Math 254: Introduction to Linear Algebra

Notes #2.2 — Linear Transformations in Geometry

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Outline

SLOs 2.2 Linear Transformations in Geometry

After this lecture you should:

- Know and be able to recognize the *Matrix Forms* for:
  - scaling,
  - rotation,
  - reflection,
  - shear.
- Be the Inter-Galactic Grand Emperor* of *Orthogonal Projections* —
  - know the formula for projection onto a line, and the geometric interpretation
- Be able to perform *Reflections Across a Line*
  - be able to derive the reflection formula using the orthogonal projection formula

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* Yes, it is important!

[FOCUS :: MATH] Challenge Question Just for "fun" 1 of 2

Last time we defined

Theorem (Linear Transforms)

A transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is linear if and only if

- **Vector Addition** —
  $T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w})$, $\quad \forall \vec{v}, \vec{w} \in \mathbb{R}^m$, *and*
- **Scalar Multiplication** —
  $T(k\vec{v}) = kT(\vec{v})$, $\quad \forall \vec{v} \in \mathbb{R}^m$, *and* $\forall k \in \mathbb{R}$.

by it is not necessary to restrict this definition to vectors. We can say:

Theorem (Linear Transforms (Generalized))

A transformation $T : V \mapsto W$ is linear if and only if

- **Addition** —
  $T(v_1 + v_2) = T(v_1) + T(v_2)$, $\quad \forall v_1, v_2 \in V$, *and*
- **Scalar Multiplication** —
  $T(k\, v) = k\, T(v)$, $\quad \forall v \in V$, *and* $\forall k \in \mathbb{R}$.

[Focus :: Math] Challenge Question — Just for "fun" — 2 of 2

Challenge Question

Keeping the generalized linear transform in mind, can you think of an example where V and W are NOT vector spaces ($\mathbb{R}^n$, $\mathbb{R}^m$)?

What is a "Challenge Question?"

It is a question which stretches beyond what we "know" at this stage in the class. Some challenge questions will be "answered" later in the semester, and some in future class(es), *e.g.* Math 524 and Math 543.

Will "Challenge Questions" show up on the tests/homework?

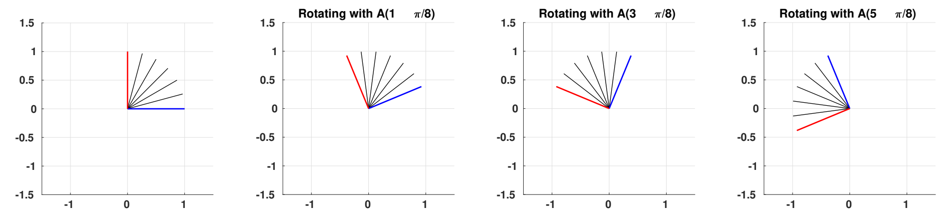
No... Well, if a question is answered later in the semester, it is fair game. (but not until then)

The Geometry of Linear Transforms — Rotations

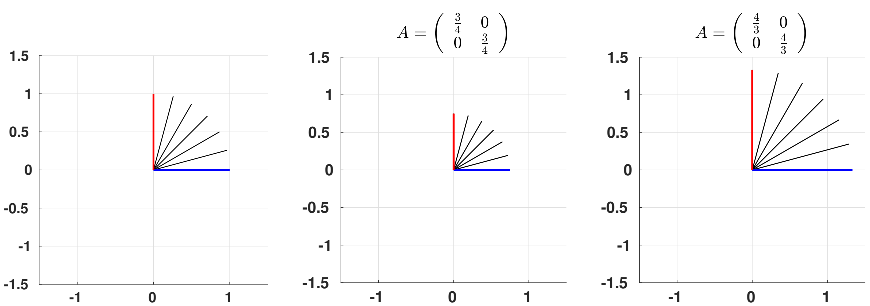
We have seen [Notes#2.1; associated movies] that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ gives a counter-clockwise rotation by $\pi/2$ (90°); in general, a matrix of the form $A(\theta) \in \mathbb{R}^2$:

$$A(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad A(\theta)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

defines a counter-clockwise rotation by θ:

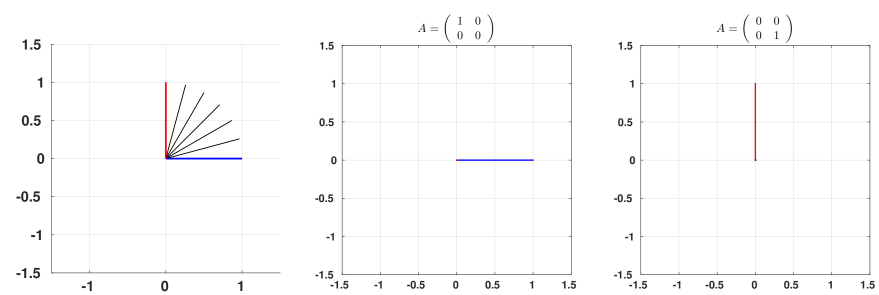


The Geometry of Linear Transforms — Scaling



When A is a multiple of the identity matrix, $\alpha\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then all vectors are *scaled* by the factor α

The Geometry of Linear Transforms — Projection

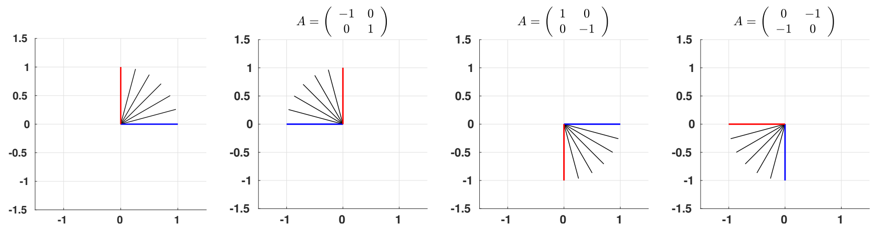


When $A \in \mathbb{R}^{n\times n}$, and $\text{rank}(A) < n$; the linear transformation $A\vec{x}$ is a *projection* onto a *subspace* of $\mathbb{R}^n$. Here $n = 2$ and $\text{rank}(A) = 1$:

— *(i)* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projects onto the x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$;

— *(ii)* $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects onto the y-axis: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$.

The Geometry of Linear Transforms — Reflection



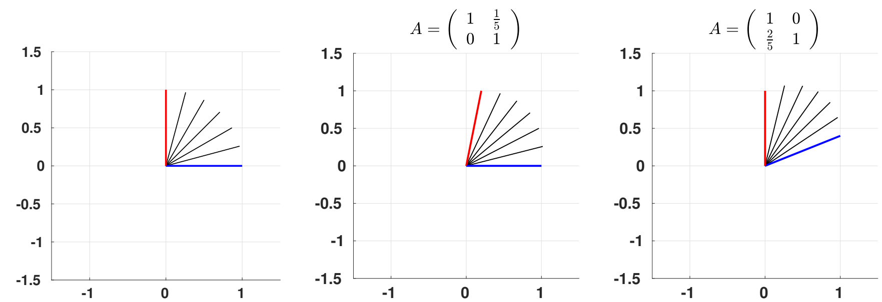
Here we see examples of reflections;

— *(i)* $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects about the y-axis; and

— *(ii)* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects about the x-axis; and

— *(iii)* $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ reflects about the line $y = -x$.

The Geometry of Linear Transforms — Shear



Here we see examples of shear;

— *(i)* $\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$ gives horizontal shear; and

— *(ii)* $\begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix}$ gives vertical shear.

The Geometry of Linear Transforms — Combinations

All these operations (+ clock-wise rotation) can be combined in a multitude of ways; the *most commonly appearing* combination being scaling+rotation, *e.g.*

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0.5\cos\theta & -0.5\sin\theta \\ 0.5\sin\theta & 0.5\cos\theta \end{bmatrix}$$

In this case, **order does not matter**; we can rotate-then-scale, or scale-then-rotate, or scale-and-rotate-at-the-same-time

The scaling and rotation matrices **commute**.

Scaling

Scaling

$\forall k > 0$, the matrix $M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ defines a scaling by k:

$$M\vec{x} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\vec{x}.$$

We call this a **dilation** (enlargement) when $k > 1$, and a **contraction** when $0 < k < 1$; when $k = 0$ you get a contraction to a point $\vec{0}$; when $k < 0$ you get a reflection in each coordinate plane followed by a scaling by $|k|$.

Scaling generalizes to $\mathbb{R}^n$ in the most straight-forward way; scaling matrices are of the form $k\, I_n$, where I_n is the identity matrix of size n.

Rotations

Theorem (Rotations)

The matrix of a counter-clockwise rotation in $\mathbb{R}^2$ through an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Note that this is a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2+b^2=1$. Conversely, any matrix of this form represents a rotation.

For clock-wise rotations, change $\theta \to -\theta$.

Combined Rotations and Scaling

Theorem (Rotation Combined with a Scaling)

A matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ represents a rotation combined with a scaling, with $r=\sqrt{a^2+b^2}$, and $\tan\theta = b/a$ we can write the matrix in the equivalent form(s)

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Shear

Theorem (Horizontal and Vertical Shears)

The matrix of a horizontal shear is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and the matrix of a vertical shear is of the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where k is any constant.

"[Mechanical shear is] a strain in the structure of a substance produced by pressure, when its layers are laterally shifted in relation to each other." — Google.

More info: — Math, Engineering, Physics, Geology (Earthquakes), Aviation...
`https://en.wikipedia.org/wiki/Shear`
`https://en.wikipedia.org/wiki/Shear_matrix`

Orthogonal Projections

Ponder a line $L = \{c_1x_1 + c_2x_2 = 0 : x_1, x_2 \in \mathbb{R}\}$ in the plane $(\mathbb{R}^2)$; any vector $\vec{x} \in \mathbb{R}^2$ can we written uniquely as

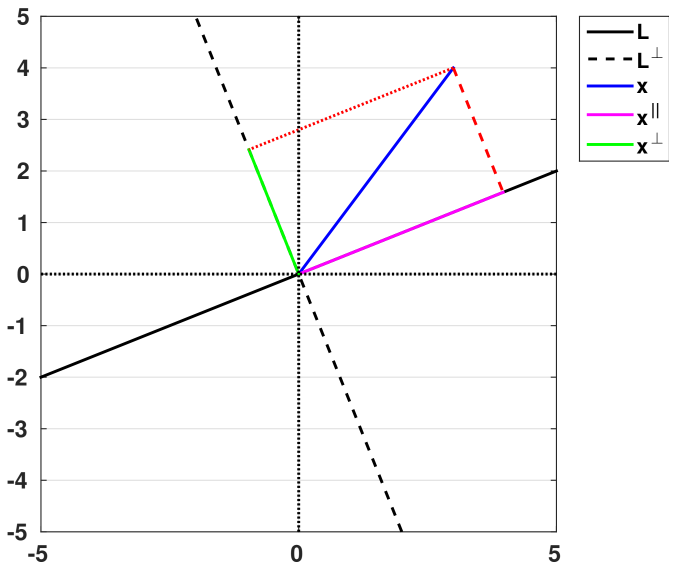
$$\vec{x} = \vec{x}^{\|} + \vec{x}^{\perp},$$

where $\vec{x}^{\|}$ is parallel to the line L, and $\vec{x}^{\perp}$ is orthogonal (perpendicular) to L.

The transformation $T(\vec{x}) = \vec{x}^{\|}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called the **orthogonal projection of $\vec{x}$ onto L**; sometimes denoted by $\mathrm{proj}_L(\vec{x})$.

The projection is essentially the *shadow* $\vec{x}$ casts on L if we shine a light on L (where are the light-rays are perfectly orthogonal to L).

Orthogonal Projections — Figure



Orthogonal Projections — Formulas

We can describe the Orthogonal Projection using the dot product...

First, let $\vec{w} \neq \vec{0}$ be any vector parallel to L. We must have

$$\vec{x}^{||} = k\vec{w},$$

for some $k \in \mathbb{R}$. The "leftovers" are

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{||} = \vec{x} - k\vec{w},$$

but $\vec{x}^{\perp}$ must be perpendicular to L; so that [DEFINITION OF ORTHOGONALITY]

$$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0.$$

Let's digest that for 10^{-10} seconds...

Orthogonal Projections — Formulas

Using the [DISTRIBUTIVE PROPERTY] of the dot product:

$$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0 \quad \Leftrightarrow \quad \vec{x} \cdot \vec{w} - k(\vec{w} \cdot \vec{w}) = 0,$$

which leads to an expression for k:

$$k = \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}.$$

We conclude with the

Formula for the Orthogonal Projection onto a line, L

$$\vec{x}^{||} = \mathrm{proj}_L(\vec{x}) = k\vec{w} = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}, \quad \text{where } \vec{w} \text{ is } \textit{any} \text{ point on } L.$$

Orthogonal Projections — Formulas

Note that $\vec{w} \cdot \vec{w}$ is [DEFINITION OF VECTOR LENGTH] just $\|\vec{w}\|^2$.

If we build the projection with a vector of length 1 (unit vector, $\|\vec{u}\| = 1$), the projection formula simplifies to

$$\vec{x}^{||} = \mathrm{proj}_L(\vec{x}) = k\vec{u} = (\vec{x} \cdot \vec{u})\,\vec{u}.$$

You can always "make" a unit vector for this purpose, by re-scaling $\vec{w}$ to be length 1:

$$\vec{u} = \frac{1}{\|\vec{w}\|}\vec{w}$$

Orthogonal Projections — Yeah, it's Linear! — Formulas

$$\vec{x}^{\|} = \mathrm{proj}_L(\vec{x}) = k\vec{u} = (\vec{x}\cdot\vec{u})\,\vec{u} = (x_1u_1 + x_2u_2)\begin{bmatrix}u_1\\u_2\end{bmatrix}$$

$$= \begin{bmatrix}x_1u_1^2 + x_2u_1u_2\\ x_1u_1u_2 + x_2u_2^2\end{bmatrix} = \underbrace{\begin{bmatrix}u_1^2 & u_1u_2\\ u_1u_2 & u_2^2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{\vec{x}}.$$

We can express the projection as a matrix-vector multiplication; therefore it is a linear transformation.

Orthogonal Projections — Full Definition

Definition (Orthogonal Projections)

Consider a line $L = \{c_1x_1 + c_2x_2 = 0 : x_1, x_2 \in \mathbb{R}\}$ in the plane ($\mathbb{R}^2$); any vector $\vec{x} \in \mathbb{R}^2$ can we written uniquely as

$$\vec{x} = \vec{x}^{\|} + \vec{x}^{\perp},$$

where $\vec{x}^{\|}$ is parallel to the line L, and $\vec{x}^{\perp}$ is orthogonal (perpendicular) to L.

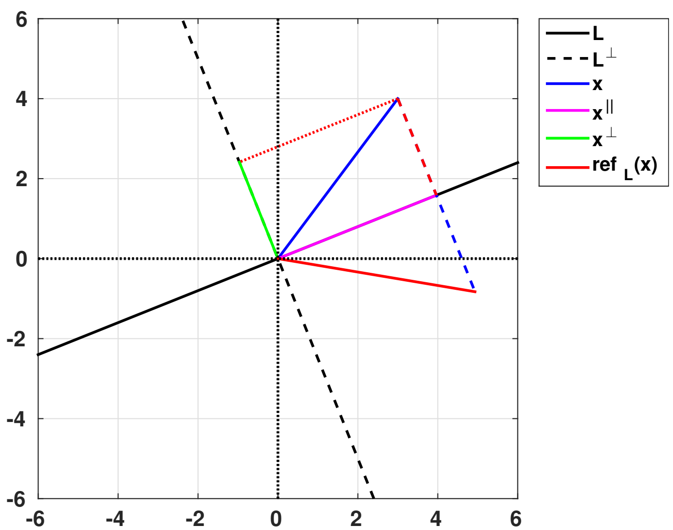
The transformation $T(\vec{x}) = \vec{x}^{\|}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called the **orthogonal projection of** $\vec{x}$ **onto** L; sometimes denoted by $\mathrm{proj}_L(\vec{x})$. If $\vec{w} \neq \vec{0}$ is any vector parallel to L, then

$$\vec{x}^{\|} = \mathrm{proj}_L(\vec{x}) = k\vec{w} = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}.$$

The transformation is linear, with matrix

$$A = \frac{1}{w_1^2 + w_2^2}\begin{bmatrix}w_1^2 & w_1w_2\\ w_1w_2 & w_2^2\end{bmatrix}$$

Reflection across L — Figure



Hey, Reflections are "Easy" if we know Projections!

We realize that

$$\vec{x} = \vec{x}^{\|} + \vec{x}^{\perp} \quad\Leftrightarrow\quad \vec{x}^{\|} = \vec{x} - \vec{x}^{\perp} \quad\Leftrightarrow\quad -\vec{x}^{\perp} = \vec{x}^{\|} - \vec{x};$$

where

- $\vec{x}^{\|}$ is the part of $\vec{x}$ in the direction of L, — $\mathrm{proj}_L(\vec{x})$.
- $\vec{x}^{\perp}$ is the part of $\vec{x}$ in the direction orthogonal to L.

$\vec{x}$ reflected in L must be the same distance "on the other size" of L, that is

$$\mathrm{ref}_L(\vec{x}) \;=\; \vec{x}^{\|} - \vec{x}^{\perp} \;=\; \vec{x} - 2\vec{x}^{\perp} \;=\; 2\vec{x}^{\|} - \vec{x}.$$

Reflections — Full Definition

Definition (Reflections)

Consider a line $L = \{c_1 x_1 + c_2 x_2 = 0 : x_1, x_2 \in \mathbb{R}\}$ in the plane $(\mathbb{R}^2)$, and let $\vec{x} = \vec{x}^{\|} + \vec{x}^{\perp}$ be a vector in $\mathbb{R}^2$. The linear transformation $T(\vec{x}) = \vec{x}^{\|} - \vec{x}^{\perp}$ is called the **reflection of $\vec{x}$ about** L, denoted by

$$\mathrm{ref}_L(\vec{x}) = \vec{x}^{\|} - \vec{x}^{\perp}.$$

We can relate $\mathrm{ref}_L(\vec{x})$ to $\mathrm{proj}_L(\vec{x})$: (here $\vec{u} \in L : \|\vec{u}\| = 1$)

$$\mathrm{ref}_L(\vec{x}) = 2\,\mathrm{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}.$$

The Reflection matrix

$$S = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$$

is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Conversely, any matrix of this form represents a reflection about a line.

Projections and Reflections in 3D, and Beyond...

in $\mathbb{R}^3$ we can "fake" it...

Nothing strange happens when you go to higher dimensions...

Let L be a line in $\mathbb{R}^3$, and let $\vec{u}$ be a unit vector parallel to L; again we can write $\vec{x} = \vec{x}^{\|} + \vec{x}^{\perp}$; and

$$\mathrm{proj}_L(\vec{x}) = \vec{x}^{\|} = (\vec{x}\cdot\vec{u})\vec{u}$$

Now, $V = L^{\perp}$ is the *plane* thru the origin which is orthogonal to L. Writing down the projections to, and reflections across V is fairly straight-forward

$$\begin{aligned}
\mathrm{proj}_V(\vec{x}) &= \vec{x} - \mathrm{proj}_L(\vec{x}) = \vec{x} - (\vec{x}\cdot\vec{u})\vec{u} \\
\mathrm{ref}_L(\vec{x}) &= \mathrm{proj}_L(\vec{x}) - \mathrm{proj}_V(\vec{x}) = 2\mathrm{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x} \\
\mathrm{ref}_V(\vec{x}) &= \mathrm{proj}_V(\vec{x}) - \mathrm{proj}_L(\vec{x}) = -\mathrm{ref}_L(\vec{x}) = \vec{x} - 2(\vec{x}\cdot\vec{u})\vec{u}
\end{aligned}$$

Projections and reflections in higher dimensions relate to each other just like they do in 2 dimensions — that should save some brain-space...

Suggested Problems 2.2

Available on Learning Glass videos:
2.2 — 1, 6, 7, 9, 12, 13, 17, 26

Lecture – Book Roadmap

Lecture	Book, [GS5–]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3
1.3	§1.1, §1.2, §1.3, §2.1, §2.3
1.4	§1.1–§1.3, §2.1–§2.3
2.1	§8.1, §8.2*, §2.5*
2.2	§8.1, §8.2*, §4.2*, §4.4*

§2.5* (p.86–88) "Calculating A^{-1} by Gauss-Jordan Elimination"

§4.2* (p.207) "Projection Onto a Line" – (p.210) end of "Example 2"

§4.4* Example 1, Example 3

§8.2* We will talk about "Basis" / "Bases" soon... don't worry about those concepts... yet.

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(2.2.1)

(2.2.1) Sketch the image of the "L," described by the two vectors

$$\begin{bmatrix}1\\0\end{bmatrix}, \quad \begin{bmatrix}0\\2\end{bmatrix}.$$

under the linear transformation

$$T(\vec{x}) = \begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}\vec{x}.$$

(2.2.6), (2.2.7)

(2.2.6) Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix}2\\1\\2\end{bmatrix}$. Find the orthogonal projection of the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$. onto L.

(2.2.7) Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix}2\\1\\2\end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$. about the line L.

(2.2.9), (2.2.12)

(2.2.9) Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\vec{x}.$$

(2.2.12) Consider a reflection matrix A and a vector $\vec{x}$ in $\mathbb{R}^2$. We define $\vec{v} = \vec{x} + A\vec{x}$, and $\vec{w} = \vec{x} - A\vec{x}$.

- **a.** Using the definition of a reflection, express $A(A\vec{x})$ in terms of $\vec{x}$
- **b.** Express $A\vec{v}$ in terms of $\vec{v}$
- **c.** Express $A\vec{w}$ in terms of $\vec{w}$
- **d.** If the vectors $\vec{v}$ and $\vec{w}$ are both non-zero, what it the angle between them?
- **e.** If the vector $\vec{v}$ is non-zero, what is the relation between $\vec{v}$ and the line L of reflection?

Draw a sketch showing $\vec{x}$, $A\vec{x}$, $A(A\vec{x})$, $\vec{v}$, $\vec{w}$, and the line L.

(2.2.13), (2.2.17)

(2.2.13) Suppose a line L in $\mathbb{R}^2$ contains the unit vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Find the matrix A of the linear transformation $T(\vec{x}) = \mathrm{ref}_L(\vec{x})$. Give the entries of A in terms of u_1 and u_2. Show that A is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$.

(2.2.17) Consider a matrix A of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Find two non-zero perpendicular vectors $\vec{v}$ and $\vec{w}$ such that $A\vec{v} = \vec{v}$, and $A\vec{w} = -\vec{w}$ — write the entries of $\vec{v}$ and $\vec{w}$ in terms of a and b) Conclude that $T(\vec{x}) = A\vec{x}$ represents a reflection about the line L spanned by $\vec{v}$.

(2.2.26)

(2.2.26) Find the...

a. scaling matrix A that transforms $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ into $\begin{bmatrix} 8 \\ -4 \end{bmatrix}$

b. orthogonal projection matrix B that transforms $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

c. rotation matrix C that transforms $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ into $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

d. shear matrix D that transforms $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$

e. reflection matrix E that transforms $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$