

Math 254: Introduction to Linear Algebra

Notes #3.1 — Image & Kernel of a Linear Transform

Peter Blomgren
(blomgren@sdsu.edu)

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

<http://terminus.sdsu.edu/>

Spring 2022
(Revised: February 23, 2022)



Outline

- 1 Student Learning Objectives
 - SLOs: Image & Kernel of a Linear Transform
- 2 Subspaces of \mathbb{R}^n and Their Dimensions
 - Image & Kernel of a Linear Transformation
- 3 Suggested Problems
 - Suggested Problems 3.1
 - Lecture–Book Roadmap
- 4 Supplemental Material
 - Metacognitive Reflection
 - Problem Statements 3.1



Student Learning Objectives

SLOs: Image & Kernel of a Linear Transformation

SLOs 3.1

Image & Kernel of a Linear Transformation

After this lecture you should:

- Be able to **identify the image** of a linear transformation (and its associated matrix) — $\text{im}(A)$
- Be able to **identify the kernel** of a linear transformation (and its associated matrix) — $\text{ker}(A)$
- Know what the **span** of a set of vectors is.
- Know when $\text{ker}(A) = \{\vec{0}\}$? — and the implications [THE CHARACTERISTICS OF INVERTIBLE MATRICES]



Fair Warning

Things get quite “math-y” starting now.



Subspaces of \mathbb{R}^n and Their Dimensions Suggested Problems

Image & Kernel of a Linear Transformation

Image of a Linear Transformation

Definition (Image of a Function (Linear Transformation))

The **image** of a function consists of all the values the function takes in its target space. If $f : X \mapsto Y$, then

$$\begin{aligned}\text{image}(f) &= \{f(x) : x \in X\} \\ &= \{b \in Y : b = f(x), \text{ for some } x \in X\}.\end{aligned}$$

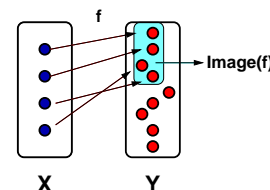


Fig: © 2019 Peter Blomgren

Figure: X is the *domain* of f ; Y the *target space* of f ; and the shaded subset of Y is the *image* of f .



Notational Hazard!



Notational Warning: "Range"

Sometimes you see the term **range** in the literature; and depending on who is speaking (writing), it may refer to what we call the *image*, or (occasionally) the entire *target space*.

(In most literature range and image are the same.)



Example $e^x : \mathbb{R} \mapsto \mathbb{R}$

[NOT A LINEAR TRANSFORMATION]

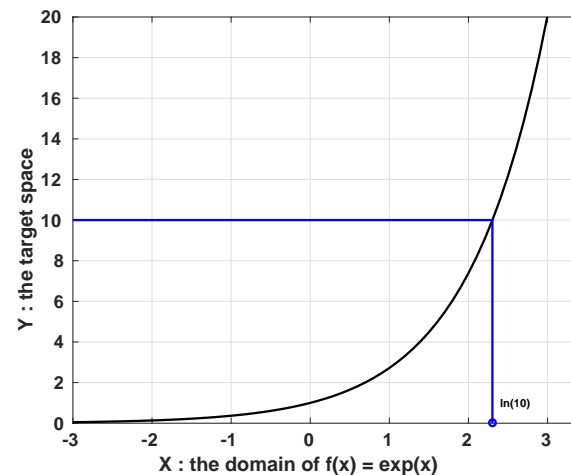


Figure: The image of $f(x) = e^x$ from \mathbb{R} to \mathbb{R} consists of \mathbb{R}^+ (all positive real numbers). Every positive number $b \in \mathbb{R}^+$ can be written as $b = e^{\ln(b)} = f(\ln(b))$.

[Figure: Copyright © 2019 Peter Blomgren]



Example: $f(t) = [\cos(t) \quad \sin(t)]^T$

[NOT A LINEAR TRANSFORMATION]



Figure: The image of $f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ from \mathbb{R} to \mathbb{R}^2 consists of the unit circle centered at the origin; f is called the *parametrization* of the unit circle.

[Figure: Copyright © 2019 Peter Blomgren]



Example: $f(t) = [\cos(t) \quad \sin(t) \quad \cos(2t)]^T$

[NOT A LINEAR TRANSFORMATION]

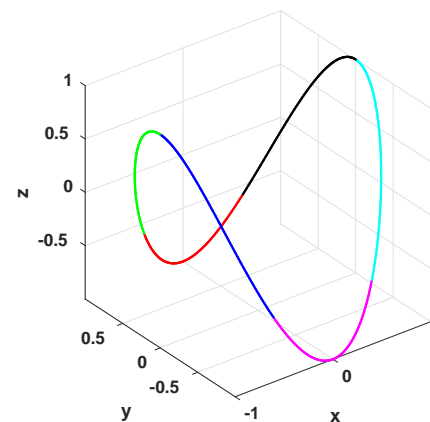


Figure: The image of $f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ \cos(2t) \end{bmatrix}$ from \mathbb{R} to \mathbb{R}^3 consists of the figure above. (Here projected from 3D to 2D for your viewing pleasure!)

[Figure: Copyright © 2019 Peter Blomgren]



Image of an Invertible Function

Image of an Invertible Function

- If the function $f : X \mapsto Y$ is *invertible*, then the image of f is (all of) Y . " $\forall b \in Y \exists x \in X : b = f(x)$."
- In this case $x = f^{-1}(b)$:

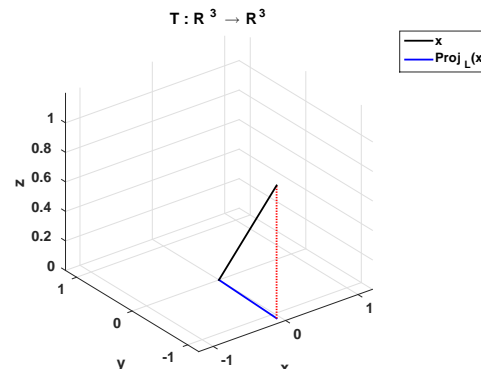
$$b = f(f^{-1}(b))$$

See also [NOTES#2.4].



Image of the Projection onto the x - y -Plane

Consider $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ that projects a vector \vec{v} orthogonally onto the x - y -plane:



\vec{x}
 $\text{Proj}_L(\vec{x})$

Figure: The image of T is the x - y -plane in \mathbb{R}^3 , consisting of all vectors of the form

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

[Figure: Copyright © 2019 Peter Blomgren]

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \Leftrightarrow T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$



$$T(\vec{x}) = A\vec{x}$$

$$\mathbb{R}^2 \mapsto \mathbb{R}^2$$

Consider $T(\vec{x}) = A\vec{x}$, with $\vec{x} \in \mathbb{R}^2$, and $A \in \mathbb{R}^{2 \times 2}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

The image is described by

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = (x_1 + 3x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

which is the line of all scalings of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

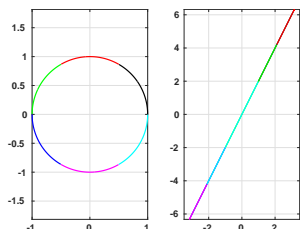


Figure: The unit circle in domain space $X = \mathbb{R}^2$, and the image of the unit circle in target space $Y = \mathbb{R}^2$. Note: we can fill $X = \mathbb{R}^2$ with circles of radii $r \in [0, \infty)$, so the image of T can be described by all scalings of the *image* of the unit circle; since $T(k\vec{x}) = Ak\vec{x} = kA\vec{x} = kT(\vec{x})$.

[Figure: Copyright © 2019 Peter Blomgren]



$$T(\vec{x}) = A\vec{x}$$

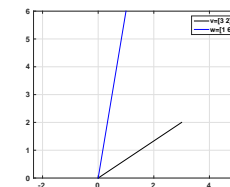
$$\mathbb{R}^2 \mapsto \mathbb{R}^2$$

Consider $T(\vec{x}) = A\vec{x}$, with $\vec{x} \in \mathbb{R}^2$, and $A \in \mathbb{R}^{2 \times 2}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}.$$

The image is described by

$$\begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



which fills out all of \mathbb{R}^2 since $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ are not parallel.

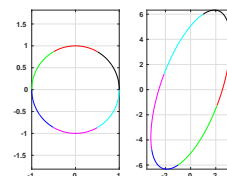


Figure: The unit circle in domain space $X = \mathbb{R}^2$, and the image of the unit circle in target space $Y = \mathbb{R}^2$. Note: we can fill $X = \mathbb{R}^2$ with circles of radii $r \in [0, \infty)$, so the image of T can be described by all scalings of the *image* of the unit circle; since $T(k\vec{x}) = Ak\vec{x} = kA\vec{x} = kT(\vec{x})$.

[Figure: Copyright © 2019 Peter Blomgren]



The Span

Describing the Linear Transformation

Definition (The Span)

Consider the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m \in \mathbb{R}^n$. The set of all linear combinations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m, \quad c_1, \dots, c_m \in \mathbb{R}$$

of the vectors is called their **span**:

$$\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m : c_1, c_2, \dots, c_m \in \mathbb{R}\}.$$



Image of a Linear Transformation — Image of A / Column Space of A

Theorem (Image of a Linear Transformation)

The image of a linear transformation $T(\vec{x}) = A\vec{x}$ is the span of the column vectors of A . We denote the image of T by $\text{im}(T)$ or $\text{im}(A)$.

 Notational Hazard (Language) 

Since $\text{im}(A)$ is the span of the columns of A , it is sometimes referred to as the **Column Space of A** , denoted $C(A)$ [GS5-3.1].



Describing the Linear Transformation

The theorem pretty much proves itself; it follows directly from how we multiply vectors and matrices:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m.$$

The vertical bars are there to illustrate that we are expressing the matrix column-wise, using the \vec{v} -vectors.



Properties of $\text{im}(T)$

Theorem (Properties of the Image)

The image of a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ has the following properties:

- a. The zero vector $\vec{0}$ in \mathbb{R}^n is in the image of T .
- b. The image of T is **closed under addition**: if \vec{v}_1 and \vec{v}_2 are in the image of T , then so is $\vec{v}_1 + \vec{v}_2$.
- c. The image of T is **closed under scalar multiplication**: if $\vec{v} \in \text{im}(T)$ and $k \in \mathbb{R}$, then $k\vec{v} \in \text{im}(T)$.



[PROOF IN THE SUPPLEMENTAL SLIDES]



[Proof] Properties of $\text{im}(T)$

[FOCUS :: MATH]

Properties of the Image.

- a. $\vec{0} = A\vec{0} = T(\vec{0})$.
- b. $\exists \vec{w}_1, \vec{w}_2 \in \mathbb{R}^m: \vec{v}_1 = T(\vec{w}_1), \vec{v}_2 = T(\vec{w}_2)$. Then $\vec{v}_1 + \vec{v}_2 = T(\vec{w}_1) + T(\vec{w}_2) \stackrel{\text{L.T.}}{=} T(\vec{w}_1 + \vec{w}_2) \Rightarrow (\vec{v}_1 + \vec{v}_2) \in \text{im}(T)$.
- c. If $\vec{v} = T(\vec{w})$, then $k\vec{v} = kT(\vec{w}) \stackrel{\text{L.T.}}{=} T(k\vec{w}) \Rightarrow k\vec{v} \in \text{im}(T)$. □

[b.] + [c.] $\Rightarrow \text{im}(T)$ is closed under linear combinations.



[Parsing the Proof] Properties of $\text{im}(T)$

[FOCUS :: MATH]

Properties of the Image.

- a. $\vec{0} = A\vec{0} = T(\vec{0})$.
 - This follows straight from how we compute matrix-vector products; given $A \in \mathbb{R}^{n \times m}$, and $T(\vec{x}) = A\vec{x}$, we immediately get $A\vec{0}_m = \vec{0}_n$, where the subscript on the $\vec{0}$ -vector indicates its number of components.
 - b. $\exists \vec{w}_1, \vec{w}_2 \in \mathbb{R}^m: \vec{v}_1 = T(\vec{w}_1), \vec{v}_2 = T(\vec{w}_2)$.
 - Since \vec{v}_1 and \vec{v}_2 are in the image; there must exist (“ \exists ”) vectors \vec{w}_1 , and \vec{w}_2 so that $\vec{v}_1 = T(\vec{w}_1), \vec{v}_2 = T(\vec{w}_2)$ {some input must generate the output!}
- Then $(\vec{v}_1 + \vec{v}_2) \stackrel{!}{=} T(\vec{w}_1) + T(\vec{w}_2) \stackrel{\text{L.T.}}{=} T(\vec{w}_1 + \vec{w}_2) \Rightarrow (\vec{v}_1 + \vec{v}_2) \in \text{im}(T)$.
- First we write the vector we want to show is in the image ($\vec{v}_1 + \vec{v}_2$); then
 - (“ $\stackrel{!}{=}$ ”) we use the fact that each vector is in the image; followed by
 - (“ $\stackrel{\text{L.T.}}{=}$ ”) the fact that T is a linear transformation; and we can conclude
 - (“ \Rightarrow ”) that we wrote $(\vec{v}_1 + \vec{v}_2)$ as the linear transformation of some vector $\vec{w}^* = (\vec{w}_1 + \vec{w}_2)$, which makes $\vec{v}^* = (\vec{v}_1 + \vec{v}_2)$ a member of the image. □



[Parsing the Proof] Properties of $\text{im}(T)$

[FOCUS :: MATH]

Properties of the Image.

- c. If $\vec{v} = T(\vec{w})$, then $k\vec{v} = kT(\vec{w}) \stackrel{\text{L.T.}}{=} T(k\vec{w}) \Rightarrow k\vec{v} \in \text{im}(T)$.
 - This is very similar to part ., given a vector \vec{v} in the image; there must be a vector \vec{w} in the domain, so that $\vec{v} = T(\vec{w})$
 - We want to show that $k\vec{v}$ is in the image; so we use $k\vec{v} = kT(\vec{w})$,
 - then the fact that T is a linear transformation: $kT(\vec{w}) = T(k\vec{w})$;
 - and conclude as in part b. □



The Kernel of a Linear Transformation

Definition (Kernel / Null Space)

The **kernel** (aka “null space”) of a linear transformation $T(\vec{x}) = A\vec{x}$ from \mathbb{R}^m to \mathbb{R}^n consists of all zeros of the transformation; that is, the solutions of the equation $T(\vec{x}) = A\vec{x} = \vec{0}$.

In other words, the kernel of T is the solution of the set of linear equations

$$A\vec{x} = \vec{0}$$

We denote the kernel of T by $\ker(T)$ or $\ker(A)$.

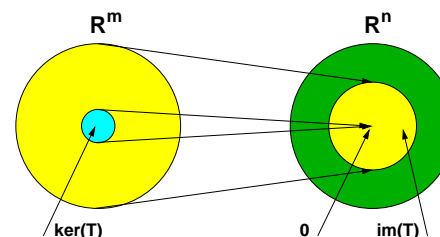


Figure: $\ker(T)$ are the elements in the domain that are transformed to 0 in the output space; the rest of the domain “paints” $\text{im}(T)$. Notice that there may be element of the output space that are NOT part of $\text{im}(T)$.

[Figure: Copyright © 2019 Peter Blomgren]



$$T : \mathbb{R}^m \mapsto \mathbb{R}^n$$

$$\text{im}(T) \subset \mathbb{R}^n$$

$$\ker(T) \subset \mathbb{R}^m$$

For the linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$,

- $\text{im}(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{R}^m\}$ is a subset of the *target space* \mathbb{R}^n of T ;
- $\ker(T) = \{\vec{x} \in \mathbb{R}^m : T(\vec{x}) = \vec{0}\}$ is a subset of the *domain*.

WARNING Notational Hazard (Language) **WARNING**

[GS5-3.2] uses the notation $N(A)$ for the null space (and [GS5-3.1] $C(A)$ for the image / column space). We will use $\ker(A)$ and $\text{im}(A)$ exclusively.

A more common notational variant for the kernel is $\text{null}(A)$.



Example: $\mathbb{R}^3 \mapsto \mathbb{R}^3$

Projection onto the x - y plane in \mathbb{R}^3

Consider, again, the linear transformation:

$$T \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \Leftrightarrow T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Clearly,

$$T \left(\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \forall z \in \mathbb{R}.$$

Therefore,

$$\ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : \forall z \in \mathbb{R} \right\}, \quad \text{also } \text{im}(T) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : \forall x, y \in \mathbb{R} \right\}.$$



Example: $\mathbb{R}^5 \mapsto \mathbb{R}^4$

Find $\ker(A)$

Consider $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$

Let's find the *kernel* (solve $A\vec{x} = \vec{0}$)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array} \right]$$



Example: $\mathbb{R}^5 \mapsto \mathbb{R}^4$

 $\text{rref}([A|\vec{b}])$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & -3 & 12 & -15 & 0 \\ 0 & 0 & -5 & 20 & -25 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$



Example: $\mathbb{R}^5 \mapsto \mathbb{R}^4$

$$\left[\begin{array}{ccccc|c} \textcircled{1} & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & \textcircled{1} & -4 & 5 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} \text{rank}(A) \\ \text{number-of-leading-variables} \\ \text{number-of-free-variables} \end{array} \quad \begin{array}{l} = 2 \\ = 2 \\ = 3 \end{array}$$

Now, the equations

$$\begin{cases} x_1 = -2x_2 - 3x_4 + 4x_5 \\ x_3 = 4x_4 - 5x_5 \end{cases}$$

describe the kernel. As usual we let $\{x_2 = s, x_4 = t, x_5 = u\}$, and write:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 3t + 4u \\ s \\ 4t - 5u \\ t \\ u \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Example: $\mathbb{R}^5 \mapsto \mathbb{R}^4$

Given that the parameters, $\{s, t, u\}$ are allowed to independently vary over $(-\infty, \infty)$, we are interested in all combinations of the 3 vectors...

Using the previously defined concept of *span*, we write

$$\ker(T) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$



Properties of the Kernel

Theorem (Some Properties of the Kernel)

Consider the linear transform $T : \mathbb{R}^m \mapsto \mathbb{R}^n$,

- The zero vector $\vec{0}$ in \mathbb{R}^m is in $\ker(T)$.
- The kernel is **closed under addition**.
- The kernel is **closed under scalar multiplication**.

The proofs for these properties are small modifications of the proofs of the analogous properties for the Image (SEE THE EXTENDED NOTES)... and are left as an exercise.

When is $\ker(A) = \{\vec{0}\}$?Theorem (When is $\ker(A) = \{\vec{0}\}$?)

- Consider an $(n \times m)$ matrix A . Then $\ker(A) = \{\vec{0}\}$ if and only if $\text{rank}(A) = m$.
- Consider an $(n \times m)$ matrix A . If $\ker(A) = \{\vec{0}\}$, then $m \leq n$. Equivalently, if $m > n$, then there are non-zero vectors in the kernel of A .
- For a square matrix A , we have $\ker(A) = \{\vec{0}\}$ if and only if A is **invertible**.



Characteristics of Invertible Matrices

IMPORTANT!

Equivalent Statements: Invertible Matrices

For an $(n \times n)$ matrix A , the following statements are equivalent; that is for a given A , they are either all true or all false:

- i. A is invertible ($\exists A^{-1}$)
- ii. The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}, \forall \vec{b} \in \mathbb{R}^n$
- iii. $\text{rref}(A) = I_n$
- iv. $\text{rank}(A) = n$
- v. $\text{im}(A) = \mathbb{R}^n$
- vi. $\text{ker}(A) = \{\vec{0}\}$

We will add to this list throughout the semester: [NOTES#2.4]^v, [NOTES#3.3], and [NOTES#7.1].



Suggested Problems 3.1

Available on Learning Glass videos:

3.1 — 1, 7, 11, 14, 15, 17, 23, 24, 29, 39



Lecture – Book Roadmap

Lecture	Book, [GS5-]
3.1	§3.1, §3.2, §3.3
3.2	§3.1, §3.2, §3.3, §3.4
3.3	§3.1, §3.2, §3.3, §3.4, §3.5
3.4	



Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		



(3.1.1), (3.1.7), (3.1.11)

(3.1.1) Find vectors that span the *kernel* of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(3.1.7) Find vectors that span the *kernel* of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

(3.1.11) Find vectors that span the *kernel* of

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$$



(3.1.14), (3.1.15), (3.1.17)

(3.1.14) Find vectors that span the *image* of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

(3.1.15) Find vectors that span the *image* of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(3.1.17) Describe the *image* of the transformation $T(\vec{x}) = A\vec{x}$ geometrically (e.g. as a line, a plane, etc. in \mathbb{R}^2 or \mathbb{R}^3 .)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$



(3.1.23), (3.1.24)

(3.1.23) Describe the *image* and *kernel* of the transformation $T(\vec{x}) = A\vec{x}$ geometrically, where

$$T(\vec{x}) = \left\{ \begin{array}{l} \text{REFLECTION ABOUT THE LINE} \\ \{y = x/3\} \text{ IN } \mathbb{R}^2 \end{array} \right\}.$$

(3.1.24) Describe the *image* and *kernel* of the transformation $T(\vec{x}) = A\vec{x}$ geometrically, where

$$T(\vec{x}) = \left\{ \begin{array}{l} \text{ORTHOGONAL PROJECTION ONTO} \\ \text{THE PLANE } \{x + 2y + 3z = 0\} \text{ IN } \mathbb{R}^3 \end{array} \right\}.$$



(3.1.29), (3.1.39)

(3.1.29) Give an example of a function whose image is the unit sphere

$$\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\} \text{ in } \mathbb{R}^3.$$

(3.1.39) Consider a square matrix A :

- What is the relationship among $\ker(A)$ and $\ker(A^2)$? Are they necessarily equal?? Is one of them necessarily contained in the other? More generally what can you say about $\ker(A)$, $\ker(A^2)$, $\ker(A^3)$, ...?
- What can you say about $\text{im}(A)$, $\text{im}(A^2)$, $\text{im}(A^3)$, ...?

