \\ \text{blue arrow} \end{array} \quad \begin{array}{l} -1 \times \text{row} \# 3 \\ -\frac{1}{6} \times \text{row} \# 3 \\ \end{array}$$

## Online Shopping

## Looking for the Decoding Matrix (Again)

$$\left[ \begin{array}{cccc|cccc} 1 & 4 & 0 & 0 & -\frac{2y_1}{5} & -\frac{7y_2}{20} & -\frac{3y_3}{20} & -\frac{11y_4}{40} & -\frac{2}{5} & -\frac{7}{20} & -\frac{3}{20} & -\frac{11}{40} \\ 0 & 1 & 0 & 0 & \frac{y_3}{20} & -\frac{y_2}{20} & -\frac{y_1}{5} & -\frac{3y_4}{40} & -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} & -\frac{19y_2}{40} & -\frac{21y_3}{40} & +\frac{43y_4}{80} & \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} & -\frac{2y_1}{5} & +\frac{9y_3}{40} & -\frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array} \right] \begin{array}{l} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \begin{array}{l} \\ \\ \\ -4 \times \text{row} \#2 \end{array}$$

$$\left[ \begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{2y_1}{5} & -\frac{3y_2}{20} & -\frac{7y_3}{20} & +\frac{y_4}{40} & \frac{2}{5} & -\frac{3}{20} & -\frac{7}{20} & \frac{1}{40} \\ 0 & 1 & 0 & 0 & \frac{y_3}{20} & -\frac{y_2}{20} & -\frac{y_1}{5} & -\frac{3y_4}{40} & -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ 0 & 0 & 1 & 0 & \frac{3y_1}{5} & -\frac{19y_2}{40} & -\frac{21y_3}{40} & +\frac{43y_4}{80} & \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ 0 & 0 & 0 & 1 & \frac{11y_2}{40} & -\frac{2y_1}{5} & +\frac{9y_3}{40} & -\frac{7y_4}{80} & -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{array} \right]$$

Which tells us (again!)

$$\underbrace{\begin{bmatrix} \frac{2}{5} & -\frac{3}{20} & -\frac{7}{20} & \frac{1}{40} \\ -\frac{1}{5} & -\frac{1}{20} & \frac{1}{20} & -\frac{3}{40} \\ \frac{3}{5} & -\frac{19}{40} & -\frac{21}{40} & \frac{43}{80} \\ -\frac{2}{5} & \frac{11}{40} & \frac{9}{40} & -\frac{7}{80} \end{bmatrix}}_B \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

## Summarizing Language and Notation : Linear Transformations

## Definition (Linear Transformations)

A function  $T : \mathbb{R}^m \mapsto \mathbb{R}^n$  is called a **linear transformation** if there exists an  $(n \times m)$  matrix  $A$  such that (MATHSPEAK: “if  $\exists A \in \mathbb{R}^{n \times m}$  :” )

$$T(\vec{x}) = A\vec{x}$$

for all  $\vec{x} \in \mathbb{R}^m$ . (MATHSPEAK: “ $\forall \vec{x} \in \mathbb{R}^m$ ”)

A linear transformation is a special type of *function*. The input/argument is a vector (in  $\mathbb{R}^m$ ), and the output/result is a vector (in  $\mathbb{R}^n$ ); usually we write “ $\vec{y} = A\vec{x}$ .”

## Summarizing Language and Notation : Inverse

Inverse Linear Transformation / Inverse of a Matrix;  $A^{-1}$ 

If  $\vec{y} = A\vec{x}$  is a linear transformation, and  $\vec{x} = B\vec{y}$  is the inverse linear transformation; then we say that  $B$  is the **inverse** of  $A$ , and we write

$$B = A^{-1}.$$

**Note:** Not all linear transformations have inverses; a linear transformation of the form  $A\vec{x}$  has an inverse **if and only if**  $\text{rref}(A) = I_n$  (the  $(n \times n)$  identity matrix).





## Qualifying “Linear” vs. “Affine”

In the realm of Linear Algebra, we have:

## Definition (Linear Function)

A **Linear Function** takes the form

$$\vec{y} = A\vec{x},$$

so that  $A\vec{0} = \vec{0}$ .

## Definition (Affine Function)

An **Affine Function** takes the form

$$\vec{y} = A\vec{x} + \vec{b},$$

so that  $A\vec{0} + \vec{b} = \vec{b}$ .

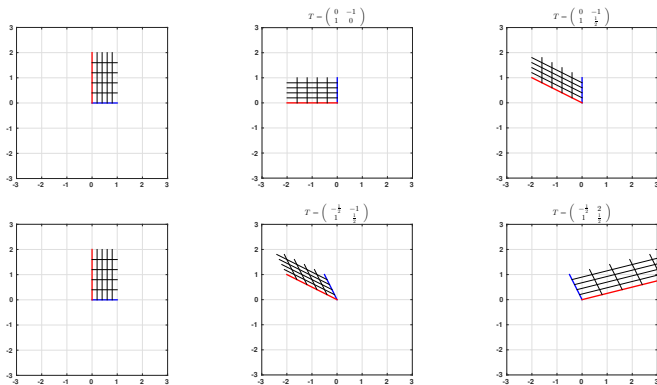
## Linear Transformation Examples

## 2D

## See Movies

We consider 4 linear transformations of the vectors

$$\vec{v} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



## Linear Transformation Examples

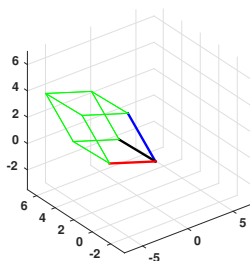
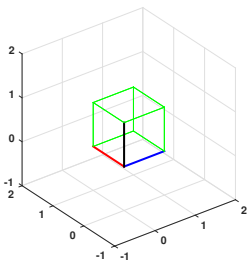
## 3D

## See Movies

We consider a linear transformation of the vectors

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T = \begin{pmatrix} -1 & -2 & -3 \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{pmatrix}$$



## Columns of the Matrix of a Linear Transformation

### Theorem (The Columns of the Matrix of a Linear Transformation)

Consider a linear transformation  $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ . Then, the matrix of  $T$  is

$$A = \left[ \begin{array}{c|c|c|c} & & & \\ T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_m) \\ & & & \end{array} \right],$$

where  $\vec{e}_i \in \mathbb{R}^m$  is the vector of all zeros, except entry  $\#i$  which is 1.

Consider

$$A\vec{e}_i = \left[ \begin{array}{c|c|c|c|c|c} & & & & & \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_i & \dots & \vec{v}_n \\ & & & & & \end{array} \right] \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{v}_i.$$

## Linearity of a Transform

## Theorem (Linear Transforms)

A transformation  $T : \mathbb{R}^m \mapsto \mathbb{R}^n$  is linear *if and only if*

- **Vector Addition** — [Additivity Condition]  
 $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}), \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^m, \text{ and}$
- **Scalar Multiplication** — [Homogeneity-of-Degree-1 Condition]  
 $T(k\vec{v}) = kT(\vec{v}), \quad \forall \vec{v} \in \mathbb{R}^m, \text{ and } \forall k \in \mathbb{R}.$

This follows directly from the **Algebraic Rules for  $A\vec{x}$** :

Theorem (Algebraic Rules for  $A\vec{x}$  [NOTES#1.3])

If  $A \in \mathbb{R}^{n \times m}$ ,  $\vec{x} \in \mathbb{R}^m$ ,  $\vec{y} \in \mathbb{R}^m$ , and  $k \in \mathbb{R}$ , then

- $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
- $A(k\vec{x}) = k(A\vec{x})$



## Suggested Problems 2.1

**Available on Learning Glass videos:**

2.1 — 1, 2, 3, 4, 5, 7, 8, 13, 14, 16, 17, 41, 42



## Lecture – Book Roadmap

Lecture	Book, [GS5-]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3
1.3	§1.1, §1.2, §1.3, §2.1, §2.3
1.4	§1.1–§1.3, §2.1–§2.3
2.1	§8.1, §8.2*, §2.5*

§2.5\* (p.86–88) “Calculating  $A^{-1}$  by Gauss-Jordan Elimination”

§8.2\* We will talk about “Basis” / “Bases” soon... don’t worry about those concepts... yet.

## Metacognitive Exercise — Thinking About Thinking &amp; Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		



(2.1.1), (2.1.2)

(2.1.1) Consider the transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  defined by

$$\begin{aligned}y_1 &= 2x_2 \\y_2 &= x_2 + 2, \\y_3 &= 2x_2\end{aligned}$$

is this a **linear** transformation?

(2.1.2) Consider the transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  defined by

$$\begin{aligned}y_1 &= 2x_2 \\y_2 &= 3x_3, \\y_3 &= x_1\end{aligned}$$

is this a **linear** transformation?

(2.1.3), (2.1.4)

(2.1.3) Consider the transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  defined by

$$\begin{aligned}y_1 &= x_2 - x_3 \\y_2 &= x_1 x_3 \quad , \\y_3 &= x_1 - x_2\end{aligned}$$

is this a **linear** transformation?

(2.1.4) Find the matrix of the linear transformation

$$\begin{aligned}y_1 &= 9x_1 + 3x_2 - 3x_3 \\y_2 &= 2x_1 - 9x_2 + x_3 \\y_3 &= 4x_1 - 9x_2 - 2x_3 \\y_4 &= 5x_1 + x_2 + 5x_3\end{aligned}$$

(2.1.5), (2.1.7)

(2.1.5) Consider the transformation from  $\mathbb{R}^3$  to  $\mathbb{R}^2$  with

$$T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 7 \\ 11 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 6 \\ 9 \end{bmatrix}, \quad \text{and} \quad T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -13 \\ 17 \end{bmatrix}.$$

Find the matrix  $A$  of  $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ .

(2.1.7) Suppose  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$  are arbitrary vectors in  $\mathbb{R}^n$ . Consider the transformation from  $\mathbb{R}^m$  to  $\mathbb{R}^n$  given by

$$T \left( \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \right) = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_m \vec{v}_m.$$

Is this transformation linear? If so, find its matrix  $A$  in terms of the vectors  $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ .

(2.1.8), (2.1.13)

**(2.1.8)** Find the inverse of the linear transformation

$$\begin{aligned}y_1 &= x_1 + 7x_2 \\y_2 &= 3x_1 + 20x_2\end{aligned}$$

**(2.1.13)** Prove the following facts:

- a. The  $(2 \times 2)$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is invertible **if and only if**  $(ad - bc) \neq 0$ . (Consider the cases  $a = 0$ , and  $a \neq 0$  separately.)
- b. If  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is invertible, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(2.1.14)

**(2.1.14)**

- a. For which values of the constant  $k$  is the matrix

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}$$

invertible?

- b. For which values of the constant  $k$  are all the entries of

$$\begin{bmatrix} 2 & 3 \\ 5 & k \end{bmatrix}^{-1}$$

integers?

(2.1.16)

**(2.1.16)** Give a geometric interpretation of the linear transformation defined by the matrix

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Show the effect of the transformation on the letter  $L$ , described by the two vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Is the transformation invertible? Find the inverse if it exists, and interpret it geometrically.

(2.1.17), (2.1.41)

**(2.1.17)** Give a geometric interpretation of the linear transformation defined by the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Show the effect of the transformation on the letter  $L$ , described by the two vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Is the transformation invertible? Find the inverse if it exists, and interpret it geometrically.

**(2.1.41)** Describe all linear transformations from  $\mathbb{R}^2$  to  $\mathbb{R}$  ( $= \mathbb{R}^1$ ). What do their graphs look like?

## (2.1.42)

(2.1.42) When you represent a 3-dimensional object in the plane (on paper, whiteboard, learning glass, screen...), you have to transform spatial coordinates  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ ,

into plane coordinates  $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ . The simplest choice is a linear transformation, for example, the one given by the matrix  $\begin{bmatrix} -0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{bmatrix}$ .

- a. Use this transformation to represent the unit cube with corner points

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- b. Represent the image of the point  $\begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix}$ , in your figure in part (a).

- c. Find all the points  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ , in  $\mathbb{R}^3$  that are transformed into  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . Explain.



## Two Separate Websites



<https://piazza.com/>

- **Piazza Account**

- Mini-Quizzes
- Access using app, or web interface



<https://Canvas.SDSU.edu/>

- **GRADEBOOK**

## Goal: Your Points in the GRADEBOOK!

### How do we get there?

#### 1 Set up a **Piazza Account**

- You should have gotten an invitation sent to your [...]@sdsu.edu email address; or go to
- <https://piazza.com/sdsu/fall2021/math254blomgren>
- It is FREE
- Download the app from the iOS App Store, or Google Play

#### 2 **If you want credit:** You can register multiple email addresses to a piazza account — one of them must match your email address in Canvas.

- In Piazza go to Account/Email Settings

## Goal: Your Points in the GRADEBOOK!

### How do we get there?

- 3 Scores will be transferred after EACH LECTURE
  - The transfer is manual, so there may be a bit of a delay.
- 4 *If something is not right, please let me know!* — Email me!

If you cannot participate in the use of Piazza (due to lack of a suitable device) — let me know, and we'll figure something out.