Math 254: Introduction to Linear Algebra

Notes #3.2 — Bases and Linear Independence

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Outline

SLOs 3.2 — Image & Bases and Linear Independence

After this lecture you should:

- Know the definition of SUBSPACES; be comfortable with the concepts of the IMAGE and KERNEL of a linear transformation (and/or its associated matrix A).
- Know how (i) the SPAN, (ii) LINEAR INDEPENDENCE, and (iii) the BASIS OF A SUBSPACE are inter-related.
- Be familiar with the Equivalent properties of LINEARLY INDEPENDENT Vectors.

Equivalent Language [GS5–3.1–3.2]

- IMAGE: "Column Space (of a Matrix)."
- KERNEL: "Null Space (of a Matrix)."

Subspaces of $\mathbb{R}^n$

In a previous episode of *"Adventures in Linear Algebra"* we encountered the *image* and *kernel* of a linear transform. It turns out both have three particular properties that fit into a more general classification:

Definition (Subspaces of $\mathbb{R}^n$)

A subset W of the vector space $\mathbb{R}^n$ is called a (linear) **subspace** of $\mathbb{R}^n$ if it has the following three properties:

1. W contains the zero vector.
2. W is closed under addition^{*1}.
3. W is closed under scalar multiplication^{*2}.

^{*1} — if $\vec{w}_1, \vec{w}_2 \in W$, then $\vec{w}_1 + \vec{w}_2 \in W$; and
^{*2} — if $\vec{w} \in W$ and $\alpha \in \mathbb{R}$, then $\alpha\vec{w} \in W$.

Subspaces of $\mathbb{R}^n$

Theorem (Image and Kernel of a Linear Transform are Subspaces)

If $T(\vec{x}) = A\vec{x}$ is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, then

- $\ker(T) = \ker(A)$ *is a subspace of* $\mathbb{R}^m$, *and*
- $\mathrm{im}(T) = \mathrm{im}(A)$ *is a subspace of* $\mathbb{R}^n$.

The proof for the image, $\mathrm{im}(A)$, is in [NOTES#3.1]; we left the (very similar) proof for $\ker(A)$ as an exercise for a dark and stormy night.

However, recall the "cartoon" illustration

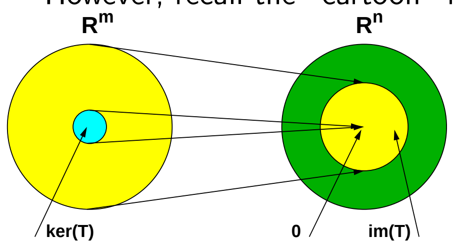

Figure: $\ker(T)$ are the elements in the domain that are transformed to 0 in the output space; the rest of the domain "paints" $\mathrm{im}(T)$. Notice that there may be element of the output space that are NOT part of $\mathrm{im}(T)$.

Example: Subspaces of $\mathbb{R}^2$

Example (Subspaces of $\mathbb{R}^2$)

There are infinitely many subspaces of $\mathbb{R}^2$; they fall into one of three categories:

- $W_0 = \{\vec{0}\}$.
- $W_1 = \{k\vec{v} : \forall k \in \mathbb{R}\}$, where $\vec{v} \in \mathbb{R}^2$ and $\vec{v} \neq \vec{0}$.
- $W_2 = \mathbb{R}^2$.

W_0 is quite straight-forward.

W_1 Once we have one non-zero vector $\vec{v}$ we must add all scalings and additions of copies of $\vec{v}$ to the space which gives the infinite line going through the origin parallel to $\vec{v}$.

W_2 If we have two non-parallel vectors $\vec{v}$ and $\vec{w}$ we must include all scalings of the parallelogram described by $\vec{0}$-$\vec{v}$-$\vec{w}$-$(\vec{v}+\vec{w})$... which fills all of $\mathbb{R}^2$.

Example: Subspaces of $\mathbb{R}^3$

Example (Subspaces of $\mathbb{R}^3$)

There are infinitely many subspaces of $\mathbb{R}^3$; they fall into one of four categories:

- $W_0 = \{\vec{0}\}$.
- $W_1 = \{k\vec{v} : \forall k \in \mathbb{R}\}$, where $\vec{v} \in \mathbb{R}^3$ — LINES THROUGH $\vec{0}$
- $W_2 = \{k\vec{v} + \ell\vec{w} : \forall k, \ell \in \mathbb{R}\}$, where $\vec{v}, \vec{w} \in \mathbb{R}^3$, and $\vec{v}$ and $\vec{w}$ are not parallel — PLANES THROUGH $\vec{0}$,
- $W_3 = \mathbb{R}^3$.

Note that the planes of type W_2 are not necessarily parallel to any (standard) coordinate axis.

Figure: In the game "labyrinth," we tilt (part) of a plane in $\mathbb{R}^3$ to move a marble from start-to-finish.

Describing a Plane in 3D... Kernel version

Example (Kernel and Image of V)

Consider the plane $V \in \mathbb{R}^3$ given by the equation $x_1 + 2x_2 + 3x_3 = 0$. Express V as the kernel of a matrix; and the image of (another) matrix.

a. First we find a matrix A so that $V = \ker(A)$:

- We can write the equation as

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

and clearly, we are looking for $\ker(A)$.

[Useful Point of View] If we are thinking about $A\vec{x}$ in terms of dot-product($\langle$ROW-OF-A$\rangle$, $\vec{x}$); we can interpret this situation as finding all $\vec{x} \perp$ all rows of A.

Describing a Plane in 3D... Image version

Example (Kernel and Image of V)

Consider the plane $V \in \mathbb{R}^3$ given by the equation $x_1 + 2x_2 + 3x_3 = 0$. Express V as the kernel of a matrix; and the image of (another) matrix.

b. Second, we find a matrix B so that $V = \mathrm{im}(B)$:

- We need **two non-parallel vectors in the plane** in order to describe it: First, let $x_3 = 0$, giving $x_1 = -2x_2$ as a possibility; then let $x_2 = 0$, giving $x_1 = -3x_3$ as a possibility. Alternatively, we can parameterize in the usual way $\{x_2 = s,\ x_3 = t\}$ and get the (same) two vectors as:

$$s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

Since V consists of all linear combinations of these vectors,

$$V = \mathrm{im}\left(\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \equiv \mathrm{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\right).$$

How Many Column Vectors Do We Need to Describe the Image / Span?

Next, consider

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}$$

Since $A \in \mathbb{R}^{3\times4}$ its image "lives in" (is a subspace of) $\mathbb{R}^3$ — $\mathrm{im}(A) \subset \mathbb{R}^3$, and kernel $\ker(A) \subset \mathbb{R}^4$.

We notice that $\vec{v}_2 = 2\vec{v}_1$, and $\vec{v}_4 = \vec{v}_1 + \vec{v}_3$; that is the vectors $\vec{v}_2$ and $\vec{v}_4$ are "redundant" as far as describing the image is concerned (we can describe them using other columns in the matrix):

$$\mathrm{im}\left(\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}\right) = \mathrm{span}\left(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\right) = \mathrm{span}\left(\vec{v}_1, \vec{v}_3\right)$$

If we have a vector $\vec{v} \in \mathbb{R}^3$:

$$\begin{aligned} \vec{v} &= \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \alpha_4\vec{v}_4 \\ &= (\alpha_1 + 2\alpha_2 + \alpha_4)\vec{v}_1 + (\alpha_3 + \alpha_4)\vec{v}_3 \end{aligned}$$

Linear Independence; Basis Key Concept!

Definition (Linear Independence; Basis)

Consider non-zero vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$.

- We say that a vector $\vec{v}_i$ is *linearly* **dependent** if it is a linear combination of the preceding vectors, $\vec{v}_1, \ldots, \vec{v}_{i-1}$
- The vectors $\vec{v}_1, \ldots, \vec{v}_m$ are **linearly independent** if none of them can be written as a linear combination of the others.
- We say that the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$ form a **basis** of V if they span V and are linearly independent.

Informally, a basis is a minimal description of a (sub)space.

A Basis for the Image

In the context of the previously considered matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

we have established that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

give us a (linearly independent) basis of $V = \mathrm{im}(A)$.

Constructing a Basis for the Image

Theorem (Basis of the Image)

To construct a basis of $\text{im}(A)$, list all the column vectors of A, and omit the linearly dependent vectors from the list.

Figure: Hauling vectors, to build a basis?

Linear Independence or Dependence?

Are the following vectors in $\mathbb{R}^7$ linearly independent?

$$\vec{v}_1 = \begin{bmatrix} 7\\0\\4\\0\\1\\9\\0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 6\\0\\7\\1\\4\\8\\0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 5\\0\\6\\2\\3\\1\\7 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 4\\5\\3\\3\\2\\2\\4 \end{bmatrix},$$

Since it is "very difficult" to write

- 1 as a linear combination of 0
- 7 as a linear combination of 0 and 0
- 5 as a linear combination of 0, 0, and 0

finding solutions to $\vec{v}_2 = \alpha\vec{v}_1$; $\vec{v}_3 = \beta_1\vec{v}_1 + \beta_2\vec{v}_2$; and $\vec{v}_4 = \gamma_1\vec{v}_1 + \gamma_2\vec{v}_2 + \gamma_3\vec{v}_3$, may prove slightly problematic?

We can conclude that these four vector are linearly independent.

Quick-Check for Linear Independence

The previous example gives us a quick-check for linear independence:

Theorem (Linear Independence and Zero Components)

Consider non-zero vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$.

If each of the vectors $\vec{v}_i$ has a non-zero entry in a component where all the preceding vectors $\vec{v}_1, \ldots, \vec{v}_{i-1}$ have a 0, then the vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent.

Note that the theorem applies to any ordering of the vectors; that is, if it is possible to sort them so that the theorem applies, then the vectors are linearly independent.

More Generally...

The previous theorem does not help for the vectors

$$\vec{v}_1 = \begin{bmatrix} 1\\2\\3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 4\\5\\6 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 7\\8\\9 \end{bmatrix}$$

Nothing obvious pops out — clearly $\vec{v}_2$ is not a scaling of $\vec{v}_1$...
Now, if $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$, then converting the augmented matrix

$$M = \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0\\ 2 & 5 & 8 & 0\\ 3 & 6 & 9 & 0 \end{array}\right]$$

into reduced-row-echelon-form, rref(M), will reveal those combinations!

More Generally...

$$M = \left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \implies \mathrm{rref}(M) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

which means that $x_3 = t$ (free variable), $x_1 = x_3$, and $x_2 = -2x_3$; *i.e.* the vectors are NOT linearly independent; $M\vec{x} = \vec{0}$ has infinitely many solutions of the form

$$\vec{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

We can write this as the expression (**linear relation**),

$$\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0},$$

where $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are the columns of M.

More Math Language

Definition (Linear Relations)

Consider vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. An equation of the form

$$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$$

is called a (linear) relation among the vectors. There is always the *trivial* relation, with $c_1 = \cdots = c_m = 0$. *Non-trivial relations* — where at least one c_k is non-zero — may or may not exist among the vectors.

Theorem (Relations and Linear Dependence)

The vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ are linearly dependent if and only if there are non-trivial relations among them.

Proof

Relations and Linear Dependence

Proof :: Relations and Linear Dependence [Fundamental Concept].

- Suppose vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly dependent, and $\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1}$. Then we can generate a nontrivial relation by
$$c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1} + (-1)\vec{v}_i = \vec{0}$$
- Conversely, if there is a non-trivial relation $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$, where i is the highest index such that $c_i \neq 0$, then we can solve for $\vec{v}_i$ and this express
$$\vec{v}_i = -\frac{c_1}{c_i}\vec{v}_1 - \cdots - \frac{c_{i-1}}{c_i}\vec{v}_{i-1}$$
this shows that $\vec{v}_i$ is a linear combination of the preceding vectors, and hence $\vec{v}_1, \ldots, \vec{v}_m$ are linearly dependent.

□

Example

Example (Find the Kernel)

Suppose the column vectors of an $(n \times m)$ matrix A are linearly independent. Find $\ker(A)$.

Solution: We are looking for

$$A\vec{x} = 0 \Leftrightarrow \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \vec{0} \Leftrightarrow x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = \vec{0}$$

now, since the columns are linearly independent, the trivial solution is the only solution ($x_1 = \cdots = x_m = 0$). Therefore $\ker(A) = \{\vec{0}\}$.

Summarizing

Theorem (Kernel and Relations)

The vectors in the kernel of an $(n \times m)$ matrix A correspond to the linear relations among the column vectors $\vec{v}_1, \ldots, \vec{v}_m$ of A: the equation

$$A\vec{x} = \vec{0} \quad \textit{means that} \quad x_1\vec{v}_1 + \cdots + x_m\vec{v}_m = \vec{0}.$$

In particular, the column vectors of A are linearly independent if and only if $\ker(A) = \{\vec{0}\}$, or equivalently, if and only if $\operatorname{rank}(A) = m$. This condition implies that $m \le n$.

We can find <u>at most</u> n linearly independent vectors in $\mathbb{R}^n$.

Example (Revisited) — Collecting All the Pieces in One Place

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

We have previously established that

$$\underbrace{\begin{bmatrix}7\\8\\9\end{bmatrix} = -\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}4\\5\\6\end{bmatrix}}_{\vec{v}_3 \text{ is } \cancel{\text{redundant}} \text{ linearly dependent}} \quad \Leftrightarrow \quad \underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}4\\5\\6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}}_{\exists \text{ non-trivial linear relation}} = \vec{0}$$

Collecting in matrix-vector form:

$$\Leftrightarrow \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix}1\\-2\\1\end{bmatrix} = \vec{0} \quad \Rightarrow \quad \underbrace{\begin{bmatrix}1\\-2\\1\end{bmatrix}}_{\text{non-trivial kernel}} \in \ker(\mathbf{A}).$$

Linear Independence — IMPORTANT SUMMARY

Equivalent Properties: Linear Independence

For a list $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ of vectors, the following statements are *equivalent* [all TRUE, or all FALSE]:

i. The vectors $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent

ii. None of the vectors in the list can be written as a linear combination of preceding vectors.

iii. None of the vectors can be written as a linear combination of the others.

iv. There is only the trivial solution to $c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$, *i.e.* $c_1 = \cdots = c_m = 0$

v. $\ker\left(\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_m\end{bmatrix}\right) = \{\vec{0}\}$

vi. $\operatorname{rank}\left(\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_m\end{bmatrix}\right) = m$

Basis and Unique Representation

Theorem (Basis and Unique Representation)

Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m$ in a subspace V of $\mathbb{R}^n$.

The vectors form a basis if and only if every vector $\vec{v}$ in V can be expressed uniquely as a linear combination

Proof

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$$

The coefficients $c_1, \ldots, c_m$ are called the **coordinates** *of $\vec{v}$ with respect to the basis $\vec{v}_1, \ldots, \vec{v}_m$.*

We will discuss **coordinates** in more details in [NOTES#3.4].

Basis and Unique Representation

Proof: BASIS $\Rightarrow$ UNIQUENESS [Fundamental Concept].

Let $\vec{v}_1, \ldots, \vec{v}_m$ be a basis of V.

Assume: we have two representations of some $\vec{v} \in V$:

$$\begin{aligned}\vec{v} &= c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \\ &= d_1\vec{v}_1 + \cdots + d_m\vec{v}_m.\end{aligned}$$

Subtracting gives

$$\vec{0} = (\vec{v} - \vec{v}) = (c_1 - d_1)\vec{v}_1 + \cdots + (c_m - d_m)\vec{v}_m.$$

Since $\vec{v}_1, \ldots, \vec{v}_m$ form a basis, they are (by definition) linearly independent, so $(c_k - d_k) = 0$, $\forall k \in \{1, \ldots, m\}$; which shows that the two representation must be the same. $\square$

Basis and Unique Representation

Proof: UNIQUENESS $\Rightarrow$ BASIS [Fundamental Concept].

Consider the subspace V of $\mathbb{R}^n$ spanned by the vectors $\vec{v}_1, \ldots, \vec{v}_m$.

Given that the representation

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m$$

is unique; let $\vec{v} = \vec{0}$, this forces $c_k = 0$ $\forall k \in \{1, \ldots, m\}$, which shows that the vectors are linearly independent; so we have a basis. $\square$

Suggested Problems 3.2

Available on Learning Glass videos:
3.2 — 1, 3, 7, 11, 17, 25, 27, 32, 34

Lecture – Book Roadmap

Lecture	Book, [GS5–]
3.1	§3.1, §3.2, §3.3
3.2	§3.1, §3.2, §3.3, §3.4
3.3	§3.1, §3.2, §3.3, §3.4, §3.5
3.4	

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(3.2.1), (3.2.3)

(3.2.1) Check whether or not the subset $\mathcal{W}$ of $\mathbb{R}^n$ is subspace:

$$\mathcal{W} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 1 \right\}.$$

(3.2.3) Check whether or not the subset $\mathcal{W}$ of $\mathbb{R}^n$ is subspace:

$$\mathcal{W} = \left\{ \begin{bmatrix} x + 2y + 3z \\ 4x + 5y + 6z \\ 7x + 8y + 9z \end{bmatrix} : x,\ y,\ z \in \mathbb{R} \right\}.$$

(3.2.7), (3.2.11)

(3.2.7) Consider a nonempty subset $\mathcal{W}$ of $\mathbb{R}^n$ that is closed under addition and under scalar multiplication. Is $\mathcal{W}$ necessarily a subspace of $\mathbb{R}^n$? Explain.

(3.2.11) Determine whether the given vectors are linearly independent:

$$\begin{bmatrix} 7 \\ 11 \end{bmatrix}, \quad \begin{bmatrix} 11 \\ 7 \end{bmatrix}.$$

(3.2.17), (3.2.25)

(3.2.17) Determine whether the given vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

(3.2.25) Find a linearly *dependent* (or *"redundant"*) column of the given matrix A, and write it as a linear combination of the preceding columns. Use this representation to write a non-trivial relation among the columns, and thus find a non-zero vector in the *kernel* of A:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

(3.2.27), (3.2.32)

(3.2.27) Find a basis for the *image* of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

(3.2.32) Find a basis for the *image* of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(3.2.34)

(3.2.34) Consider the (5×4) matrix

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix},$$

we are told that the vector

$$\vec{n}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

is in the *kernel* of A. Write $\vec{v}_4$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

Spring 2019 "Live Math" Debugged

A "2" went missing...

$$A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & \vec{v}_6 \end{array} \end{array}, \quad A \in \mathbb{R}^{5\times 6},\ \mathrm{rank}(A) = 3$$

$\vec{v}_2$, $\vec{v}_4$, and $\vec{v}_5$ are **linearly independent**. $\textsc{Basis}(\mathrm{im}(A)) = \{\vec{v}_2, \vec{v}_4, \vec{v}_5\}$.
$\vec{v}_1$, $\vec{v}_3$, and $\vec{v}_6$ are **linearly dependent**: the zero-vector is always "dependent," and $\vec{v}_3 = 2\vec{v}_2$, $\vec{v}_6 = 4\vec{v}_2 + 3\vec{v}_4 + 2\vec{v}_5$.
If we parameterize the free variables as usual $(x_1, x_3, x_6) = (s, t, u)$; then $A\vec{x} = 0$ has solutions:

$$s\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} 0 \\ -4 \\ 0 \\ -3 \\ -2 \\ 1 \end{bmatrix}, \quad \textsc{Basis}(\ker(A)) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 0 \\ -3 \\ -2 \\ 1 \end{bmatrix} \right\}.$$