Math 254: Introduction to Linear Algebra

Notes #3.1 — Image & Kernel of a Linear Transform

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Outline

1. Student Learning Objectives
   - SLOs: Image & Kernel of a Linear Transform
2. Subspaces of $\mathbb{R}^n$ and Their Dimensions
   - Image & Kernel of a Linear Transformation
3. Suggested Problems
   - Suggested Problems 3.1
   - Lecture – Book Roadmap
4. Supplemental Material
   - Metacognitive Reflection
   - Problem Statements 3.1

SLOs 3.1 Image & Kernel of a Linear Transform

After this lecture you should:

- Be able to identify the *image* of a linear transformation (and its associated matrix) — $\mathrm{im}(A)$
- Be able to identify the *kernel* of a linear transformation (and its associated matrix) — $\ker(A)$
- Know what the *span* of a set of vectors is.
- Know when $\ker(A) = \{\vec{0}\}$? — and the implications [The Characteristics of Invertible Matrices]

WARNING Fair Warning WARNING

Things get quite "math-y" starting now.

Image of a Linear Transformation

Definition (Image of a Function (Linear Transformation))

The **image** of a function consists of all the values the function takes in its target space. If $f : X \mapsto Y$, then

$$\begin{aligned} \mathrm{image}(f) &= \{ f(x) : x \in X \} \\ &= \{ b \in Y : b = f(x), \text{ for some } x \in X \}. \end{aligned}$$

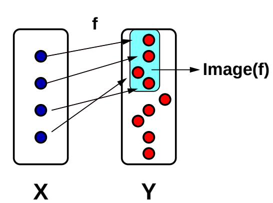


Fig: © 2019 Peter Blomgren

Figure: X is the *domain* of f; Y the *target space* of f; and the shaded subset of Y is the *image* of f.

Notational Hazard!

Notational Warning: "Range"

Sometimes you see the term **range** in the literature; and depending on who is speaking (writing), it may refer to what we call the *image*, or (occasionally) the entire *target space*.

(**In most literature range and image are the same.**)

Example $e^x : \mathbb{R} \mapsto \mathbb{R}$ [NOT A LINEAR TRANSFORMATION]

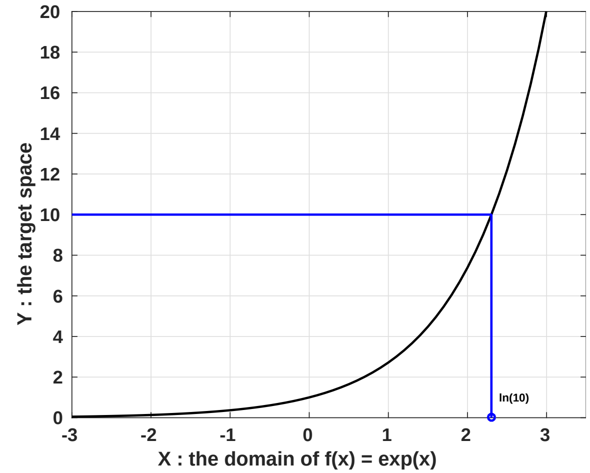


Figure: The image of $f(x) = e^x$ from $\mathbb{R}$ to $\mathbb{R}$ consists of $\mathbb{R}^+$ (all positive real numbers). Every positive number $b \in \mathbb{R}^+$ can be written as $b = e^{\ln(b)} = f(\ln(b))$.

Example: $f(t) = \begin{bmatrix} \cos(t) & \sin(t) \end{bmatrix}^T$ [NOT A LINEAR TRANSFORMATION]

Figure: The image of $f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ from $\mathbb{R}$ to $\mathbb{R}^2$ consists of the unit circle centered at the origin; f is called the *parametrization* of the unit circle.

Example: $f(t) = \begin{bmatrix} \cos(t) & \sin(t) & \cos(2t) \end{bmatrix}^T$ [NOT A LINEAR TRANSFORMATION]

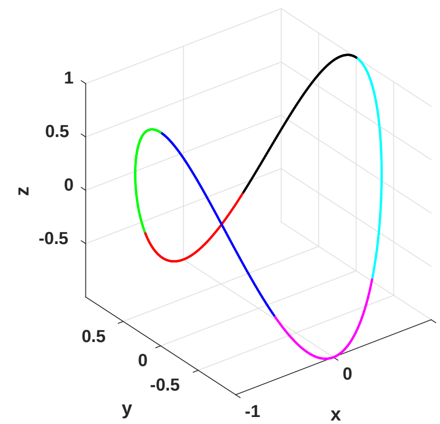


Figure: The image of $f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ \cos(2t) \end{bmatrix}$ from $\mathbb{R}$ to $\mathbb{R}^3$ consists of the figure above. (Here projected from 3D to 2D for your viewing pleasure!)

Image of an Invertible Function

Image of an Invertible Function

- If the function $f : X \mapsto Y$ is *invertible*, then the image of f is (all of) Y. "$\forall b \in Y\ \exists x \in X : b = f(x)$."
- In this case $x = f^{-1}(b)$:

$$b = f\left(f^{-1}(b)\right)$$

See also [NOTES#2.4].

Image of the Projection onto the x-y-Plane

Consider $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ that projects a vector $\vec{v}$ orthogonally onto the x-y-plane:

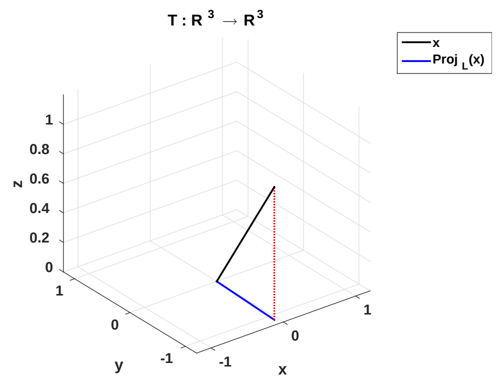


Figure: The image of T is the x-y-plane in $\mathbb{R}^3$, consisting of all vectors of the form

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

[Figure: Copyright © 2019 Peter Blomgren]

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

$T(\vec{x}) = A\vec{x}$ $\qquad \mathbb{R}^2 \mapsto \mathbb{R}^2$

Consider $T(\vec{x}) = A\vec{x}$, with $\vec{x} \in \mathbb{R}^2$, and $A \in \mathbb{R}^{2\times 2}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

The image is described by

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = (x_1 + 3x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

which is the line of all scalings of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

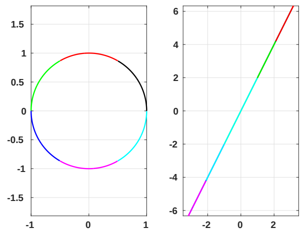


Figure: The unit circle in domain space $X = \mathbb{R}^2$, and the image of the unit circle in target space $Y = \mathbb{R}^2$. Note: we can fill $X = \mathbb{R}^2$ with circles of radii $r \in [0, \infty)$, so the image of T can be described by all scalings of the *image* of the unit circle; since $T(k\vec{x}) = Ak\vec{x} = kA\vec{x} = kT(\vec{x})$.

[Figure: Copyright © 2019 Peter Blomgren]

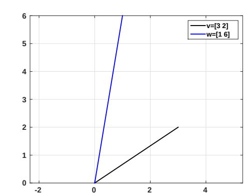
$T(\vec{x}) = A\vec{x}$ $\qquad \mathbb{R}^2 \mapsto \mathbb{R}^2$

Consider $T(\vec{x}) = A\vec{x}$, with $\vec{x} \in \mathbb{R}^2$, and $A \in \mathbb{R}^{2\times 2}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}.$$

The image is described by

$$\begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



which fills out all of $\mathbb{R}^2$ since $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ are not parallel.

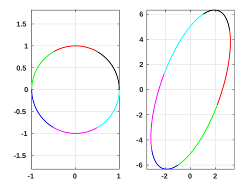


Figure: The unit circle in domain space $X = \mathbb{R}^2$, and the image of the unit circle in target space $Y = \mathbb{R}^2$. Note: we can fill $X = \mathbb{R}^2$ with circles of radii $r \in [0, \infty)$, so the image of T can be described by all scalings of the *image* of the unit circle; since $T(k\vec{x}) = Ak\vec{x} = kA\vec{x} = kT(\vec{x})$.

[Figure: Copyright © 2019 Peter Blomgren]

The Span — Describing the Linear Transformation

Definition (The Span)

Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n$. The set of all linear combinations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m, \quad c_1, \ldots, c_m \in \mathbb{R}$$

of the vectors is called their **span**:

$$\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m) = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m \;:\; c_1, c_2, \ldots, c_m \in \mathbb{R}\}.$$

Image of a Linear Transformation — Image of A / Column Space of A

Theorem (Image of a Linear Transformation)

The image of a linear transformation $T(\vec{x}) = A\vec{x}$ is the span of the column vectors of A. We denote the image of T by $\mathrm{im}(T)$ or $\mathrm{im}(A)$.

Notational Hazard (Language)

Since $\mathrm{im}(A)$ is the span of the columns of A, it is sometimes referred to as the **Column Space of** A, denoted $C(A)$ [GS5–3.1].

Describing the Linear Transformation

The theorem pretty much proves itself; it follows directly from how we multiply vectors and matrices:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \ldots & \vec{v}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \cdots + x_m\vec{v}_m.$$

The vertical bars are there to illustrate that we are expressing the matrix column-wise, using the $\vec{v}$-vectors.

Properties of $\mathrm{im}(T)$

Theorem (Properties of the Image)

The image of a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$ has the following properties:

- **a.** *The zero vector $\vec{0}$ in $\mathbb{R}^n$ is in the image of T.*
- **b.** *The image of T is* **closed under addition**: *if $\vec{v}_1$ and $\vec{v}_2$ are in the image of T, then so is $\vec{v}_1 + \vec{v}_2$.*
- **c.** *The image of T is* **closed under scalar multiplication**: *if $\vec{v} \in \mathrm{im}(T)$ and $k \in \mathbb{R}$, then $k\vec{v} \in \mathrm{im}(T)$.*

Proof

[Proof in the Supplemental Slides]

[Proof] Properties of $\mathrm{im}(T)$ [Focus :: Math]

Properties of the Image.

a. $\vec{0} = A\vec{0} = T(\vec{0})$.

b. $\exists \vec{w}_1, \vec{w}_2 \in \mathbb{R}^m$: $\vec{v}_1 = T(\vec{w}_1)$, $\vec{v}_2 = T(\vec{w}_2)$. Then $\vec{v}_1 + \vec{v}_2 = T(\vec{w}_1) + T(\vec{w}_2) \stackrel{\text{L.T.}}{=} T(\vec{w}_1 + \vec{w}_2) \Rightarrow (\vec{v}_1 + \vec{v}_2) \in \mathrm{im}(T)$.

c. If $\vec{v} = T(\vec{w})$, then $k\vec{v} = kT(\vec{w}) \stackrel{\text{L.T.}}{=} T(k\vec{w})$. $\Rightarrow k\vec{v} \in \mathrm{im}(T)$. $\square$

[b.] + **[c.]** $\Rightarrow \mathrm{im}(T)$ is closed under linear combinations.

[Parsing the Proof] Properties of $\mathrm{im}(T)$ [Focus :: Math]

Properties of the Image.

a. $\vec{0} = A\vec{0} = T(\vec{0})$.
- This follows straight from how we compute matrix-vector products; given $A \in \mathbb{R}^{n\times m}$, and $T(\vec{x}) = A\vec{x}$, we immediately get $A\vec{0}_m = \vec{0}_n$, where the subscript on the $\vec{0}$-vector indicates its number of components.

b. $\exists \vec{w}_1, \vec{w}_2 \in \mathbb{R}^m$: $\vec{v}_1 = T(\vec{w}_1)$, $\vec{v}_2 = T(\vec{w}_2)$.
- Since $\vec{v}_1$ and $\vec{v}_2$ are in the image; there must exist ("$\exists$") vectors $\vec{w}_1$, and $\vec{w}_2$ so that $\vec{v}_1 = T(\vec{w}_1)$, $\vec{v}_2 = T(\vec{w}_2)$ {*some input must generate the output!*}

Then $(\vec{v}_1 + \vec{v}_2) \stackrel{1}{=} T(\vec{w}_1) + T(\vec{w}_2) \stackrel{\text{L.T.}}{=} T(\vec{w}_1 + \vec{w}_2) \Rightarrow (\vec{v}_1 + \vec{v}_2) \in \mathrm{im}(T)$.
- First we write the vector we want to show is in the image $(\vec{v}_1 + \vec{v}_2)$; then
- ("$\stackrel{1}{=}$") we use the fact that each vector is in the image; followed by
- ("$\stackrel{\text{L.T.}}{=}$") the fact that T is a linear transformation; and we can conclude
- ("$\Rightarrow$") that we wrote $(\vec{v}_1 + \vec{v}_2)$ as the linear transformation of some vector $\vec{w}^* = (\vec{w}_1 + \vec{w}_2)$, which makes $\vec{v}^* = (\vec{v}_1 + \vec{v}_2)$ a member of the image.

$\square$

[Parsing the Proof] Properties of $\mathrm{im}(T)$ [Focus :: Math]

Properties of the Image.

c. If $\vec{v} = T(\vec{w})$, then $k\vec{v} = kT(\vec{w}) \stackrel{\text{L.T.}}{=} T(k\vec{w})$. $\Rightarrow k\vec{v} \in \mathrm{im}(T)$.
- This is very similar to part ., given a vector $\vec{v}$ in the image; there must be a vector $|vecw$ in the domain, so that $\vec{v} = T(\vec{w})$
- We want to show that $k\vec{v}$ is in the image; so we use $k\vec{v} = kT(\vec{w})$,
- then the fact that T is a linear transformation: $kT(\vec{w}) = T(k\vec{w})$;
- and conclude as in part b.

$\square$

The Kernel of a Linear Transformation

Definition (Kernel / Null Space)

The **kernel** (aka *"null space"*) of a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ consists of all zeros of the transformation; that is, the solutions of the equation $T(\vec{x}) = A\vec{x} = \vec{0}$.

In other words, the kernel of T is the solution of the set of linear equations

$$A\vec{x} = \vec{0}$$

We denote the kernel of T by $\ker(T)$ or $\ker(A)$.

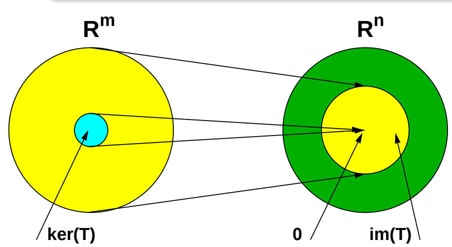


Figure: $\ker(T)$ are the elements in the domain that are transformed to 0 in the output space; the rest of the domain "paints" $\mathrm{im}(T)$. Notice that there may be element of the output space that are NOT part of $\mathrm{im}(T)$.

[Figure: Copyright © 2019 Peter Blomgren]

$T : \mathbb{R}^m \mapsto \mathbb{R}^n$ $\quad \text{im}(T) \subset \mathbb{R}^n$ $\quad \ker(T) \subset \mathbb{R}^m$

For the linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$,

- $\text{im}(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{R}^m\}$ is a subset of the *target space* $\mathbb{R}^n$ of T;
- $\ker(T) = \left\{\vec{x} \in \mathbb{R}^m : T(\vec{x}) = \vec{0}\right\}$ is a subset of the *domain*.

Notational Hazard (Language)

[GS5–3.2] uses the notation $N(A)$ for the null space (and [GS5–3.1] $C(A)$ for the image / column space). We will use $\ker(A)$ and $\text{im}(A)$ exclusively.

A more common notational variant for the kernel is $\text{null}(A)$.

Example: $\mathbb{R}^3 \mapsto \mathbb{R}^3$ — Projection onto the x-y plane in $\mathbb{R}^3$

Consider, again, the linear transformation:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Clearly,

$$T\left(\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \forall z \in \mathbb{R}.$$

Therefore,

$$\ker(T) = \left\{\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : \forall z \in \mathbb{R}\right\}, \quad \text{also } \text{im}(T) = \left\{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : \forall x, y \in \mathbb{R}\right\}.$$

Example: $\mathbb{R}^5 \mapsto \mathbb{R}^4$ — Find $\ker(A)$

Consider $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$

Let's find the *kernel* (solve $A\vec{x} = \vec{0}$)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right]$$

Example: $\mathbb{R}^5 \mapsto \mathbb{R}^4$ — $\text{rref}([A|\vec{b}])$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 0 \\ -1 & -2 & -1 & 1 & -1 & 0 \\ 4 & 8 & 5 & -8 & 9 & 0 \\ 3 & 6 & 1 & 5 & -7 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & -3 & 12 & -15 & 0 \\ 0 & 0 & -5 & 20 & -25 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Example: $\mathbb{R}^5 \mapsto \mathbb{R}^4$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{lcl} \mathrm{rank}(A) & = & 2 \\ \texttt{number-of-leading-variables} & = & 2 \\ \texttt{number-of-free-variables} & = & 3 \end{array}$$

Now, the equations

$$\begin{cases} x_1 &= -2x_2 - 3x_4 + 4x_5 \\ x_3 &= 4x_4 - 5x_5 \end{cases}$$

describe the kernel. As usual we let $\{ x_2 = s,\ x_4 = t,\ x_5 = u \}$, and write:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s - 3t + 4u \\ s \\ 4t - 5u \\ t \\ u \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Example: $\mathbb{R}^5 \mapsto \mathbb{R}^4$

Given that the parameters, $\{s, t, u\}$ are allowed to independently vary over $(-\infty, \infty)$, we are interested in all combinations of the 3 vectors...

Using the previously defined concept of *span*, we write

$$\ker(T) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Properties of the Kernel

Theorem (Some Properties of the Kernel)

Consider the linear transform $T : \mathbb{R}^m \mapsto \mathbb{R}^n$,

- **a.** *The zero vector $\vec{0}$ in $\mathbb{R}^m$ is in $\ker(T)$.*
- **b.** *The kernel is* **closed under addition.**
- **c.** *The kernel is* **closed under scalar multiplication.**

The proofs for these properties are small modifications of the proofs of the analogous properties for the Image (SEE THE EXTENDED NOTES)... and are left as an exercise.

When is $\ker(A) = \{\vec{0}\}$?

Theorem (When is $\ker(A) = \{\vec{0}\}$?)

- **a.** *Consider an $(n \times m)$ matrix A. Then $\ker(A) = \{\vec{0}\}$ if and only if $\mathrm{rank}(A) = m$.*
- **b.** *Consider an $(n \times m)$ matrix A. If $\ker(A) = \{\vec{0}\}$, then $m \leq n$. Equivalently, if $m > n$, then there are non-zero vectors in the kernel of A.*
- **c.** *For a square matrix A, we have $\ker(A) = \{\vec{0}\}$ if and only if A is* **invertible.**

Characteristics of Invertible Matrices — Important!

Equivalent Statements: Invertible Matrices

For an $(n \times n)$ matrix A, the following statements are equivalent; that is for a given A, they are either all `true` or all `false`:

- **i.** A is invertible ($\exists A^{-1}$)
- **ii.** The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, $\forall \vec{b} \in \mathbb{R}^n$
- **ii.** $\text{rref}(A) = I_n$
- **iv.** $\text{rank}(A) = n$
- **v.** $\text{im}(A) = \mathbb{R}^n$
- **vi.** $\ker(A) = \{\vec{0}\}$

We will add to this list throughout the semester: [Notes#2.4]$^{\checkmark}$, [Notes#3.3], and [Notes#7.1].

Suggested Problems 3.1

Available on Learning Glass videos:
3.1 — 1, 7, 11, 14, 15, 17, 23, 24, 29, 39

Lecture – Book Roadmap

Lecture	Book, [GS5–]
3.1	§3.1, §3.2, §3.3
3.2	§3.1, §3.2, §3.3, §3.4
3.3	§3.1, §3.2, §3.3, §3.4, §3.5
3.4	

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(3.1.1), (3.1.7), (3.1.11)

(3.1.1) Find vectors that span the *kernel* of

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(3.1.7) Find vectors that span the *kernel* of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

(3.1.11) Find vectors that span the *kernel* of

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$$

(3.1.14), (3.1.15), (3.1.17)

(3.1.14) Find vectors that span the *image* of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

(3.1.15) Find vectors that span the *image* of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

(3.1.17) Describe the *image* of the transformation $T(\vec{x}) = A\vec{x}$ geometrically (*e.g.* as a line, a plane, etc. in $\mathbb{R}^2$ or $\mathbb{R}^3$.)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

(3.1.23), (3.1.24)

(3.1.23) Describe the *image* and *kernel* of the transformation $T(\vec{x}) = A\vec{x}$ geometrically, where

$$T(\vec{x}) = \left\{ \begin{array}{c} \text{Reflection about the line} \\ \{y = x/3\} \text{ in } \mathbb{R}^2 \end{array} \right\}.$$

(3.1.24) Describe the *image* and *kernel* of the transformation $T(\vec{x}) = A\vec{x}$ geometrically, where

$$T(\vec{x}) = \left\{ \begin{array}{c} \text{Orthogonal projection onto} \\ \text{the plane } \{x + 2y + 3z = 0\} \text{ in } \mathbb{R}^3 \end{array} \right\}.$$

(3.1.29), (3.1.39)

(3.1.29) Give an example of a function whose image is the unit sphere

$$\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\} \text{ in } \mathbb{R}^3.$$

(3.1.39) Consider a square matrix A:

a. What is the relationship among $\ker(A)$ and $\ker(A^2)$? Are they necessarily equal?? Is one of them necessarily contained in the other? More generally what can you say about $\ker(A)$, $\ker(A^2)$, $\ker(A^3)$, $\ldots$?

b. What can you say about $\mathrm{im}(A)$, $\mathrm{im}(A^2)$, $\mathrm{im}(A^3)$, $\ldots$?