Math 254: Introduction to Linear Algebra

Notes #2.4 — Inverse of a Linear Transform

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Outline

SLOs 2.4 — Inverse of a Linear Transform

After this lecture you should:

- Know the connection between invertibility, the form of $\mathrm{rref}(A)$, and the rank of A:

  $A \in \mathbb{R}^{n \times n}$ is invertible $\Leftrightarrow \mathrm{rref}(A) = I_n \Leftrightarrow \mathrm{rank}(A) = n.$
- Know the definition of, and be able to compute the kernel of a matrix: $\ker(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$
- Given an invertible matrix A, perform row-operations to find the inverse (see also [Notes#2.1]):

$$\left[\, A \,\middle|\, I_n \,\right] \rightsquigarrow \left[\, I_n \,\middle|\, A^{-1} \,\right]$$

- Be able to compute the determinant, $\det(A)$ and know its Geometrical Interpretation for $A \in \mathbb{R}^{2\times 2}$.

Invertible Functions — Calculus Review

Definition (Invertible Functions)

A function $f : X \mapsto Y$ is called **invertible** if the equation $f(x) = y$ has a unique solution $x \in X$ for each $y \in Y$. In this case, the inverse $f^{-1} : Y \mapsto X$ is defined by

$$f^{-1}(y) = \{\text{the unique } x \in X \text{ such that } f(x) = y\}.$$

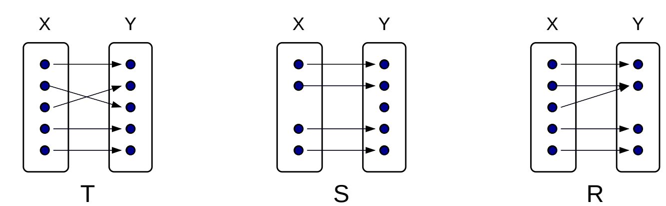


Figure: T is invertible since there is a unique $x \in X$ for each $y \in Y$;
S is not invertible since there is one y which is not "reachable" from X;
R is not invertible since there is one y for which $R(x) = y$ has two solutions.

[Focus :: Math] "Speak" Like a Mathematician

We can also say that a function is invertible if and only if it is both "onto" (surjective) and "1-to-1" (injective).

Definition (One-to-One Function [adopted from Wikipedia])

In mathematics, an injective function or injection or one-to-one function is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its range. In other words, every element of the function's range is the image of at most one element of its domain. The term one-to-one function must not be confused with one-to-one correspondence (a.k.a. bijective function), which uniquely maps all elements in both domain and range to each other

Definition (Onto Function [adopted from Wikipedia])

In mathematics, a function f from a set X to a set Y is surjective (or onto), or a surjection, if for every element y in the range Y of f there is at least one element x in the domain X of f such that $f(x) = y$. It is not required that x be unique; the function f may map one or more elements of X to the same element of Y.

Invertible Functions — Calculus Revisited

Rewind (f and f^{-1})

The equation

$$x = f^{-1}(y) \quad \text{means that} \quad y = f(x).$$

It is true that $\forall x \in X$ and $\forall y \in Y$

$$f^{-1}(f(x)) = x, \quad \text{and} \quad f(f^{-1}(y)) = y$$

Rewind (f and g so that $f \circ g = g \circ f =$ [Identity Function])

If $g : Y \mapsto X$ such that $\forall x \in X$ and $\forall y \in Y$

$$g(f(x)) = x, \quad \text{and} \quad f(g(y)) = y,$$

then f is invertible, and $f^{-1} = g$.

Rewind (Inverse of the Inverse)

If f is invertible, then so is f^{-1}, and $(f^{-1})^{-1} = f$.

Invertible Functions — Calculus Revisited

Example (f and its inverse f^{-1})

Let

$$f(x) = \frac{x^5 - 1}{3}, \quad g(y) = \sqrt[5]{3y + 1}$$

with $x \in [0, \infty)$, and $y \in \left[-\frac{1}{3}, \infty\right)$.

Then

$$f(g(y)) = f\left(\sqrt[5]{3y+1}\right) = \frac{(\sqrt[5]{3y+1})^5 - 1}{3} = y,$$

and

$$g(f(x)) = g\left(\frac{x^5 - 1}{3}\right) = \sqrt[5]{3\frac{x^5-1}{3} + 1} = x.$$

Invertible Functions — Linear Algebra

Next, consider a linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by

$$\vec{y} = T(\vec{x}) = A\vec{x},$$

here $A \in \mathbb{R}^{n \times n}$.

The linear transformation is invertible if and only if the linear system

$$A\vec{x} = \vec{y}$$

has a unique solution $\vec{x} \in \mathbb{R}^n \ \forall \ \vec{y} \in \mathbb{R}^n$.

This is true if and only if $\mathrm{rank}(A) = n$, or equivalently if and only if

$$\mathrm{rref}(A) = I_n.$$

Invertible Matrices

Definition (Invertible Matrices, $A \mapsto A^{-1}$)

A square matrix A is said to be **invertible** if the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible. In this case the matrix of T^{-1} is denoted A^{-1}. If the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible, then its inverse is $\vec{x} = T^{-1}(\vec{y}) = A^{-1}\vec{y}$.

Theorem (Invertibility, Rank, and RREF)

An $n \times n$ matrix A is invertible if and only if

$$\mathrm{rref}(A) = I_n$$

or, equivalently, if and only if

$$\mathrm{rank}(A) = n.$$

Invertibility and Linear Systems

Recasting some of the results from [NOTES#1.3] into our new "language:"

Theorem (Invertibility and Linear Systems)

Let $A \in \mathbb{R}^{n\times n}$:

a. *Consider a vector $\vec{b} \in \mathbb{R}^n$. If A is invertible, then the system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$. If A is non-invertible, the system $A\vec{x} = \vec{b}$ has either infinitely many solutions, or no solutions.*

b. *Consider the special case when $\vec{b} = \vec{0}$. The system $A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as a solution. If A is invertible, then this is the only solution; otherwise it has infinitely many solutions. — The collection of all vectors $\{\, \vec{x} \,:\, A\vec{x} = \vec{0} \,\}$ is called the **null space** or **kernel** of A, denoted $\ker(A)$.*

We will discuss the **null space** / **kernel** extensively in the next four lectures (after the midterm).

Characteristics of Invertible Matrices — IMPORTANT!

Equivalent Statements: Invertible Matrices

For an $n \times n$ matrix A, the following statements are equivalent; that is for a given A, they are either all `true` or all `false`:

- **i.** A is invertible ($\exists\, A^{-1}$)
- **ii.** The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, $\forall \vec{b} \in \mathbb{R}^n$
- **ii.** $\mathrm{rref}(A) = I_n$
- **iv.** $\mathrm{rank}(A) = n$

We will add to this list throughout the semester: [NOTES#3.1], [NOTES#3.3], and [NOTES#7.1].

Finding A^{-1}...

Theorem (Finding the Inverse of a Matrix)

*To find the **inverse** of and $n \times n$ matrix, form the $n \times (2n)$ matrix*

$$\left[\begin{array}{c|c} A & I_n \end{array}\right]$$

and compute

$$\mathrm{rref}\left(\left[\begin{array}{c|c} A & I_n \end{array}\right]\right)$$

- *If $\mathrm{rref}\left(\left[\begin{array}{c|c} A & I_n \end{array}\right]\right) = \left[\begin{array}{c|c} I_n & B \end{array}\right]$, then A is invertible, and $A^{-1} = B$.*
- *If $\mathrm{rref}\left(\left[\begin{array}{c|c} A & I_n \end{array}\right]\right)$ is of another form, then A is not invertible.*

Note: The best way to establish whether a matrix is invertible is to try to compute the inverse using the method above. If successful, you have A^{-1}; otherwise you know that A is not invertible.

Example (Computation of the Matrix Inverse)

See also [Notes#2.1]

Start with $\left[\begin{array}{c|c} A & I_3 \end{array}\right]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right]$$

Eliminate Column#1:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right]$$

Eliminate Column#2:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right]$$

Example (Computation of the Matrix Inverse) 2/2

Normalize Row #3 (Divide by -1):

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

Eliminate Column#3:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

We arrive at $\left[\begin{array}{c|c} I_3 & A^{-1} \end{array}\right]$

Multiplying by the Inverse

Theorem (Product of a Matrix and its Inverse)

For an invertible matrix $A \in \mathbb{R}^{n\times n}$,

$$A^{-1}A = AA^{-1} = I_n$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} [1,1,1]\cdot[10,-2,-7] & [1,1,1]\cdot[-6,1,5] & [1,1,1]\cdot[1,0,-1] \\ [2,3,2]\cdot[10,-2,-7] & [2,3,2]\cdot[-6,1,5] & [2,3,2]\cdot[1,0,-1] \\ [3,8,2]\cdot[10,-2,-7] & [3,8,2]\cdot[-6,1,5] & [3,8,2]\cdot[1,0,-1] \end{bmatrix} = I_3$$

$$A^{-1}A = \begin{bmatrix} [10,-6,1]\cdot[1,2,3] & [10,-6,1]\cdot[1,3,8] & [10,-6,1]\cdot[1,2,2] \\ [-2,1,0]\cdot[1,2,3] & [-2,1,0]\cdot[1,3,8] & [-2,1,0]\cdot[1,2,2] \\ [-7,5,-1]\cdot[1,2,3] & [-7,5,-1]\cdot[1,3,8] & [-7,5,-1]\cdot[1,2,2] \end{bmatrix} = I_3$$

The Inverse of the Product of Invertible Matrices

Let $A \in \mathbb{R}^{n\times n}$, and $B \in \mathbb{R}^{n\times n}$ be two invertible matrices; *i.e.* $A^{-1} \in \mathbb{R}^{n\times n}$ and $B^{-1} \in \mathbb{R}^{n\times n}$ exists.

Does $(BA)^{-1}$ exist as well?

Linear Transform	$\vec{y} = BA\vec{x}$
multiply by B^{-1} from the left	$B^{-1}\vec{y} = B^{-1}BA\vec{x}$
	$B^{-1}\vec{y} = A\vec{x}$
multiply by A^{-1} from the left	$A^{-1}B^{-1}\vec{y} = A^{-1}A\vec{x}$
Inverse Linear Transform	$A^{-1}B^{-1}\vec{y} = \vec{x}$

The Inverse of the Product of Invertible Matrices

Theorem (The Inverse of a Product of Matrices)

If A and B are invertible $n \times n$ matrices, then BA (AB) is invertible as well, and

$$(BA)^{-1} = A^{-1}B^{-1}, \quad (AB)^{-1} = B^{-1}A^{-1}$$

La soupe à l'alphabet:

$$\begin{array}{rcccccl} A^{-1}B^{-1}\,BA &=& A^{-1}I_nA &=& A^{-1}A &=& I_n \\ BA\,A^{-1}B^{-1} &=& BI_nB^{-1} &=& BB^{-1} &=& I_n \\ B^{-1}A^{-1}\,AB &=& B^{-1}I_nB &=& B^{-1}B &=& I_n \\ AB\,B^{-1}A^{-1} &=& AI_nA^{-1} &=& AA^{-1} &=& I_n \end{array}$$

Invertibility Criterion

Theorem (Invertibility Criterion)

Let A and B be two $n \times n$ matrices such that

$$BA = I_n.$$

Then

- **a.** *A and B are both invertible*
- **b.** *$A^{-1} = B$, and $B^{-1} = A$, and*
- **c.** *$AB = I_n$*

We can use this result repeatedly to navigate thru long matrix products...

Example (Inverse of a "Chain" Product of Matrices) La Sopa del Alfabeto!!!

Ponder the invertible matrices $A, B, C, D, E \in \mathbb{R}^{n\times n}$; lets find C and C^{-1} in terms of the other matrices...

$$\begin{array}{l||rcl} \text{Given} & ABCDE &=& I_n \\ \hline & A^{-1}ABCDEE^{-1} &=& A^{-1}E^{-1} \\ & BCD &=& A^{-1}E^{-1} \\ & B^{-1}BCDD^{-1} &=& B^{-1}A^{-1}E^{-1}D^{-1} \\ C & C &=& B^{-1}A^{-1}E^{-1}D^{-1} \\ \hline & C^{-1}B^{-1}A^{-1}ABCDE &=& C^{-1}B^{-1}A^{-1} \\ & DE &=& C^{-1}B^{-1}A^{-1} \\ & DEAB &=& C^{-1}B^{-1}A^{-1}AB \\ C^{-1} & DEAB &=& C^{-1} \end{array}$$

Inverse and Determinant in the 2×2 Case

It is true that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{then} \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

under the condition that $(ad - bc) \neq 0$.

The quantity $(ad - bc)$ is the determinant of A, denoted $\det(\mathbf{A})$.

(We will return to this quantity for general $n \times n$ matrices later.)

Formulas for the inverse (using the determinant) generalize poorly to higher dimensions, check out the next slide for the result in 3×3 case...

Inverse and Determinant — in the 3×3 Case

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} ej - fh & ch - bj & bf - ce \\ fg - dj & aj - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix},$$

where $\det(A) = bfg - afh + cdh - ceg + aej - bdj$.

Maybe **not** something you want to try to memorize?

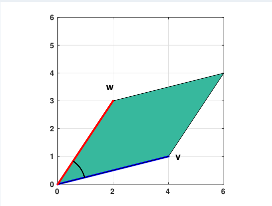
Geometrically Interpreting the Determinant in 2D

Theorem (Geometrical Interpretation of $\det(A)$, for $A \in \mathbb{R}^{2\times 2}$)

If $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$ is a 2×2 matrix with non-zero column vectors $\vec{v}$ and $\vec{w}$, then

$$\det(A) = \det\left(\begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}\right) = \sin\theta \, \|v\| \, \|w\|,$$

where θ is the angle from $\vec{v}$ to $\vec{w}$, with $\theta \in (-\pi, \pi]$. It follows that:



- $|\det(A)| = |\sin\theta| \, \|v\| \, \|w\|$ *is the* **area of the parallelogram** *spanned by $\vec{v}$ and $\vec{w}$.*
- $\det(A) = 0$ *if $\vec{v}$ and $\vec{w}$ are parallel, i.e. $\theta = 0$, or $\theta = \pi$.*
- $\det(A) > 0$ *if $\theta \in (0, \pi)$.*
- $\det(A) < 0$ *if $\theta \in (-\pi, 0)$.*

Example (2×2 Determinant : Computed 2 Ways)

Let

$$\vec{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Then

$$\|\vec{v}\| = \sqrt{17}, \quad \|\vec{w}\| = \sqrt{13}$$

$$\begin{aligned} \det\left(\begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}\right) &= \sin(0.737815\ldots)\sqrt{17}\sqrt{13} = 10 \\ &= 4 \cdot 3 - 2 \cdot 1 = 12 - 2 = 10 \end{aligned}$$

$$\begin{aligned} \det\left(\begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}\right) &= \sin(-0.737815\ldots)\sqrt{17}\sqrt{13} = -10. \\ &= 2 \cdot 1 - 4 \cdot 2 = 2 - 12 = -10 \end{aligned}$$

Suggested Problems 2.4

Available on Learning Glass videos:
2.4 — 1, 3, 9, 16, 17, 29, 31, 35

Lecture – Book Roadmap

Lecture	Book, [GS5–]
1.1	§2.2
1.2	§1.1, §1.3, §2.1, §2.3
1.3	§1.1, §1.2, §1.3, §2.1, §2.3
1.4	§1.1–§1.3, §2.1–§2.3
2.1	§8.1, §8.2*, §2.5*
2.2	§8.1, §8.2*, §4.2*, §4.4*
2.3	§2.4
2.4	§2.5

§2.5* (p.86–88) "Calculating A^{-1} by Gauss-Jordan Elimination"
§4.2* (p.207) "Projection Onto a Line" – (p.210) end of "Example 2"
§4.4* Example 1, Example 3
§8.2* We will talk about "Basis" / "Bases" soon... don't worry about those concepts... yet.

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

Live Math Fall 2019 — Projections and Reflections 1 of 2

Given

$$\vec{x} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \quad \vec{w}_1 = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix}1\\-1\\1\\-1\\1\end{bmatrix}, \quad \begin{array}{l} L_1 = \{k_1\vec{w}_1 : k_1 \in \mathbb{R}\} \\ L_2 = \{k_2\vec{w}_2 : k_2 \in \mathbb{R}\}\end{array}$$

compute $\mathrm{proj}_{L_1}(\vec{x})$, $\mathrm{proj}_{L_2}(\vec{x})$, $\mathrm{ref}_{L_1}(\vec{x})$, and $\mathrm{ref}_{L_2}(\vec{x})$.

$$\mathrm{proj}_{L_1}(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{w}_1}{\vec{w}_1\cdot\vec{w}_1}\right)\vec{w}_1 = \left\{ \begin{array}{lcl} \vec{x}\cdot\vec{w}_1 & = & 1+2+3+4+5=15 \\ \vec{w}_1\cdot\vec{w}_1 & = & 1+1+1+1+1=5 \end{array}\right\} = \frac{15}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\3\\3\\3\\3\end{bmatrix}$$

Live Math Fall 2019 — Projections and Reflections 2 of 2

$$\mathrm{proj}_{L_2}(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{w}_2}{\vec{w}_2\cdot\vec{w}_2}\right)\vec{w}_2 = \left\{ \begin{array}{lcl} \vec{x}\cdot\vec{w}_2 & = & 1-2+3-4+5=3 \\ \vec{w}_2\cdot\vec{w}_2 & = & 1+1+1+1+1=5 \end{array}\right\} = \frac{3}{5}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}3/5\\-3/5\\3/5\\-3/5\\3/5\end{bmatrix}$$

$$\mathrm{ref}_{L_1}(\vec{x}) = 2\mathrm{proj}_{L_1}(\vec{x}) - \vec{x} = 2\begin{bmatrix}3\\3\\3\\3\\3\end{bmatrix} - \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \begin{bmatrix}5\\4\\3\\2\\1\end{bmatrix}$$

$$\mathrm{ref}_{L_2}(\vec{x}) = 2\mathrm{proj}_{L_2}(\vec{x}) - \vec{x} = 2\begin{bmatrix}3/5\\-3/5\\3/5\\-3/5\\3/5\end{bmatrix} - \frac{5}{5}\begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1\\-16\\-9\\-26\\-19\end{bmatrix}$$

(2.4.1), (2.4.3)

(2.4.1) Decide whether the matrix is invertible; if it is, find the inverse.

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$$

(2.4.3) Decide whether the matrix is invertible; if it is, find the inverse.

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

(2.4.9), (2.4.16)

(2.4.9) Decide whether the matrix is invertible; if it is, find the inverse.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(2.4.16) Decide whether the linear transformation is invertible; if it is, find the inverse transformation.

$$\begin{array}{rcrcr} y_1 & = & 3x_1 & + & 5x_2 \\ y_2 & = & 5x_1 & + & 8x_2 \end{array}$$

(2.4.17), (2.4.29)

(2.4.17) Decide whether the linear transformation is invertible; if it is, find the inverse transformation.

$$\begin{array}{rcrcr} y_1 & = & x_1 & + & 2x_2 \\ y_2 & = & 4x_1 & + & 8x_2 \end{array}$$

(2.4.29) For which values of the constant k is the following matrix invertible?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix}$$

(2.4.31)

(2.4.31) For which values of the constants a, b, and c is the following matrix invertible?

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

(2.4.35)

(2.4.35)

a. Consider the upper triangular matrix

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

For which values of a, b, c, d, e, and f is A invertible?

b. More generally, when is an upper triangular matrix (of size $n \times n$) invertible?

c. If an upper triangular matrix is invertible, is its inverse also and upper triangular matrix?

d. Repeat questions **b.** and **c.** for lower triangular matrices.