

Math 254: Introduction to Linear Algebra

Notes #2.2 — Linear Transformations in Geometry

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Spring 2022
(Revised: January 18, 2022)

Outline

- 1 Student Learning Objectives
 - SLOs: Linear Transformations in Geometry
 - Challenge Questions :: Going Deeper
- 2 Linear Transformations in Geometry
 - Introduction by Figures
 - Collecting and Formalizing
- 3 Orthogonal Projections, and Reflections
 - Orthogonal Projections
 - Reflections
- 4 Suggested Problems
 - Suggested Problems 2.2
 - Lecture – Book Roadmap
- 5 Supplemental Material
 - Metacognitive Reflection
 - Problem Statements 2.2

SLOs 2.2

Linear Transformations in Geometry

After this lecture you should:

- Know and be able to recognize the *Matrix Forms* for:
 - scaling,
 - rotation,
 - reflection,
 - shear.
- Be the Inter-Galactic Grand Emperor* of *Orthogonal Projections* —
 - know the formula for projection onto a line, and the geometric interpretation
- Be able to perform *Reflections Across a Line*
 - be able to derive the reflection formula using the orthogonal projection formula

* Yes, it is important!

[FOCUS :: MATH] Challenge Question

Just for “fun”

1 of 2

Last time we defined

Theorem (Linear Transforms)

A transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ is linear if and only if

- **Vector Addition** —

$$T(\vec{v} + \vec{w}) = T(\vec{v}) + T(\vec{w}), \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^m, \text{ and}$$

- **Scalar Multiplication** —

$$T(k\vec{v}) = kT(\vec{v}), \quad \forall \vec{v} \in \mathbb{R}^m, \text{ and } \forall k \in \mathbb{R}.$$

by it is not necessary to restrict this definition to vectors. We can say:

Theorem (Linear Transforms (Generalized))

A transformation $T : V \mapsto W$ is linear if and only if

- **Addition** —

$$T(v_1 + v_2) = T(v_1) + T(v_2), \quad \forall v_1, v_2 \in V, \text{ and}$$

- **Scalar Multiplication** —

$$T(kv) = kT(v), \quad \forall v \in V, \text{ and } \forall k \in \mathbb{R}.$$

Challenge Question

Keeping the generalized linear transform in mind, can you think of an example where V and W are NOT vector spaces (\mathbb{R}^n , \mathbb{R}^m)?

What is a “Challenge Question?”

It is a question which stretches beyond what we “know” at this stage in the class. Some challenge questions will be “answered” later in the semester, and some in future class(es), e.g. Math 524 and Math 543.

Will “Challenge Questions” show up on the tests/homework?

No... Well, if a question is answered later in the semester, it is fair game. (but not until then)

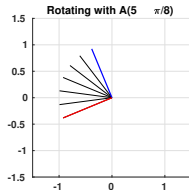
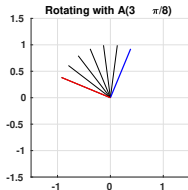
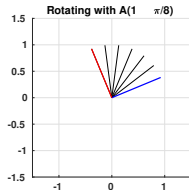
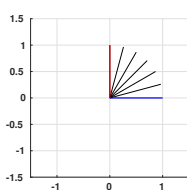
The Geometry of Linear Transforms

Rotations

We have seen [NOTES#2.1; ASSOCIATED MOVIES] that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ gives a counter-clockwise rotation by $\pi/2$ (90°); in general, a matrix of the form $A(\theta) \in \mathbb{R}^2$:

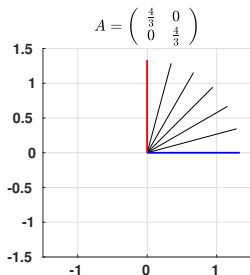
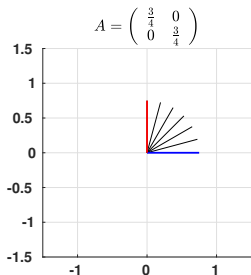
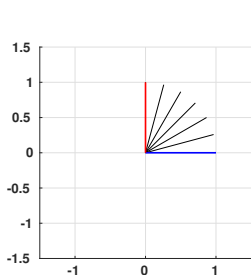
$$A(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \quad A(\theta) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}$$

defines a counter-clockwise rotation by θ :



The Geometry of Linear Transforms

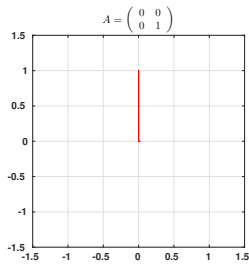
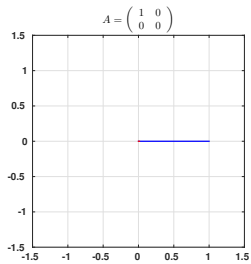
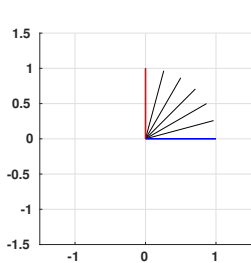
Scaling



When A is a multiple of the identity matrix, $\alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then all vectors are *scaled* by the factor α

The Geometry of Linear Transforms

Projection



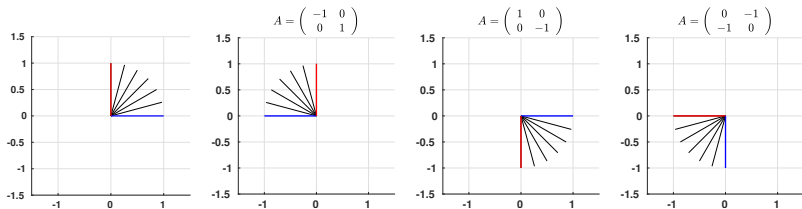
When $A \in \mathbb{R}^{n \times n}$, and $\text{rank}(A) < n$; the linear transformation $A\vec{x}$ is a *projection* onto a *subspace* of \mathbb{R}^n . Here $n = 2$ and $\text{rank}(A) = 1$:

— (i) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projects onto the x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$;

— (ii) $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects onto the y-axis: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$.

The Geometry of Linear Transforms

Reflection

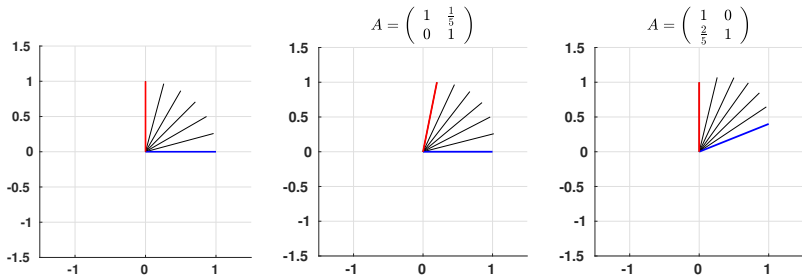


Here we see examples of reflections;

- (i) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ reflects about the y -axis; and
- (ii) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects about the x -axis; and
- (iii) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ reflects about the line $y = -x$.

The Geometry of Linear Transforms

Shear



Here we see examples of shear;

- (i) $\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$ gives horizontal shear; and
- (ii) $\begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix}$ gives vertical shear.

The Geometry of Linear Transforms

Combinations

All these operations (+ clock-wise rotation) can be combined in a multitude of ways; the *most commonly appearing* combination being scaling+rotation, e.g.

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 0.5 \cos \theta & -0.5 \sin \theta \\ 0.5 \sin \theta & 0.5 \cos \theta \end{bmatrix}$$

In this case, **order does not matter**; we can rotate-then-scale, or scale-then-rotate, or scale-and-rotate-at-the-same-time

The scaling and rotation matrices **commute**.

Scaling

Scaling

$\forall k > 0$, the matrix $M = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ defines a scaling by k :

$$M\vec{x} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\vec{x}.$$

We call this a **dilation** (enlargement) when $k > 1$, and a **contraction** when $0 < k < 1$; when $k = 0$ you get a contraction to a point $\vec{0}$; when $k < 0$ you get a reflection in each coordinate plane followed by a scaling by $|k|$.

Scaling generalizes to \mathbb{R}^n in the most straight-forward way; scaling matrices are of the form kI_n , where I_n is the identity matrix of size n .

Rotations

Theorem (Rotations)

The matrix of a counter-clockwise rotation in \mathbb{R}^2 through an angle θ is

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

*Note that this is a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2 + b^2 = 1$.
Conversely, any matrix of this form represents a rotation.*

For clock-wise rotations, change $\theta \rightarrow -\theta$.

Combined Rotations and Scaling

Theorem (Rotation Combined with a Scaling)

A matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ represents a rotation combined with a scaling, with $r = \sqrt{a^2 + b^2}$, and $\tan \theta = b/a$ we can write the matrix in the equivalent form(s)

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r \cos \theta & -r \sin \theta \\ r \sin \theta & r \cos \theta \end{bmatrix} = r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

Shear

Theorem (Horizontal and Vertical Shears)

The matrix of a horizontal shear is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and the matrix of a vertical shear is of the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where k is any constant.

“[Mechanical shear is] a strain in the structure of a substance produced by pressure, when its layers are laterally shifted in relation to each other.” — Google.

More info: — Math, Engineering, Physics, Geology (Earthquakes), Aviation...
<https://en.wikipedia.org/wiki/Shear>
https://en.wikipedia.org/wiki/Shear_matrix

Orthogonal Projections

Ponder a line $L = \{c_1x_1 + c_2x_2 = 0 : x_1, x_2 \in \mathbb{R}\}$ in the plane (\mathbb{R}^2); any vector $\vec{x} \in \mathbb{R}^2$ can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

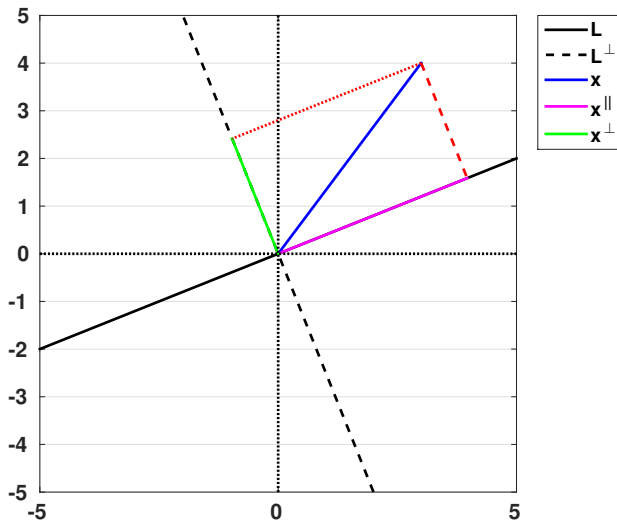
where \vec{x}^{\parallel} is parallel to the line L , and \vec{x}^{\perp} is orthogonal (perpendicular) to L .

The transformation $T(\vec{x}) = \vec{x}^{\parallel}$ from \mathbb{R}^2 to \mathbb{R}^2 is called the **orthogonal projection of \vec{x} onto L** ; sometimes denoted by $\text{proj}_L(\vec{x})$.

The projection is essentially the *shadow* \vec{x} casts on L if we shine a light on L (where the light-rays are perfectly orthogonal to L).

Orthogonal Projections

Figure



Orthogonal Projections

Formulas

We can describe the Orthogonal Projection using the dot product...

First, let $\vec{w} \neq \vec{0}$ be any vector parallel to L . We must have

$$\vec{x}^{\parallel} = k\vec{w},$$

for some $k \in \mathbb{R}$. The “leftovers” are

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \vec{x} - k\vec{w},$$

but \vec{x}^{\perp} must be perpendicular to L ; so that [DEFINITION OF ORTHOGONALITY]

$$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0.$$

Let's digest that for 10^{-10} seconds...

Orthogonal Projections

Formulas

Using the [DISTRIBUTIVE PROPERTY] of the dot product:

$$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0 \quad \Leftrightarrow \quad \vec{x} \cdot \vec{w} - k(\vec{w} \cdot \vec{w}) = 0,$$

which leads to an expression for k :

$$k = \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}.$$

We conclude with the

Formula for the Orthogonal Projection onto a line, L

$$\vec{x}^{\parallel} = \text{proj}_L(\vec{x}) = k\vec{w} = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}, \quad \text{where } \vec{w} \text{ is any point on } L.$$

Orthogonal Projections

Formulas

Note that $\vec{w} \cdot \vec{w}$ is [DEFINITION OF VECTOR LENGTH] just $\|\vec{w}\|^2$.

If we build the projection with a vector of length 1 (unit vector, $\|\vec{u}\| = 1$), the projection formula simplifies to

$$\vec{x}^{\parallel} = \text{proj}_L(\vec{x}) = k\vec{u} = (\vec{x} \cdot \vec{u})\vec{u}.$$

You can always “make” a unit vector for this purpose, by re-scaling \vec{w} to be length 1:

$$\vec{u} = \frac{1}{\|\vec{w}\|} \vec{w}$$

Orthogonal Projections

Yeah, it's Linear!

Formulas

$$\begin{aligned} \vec{x}^{\parallel} &= \text{proj}_L(\vec{x}) = k\vec{u} = (\vec{x} \cdot \vec{u}) \vec{u} = (x_1 u_1 + x_2 u_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 u_1^2 + x_2 u_1 u_2 \\ x_1 u_1 u_2 + x_2 u_2^2 \end{bmatrix} = \underbrace{\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}}. \end{aligned}$$

We can express the projection as a matrix-vector multiplication; therefore it is a linear transformation.

Orthogonal Projections

Full Definition

Definition (Orthogonal Projections)

Consider a line $L = \{c_1x_1 + c_2x_2 = 0 : x_1, x_2 \in \mathbb{R}\}$ in the plane (\mathbb{R}^2); any vector $\vec{x} \in \mathbb{R}^2$ can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

where \vec{x}^{\parallel} is parallel to the line L , and \vec{x}^{\perp} is orthogonal (perpendicular) to L .

The transformation $T(\vec{x}) = \vec{x}^{\parallel}$ from \mathbb{R}^2 to \mathbb{R}^2 is called the **orthogonal projection of \vec{x} onto L** ; sometimes denoted by $\text{proj}_L(\vec{x})$. If $\vec{w} \neq \vec{0}$ is any vector parallel to L , then

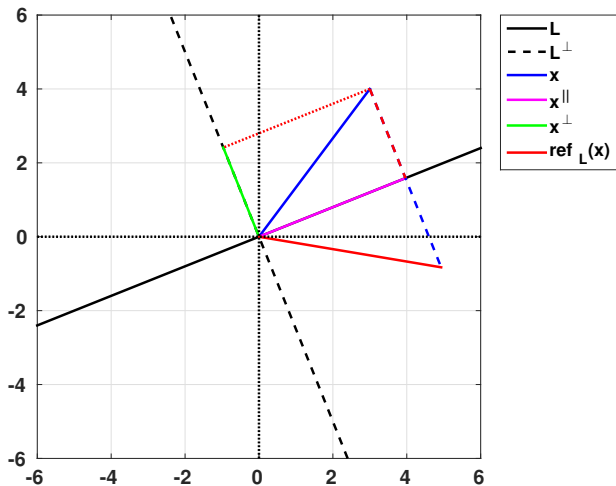
$$\vec{x}^{\parallel} = \text{proj}_L(\vec{x}) = k\vec{w} = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}.$$

The transformation is linear, with matrix

$$A = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}$$

Reflection across L

Figure



Hey, Reflections are “Easy” if we know Projections!

We realize that

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp} \quad \Leftrightarrow \quad \vec{x}^{\parallel} = \vec{x} - \vec{x}^{\perp} \quad \Leftrightarrow \quad -\vec{x}^{\perp} = \vec{x}^{\parallel} - \vec{x};$$

where

- \vec{x}^{\parallel} is the part of \vec{x} in the direction of L , — $\text{proj}_L(\vec{x})$.
- \vec{x}^{\perp} is the part of \vec{x} in the direction orthogonal to L .

\vec{x} reflected in L must be the same distance “on the other side” of L , that is

$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp} = \vec{x} - 2\vec{x}^{\perp} = 2\vec{x}^{\parallel} - \vec{x}.$$

Reflections

Full Definition

Definition (Reflections)

Consider a line $L = \{c_1x_1 + c_2x_2 = 0 : x_1, x_2 \in \mathbb{R}\}$ in the plane (\mathbb{R}^2), and let $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ be a vector in \mathbb{R}^2 . The linear transformation $T(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$ is called the **reflection of \vec{x} about L** , denoted by

$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}.$$

We can relate $\text{ref}_L(\vec{x})$ to $\text{proj}_L(\vec{x})$: (here $\vec{u} \in L : \|\vec{u}\| = 1$)

$$\text{ref}_L(\vec{x}) = 2 \text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}.$$

The Reflection matrix

$$S = \begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$$

is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Conversely, any matrix of this form represents a reflection about a line.

Projections and Reflections in 3D, and Beyond...

in \mathbb{R}^3 we can “fake” it...

Nothing strange happens when you go to higher dimensions...

Let L be a line in \mathbb{R}^3 , and let \vec{u} be a unit vector parallel to L ; again we can write $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$; and

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel} = (\vec{x} \cdot \vec{u})\vec{u}$$

Now, $V = L^{\perp}$ is the *plane* thru the origin which is orthogonal to L . Writing down the projections to, and reflections across V is fairly straight-forward

$$\begin{aligned} \text{proj}_V(\vec{x}) &= \vec{x} - \text{proj}_L(\vec{x}) = \vec{x} - (\vec{x} \cdot \vec{u})\vec{u} \\ \text{ref}_L(\vec{x}) &= \text{proj}_L(\vec{x}) - \text{proj}_V(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x} \\ \text{ref}_V(\vec{x}) &= \text{proj}_V(\vec{x}) - \text{proj}_L(\vec{x}) = -\text{ref}_L(\vec{x}) = \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u} \end{aligned}$$

Projections and reflections in higher dimensions relate to each other just like they do in 2 dimensions — that should save some brain-space...

Suggested Problems 2.2

Available on Learning Glass videos:

2.2 — 1, 6, 7, 9, 12, 13, 17, 26

Lecture – Book Roadmap

Lecture	Book, [GS5-]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3
1.3	§1.1, §1.2, §1.3, §2.1, §2.3
1.4	§1.1–§1.3, §2.1–§2.3
2.1	§8.1, §8.2*, §2.5*
2.2	§8.1, §8.2*, §4.2*, §4.4*

§2.5* (p.86–88) “Calculating A^{-1} by Gauss-Jordan Elimination”

§4.2* (p.207) “Projection Onto a Line” – (p.210) end of
“Example 2”

§4.4* Example 1, Example 3

§8.2* We will talk about “Basis” / “Bases” soon... don’t worry
about those concepts... yet.

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(2.2.1)

(2.2.1) Sketch the image of the “L,” described by the two vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

under the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}.$$

(2.2.6), (2.2.7)

(2.2.6) Let L be the line in \mathbb{R}^3 that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L .

(2.2.7) Let L be the line in \mathbb{R}^3 that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the line L .

(2.2.9), (2.2.12)

(2.2.9) Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vec{x}.$$

(2.2.12) Consider a reflection matrix A and a vector \vec{x} in \mathbb{R}^2 . We define $\vec{v} = \vec{x} + A\vec{x}$, and $\vec{w} = \vec{x} - A\vec{x}$.

- Using the definition of a reflection, express $A(A\vec{x})$ in terms of \vec{x}
- Express $A\vec{v}$ in terms of \vec{v}
- Express $A\vec{w}$ in terms of \vec{w}
- If the vectors \vec{v} and \vec{w} are both non-zero, what is the angle between them?
- If the vector \vec{v} is non-zero, what is the relation between \vec{v} and the line L of reflection?

Draw a sketch showing \vec{x} , $A\vec{x}$, $A(A\vec{x})$, \vec{v} , \vec{w} , and the line L .

(2.2.13), (2.2.17)

(2.2.13) Suppose a line L in \mathbb{R}^2 contains the unit vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Find the matrix A of the linear transformation $T(\vec{x}) = \text{ref}_L(\vec{x})$. Give the entries of A in terms of u_1 and u_2 . Show that A is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$.

(2.2.17) Consider a matrix A of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Find two non-zero perpendicular vectors \vec{v} and \vec{w} such that $A\vec{v} = \vec{v}$, and $A\vec{w} = -\vec{w}$ — write the entries of \vec{v} and \vec{w} in terms of a and b) Conclude that $T(\vec{x}) = A\vec{x}$ represents a reflection about the line L spanned by \vec{v} .

(2.2.26)

(2.2.26) Find the...

a. scaling matrix A that transforms $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ into $\begin{bmatrix} 8 \\ -4 \end{bmatrix}$

b. orthogonal projection matrix B that transforms $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

c. rotation matrix C that transforms $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ into $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

d. shear matrix D that transforms $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$

e. reflection matrix E that transforms $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$