Math 254: Introduction to Linear Algebra

Notes #3.3 — Dimension of a Subspace of $\mathbb{R}^n$

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Outline

SLOs 3.3 Dimension of a Subspace of $\mathbb{R}^n$

After this lecture you should:

- Know that the dimension of a subspace $V \subset \mathbb{R}^n$, is denoted $\dim(V)$; it is a non-negative *integer* counting the number of vectors in (all) bases for V.
- Know that for $V \subset \mathbb{R}^n$, we must have $\dim(V) \leq \dim(\mathbb{R}^n) = n$.
- Know that the trivial subspace has dimension zero: $\dim(\{\vec{0}\}) = 0$.

Common Struggles at This Point

Comment (“Basis for” vs. ”Span of” a Vector Space)

The **Basis** is the (minimal) description of the space — given a *set of* **Basis Vectors**. *Think “building materials.”*

The **Span** of the Basis Vectors (that is the collection of all linear combinations) is the space. *Think “finished product.”*

Comment (Dimension of a Space)

The dimension is always just the count of the number of vectors in the basis — *see today’s lecture!*

How many (non-zero) vectors do we need to describe a subspace?

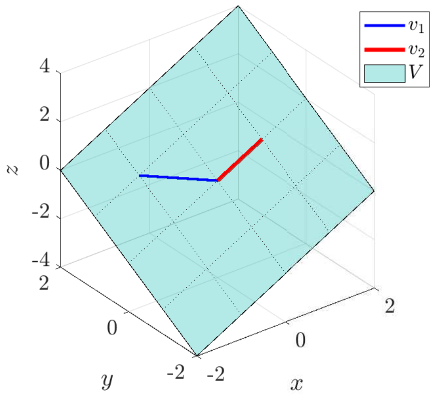


Figure: The two vectors $\vec{v}_1$ and $\vec{v}_2$ form a basis of the subspace V.

Our geometric intuition, and some vigorous hand-waving, seem to indicate that all bases of a plane V in $\mathbb{R}^3$ consist of 2 non-parallel vectors.

Maximum Number of Linearly Independent Vectors in a Subspace

Theorem (Proof in [Math 524 (Notes#2)])

Consider vectors $\vec{v}_1, \dots, \vec{v}_p$ and $\vec{w}_1, \dots, \vec{w}_q$ in a subspace of $\mathbb{R}^n$. If the vectors $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent, and the vectors $\vec{w}_1, \dots, \vec{w}_q$ span V, then $q \geq p$.

Comments:

- It is not possible to squeeze more than q linearly independent vectors into the subspace.
- When $q = 2$ (V is a plane), we can have at most 2 linearly independent vectors in the plane.
- We push the proof into the distant future (it's a bit technical and does not necessarily help our intuition; however, we show some related results....)

Number of Vectors in a Basis

Theorem (Number of Vectors in a Basis)

All bases of a subspace V of $\mathbb{R}^n$ consist of the same number of vectors.

Proof

Proof :: Number of Vectors in a Basis.

We consider 2 bases $\vec{v}_1, \dots, \vec{v}_p$ and $\vec{w}_1, \dots, \vec{w}_q$ of V. Since $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent *(they're a basis!)*, and the vectors $\vec{w}_1, \dots, \vec{w}_q$ span the space *(also a basis!)*, we have $q \geq p$ by the theorem on the previous slide.

Flipping the argument, since $\vec{w}_1, \dots, \vec{w}_q$ are linearly independent, and $\vec{v}_1, \dots, \vec{v}_p$ span the space; we must also have $p \geq q$.

We conclude that $p = q$. □

The Dimension of a Subspace

Definition (Dimension of a Subspace)

Consider a subspace V of $\mathbb{R}^n$. The number of vectors in a basis of V is called the *dimension* of V, denoted $\dim(V)$.

Example (The Dimension of $\mathbb{R}^n$)

The vectors $\vec{e}_1, \dots, \vec{e}_n$ (where $\vec{e}_k \in \mathbb{R}^n$, and only the k^{th} component is non-zero (one)) form a basis for $\mathbb{R}^n$; we call this the *standard basis*. As expected, it follows that $\dim(\mathbb{R}^n) = n$.

$$\text{Standard basis for } \mathbb{R}^4: \quad \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right\}$$

Independent Vectors and Spanning Vectors in a Subspace of $\mathbb{R}^n$

Example (The Dimension of a Plane)

A plane V in $\mathbb{R}^n$, $n \geq 2$ is two-dimensional. We need exactly two [LINEARLY INDEPENDENT] vectors to describe a plane.

Theorem (Independent Vectors and Spanning Vectors in a Subspace of $\mathbb{R}^n$)

Consider a subspace V of $\mathbb{R}^n$, with $\dim(V) = m$.

- **a.** *We can find at most m linearly independent vectors in V.*
- **b.** *We need at least m vectors to span V.*
- **c.** **If** *m vectors in V are linearly independent,* **then** *they form a basis of V.*
- **d.** **If** *m vectors in V span V,* **then** *they form a basis of V.*

Example: Bases for $\ker(A)$ and $\mathrm{im}(A)$

Example

Find a basis for the kernel, and image, of

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$

The Kernel: We solve $A\vec{x} = \vec{0}$ and get

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{rank}(A) = 2.$$

The leading ones tell what columns of A contain basis vectors for the *image* of A. [SEE THE NEXT FEW SLIDES]

Example: Bases for $\ker(A)$ and $\mathrm{im}(A)$

Standard parameterization of the free variables (x_2, x_4, x_5) yield the infinitely many non-trivial solutions for the kernel (see [NOTES#3.1]):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = s \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{w}_1} + t \underbrace{\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}_2} + r \underbrace{\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_3}$$

The vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ are linearly independent (see [LINEAR INDEPENDENCE AND ZERO COMPONENTS (NOTES#3.2)]), and span the kernel.

It follows that $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$ form a basis for the kernel of A, and $\dim(\ker(A)) = 3$.

Example: Bases for $\ker(A)$ and $\mathrm{im}(A)$

The Image: Consider

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}, \quad B = \mathrm{rref}(A) = \begin{array}{c} \begin{array}{ccccc} 3 & & -4 & & \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} & 2 & & & \end{array} \end{array}$$

We quickly see that $\vec{b}_1$, and $\vec{b}_3$ form a basis for $\mathrm{im}(B)$. —: $\vec{b}_2 = 2\vec{b}_1$, $\vec{b}_4 = 3\vec{b}_1 - 4\vec{b}_3$, and $\vec{b}_5 = -4\vec{b}_1 + 5\vec{b}_3$.
Also,

$$2\vec{a}_1 = \begin{bmatrix} 2 \\ -2 \\ 8 \\ 6 \end{bmatrix} = \vec{a}_2, \quad 3\vec{a}_1 - 4\vec{a}_3 = \begin{bmatrix} 3 \\ -3 \\ 12 \\ 9 \end{bmatrix} - \begin{bmatrix} 8 \\ -4 \\ 20 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ -8 \\ 5 \end{bmatrix} = \vec{a}_4$$

Example: Bases for $\ker(A)$ and $\mathrm{im}(A)$

The Image: Consider

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}, \quad B = \mathrm{rref}(A) = \begin{array}{c} \begin{array}{ccccc} -4 & & 5 & & \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

We quickly see that $\vec{b}_1$, and $\vec{b}_3$ form a basis for $\mathrm{im}(B)$. —: $\vec{b}_2 = 2\vec{b}_1$, $\vec{b}_4 = 3\vec{b}_1 - 4\vec{b}_3$, and $\vec{b}_5 = -4\vec{b}_1 + 5\vec{b}_3$.
and finally:

$$-4\vec{a}_1 + 5\vec{a}_3 = \begin{bmatrix} -4 \\ 4 \\ -16 \\ -12 \end{bmatrix} + \begin{bmatrix} 10 \\ -5 \\ 25 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 9 \\ -7 \end{bmatrix} = \vec{a}_5$$

Example: Bases for $\ker(A)$ and $\mathrm{im}(A)$

The Image: Consider

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}, \quad B = \mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bottom line: It is easy to see what column vectors are the basis for $\mathrm{im}(\mathrm{rref}(A))$ *[and how the other columns are formed from these]*, once we have identified them; the corresponding ones in A form a basis for $\mathrm{im}(A)$ *[and the same linear relations hold for A and $\mathrm{rref}(A)$]*; in this case the vectors

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \\ 1 \end{bmatrix}$$

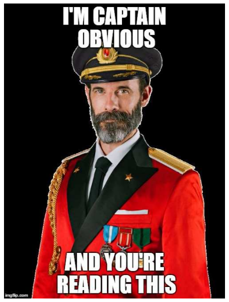
form a basis for $\mathrm{im}(A)$, and $\dim(\mathrm{im}(A)) = 2$.

i.c.

Insight :: Row-Reductions and Columns

Insight

Row-reductions DO NOT change the relations between columns.



"Captain Obvious" from https://imgflip.com/i/1klhm5, copyright/license unknown.

Using RREF to Construct a Basis of the Image

Theorem (Using RREF to Construct a Basis of the Image)

To construct a basis of the image of A, pick the column vectors of A that correspond to the columns of $\mathrm{rref}(A)$ containing leading 1's.

WARNING WARNING! WARNING

Note that you are picking columns of A (not $\mathrm{rref}(A)$). Generally $\mathrm{im}(A) \neq \mathrm{im}(\mathrm{rref}(A))$.

Dimension of the Image

Theorem (Dimension of the Image)

For any matrix A,

$$\dim(\mathrm{im}(A)) = \mathrm{rank}(A).$$

If $A \in \mathbb{R}^{n\times m}$:

- the basis of the **kernel** contains as many vectors as there are **free variables**;
- the basis of the **image** contains as many vectors as there are **leading variables**.

This means

$$\dim(\ker(\mathbf{A})) + \dim(\mathrm{im}(\mathbf{A})) = m.$$

Rank-Nullity

a.k.a. The Fundamental Theorem of Linear Transformations

Theorem (Rank-Nullity Theorem)

For any $A \in \mathbb{R}^{n\times m}$, the equation

$$\dim(\ker(A)) + \dim(\mathrm{im}(A)) = m$$

holds. $\dim(\ker(A))$ is called the nullity of A; and we have previously established that $\dim(\mathrm{im}(A)) = \mathrm{rank}(A)$. Thus

$$(\text{nullity of } A) + (\text{rank of } A) = m.$$

This is one of the corner-stone theorems of Linear Algebra.

It reappears in a more general form (The Fundamental Theorem of Linear Maps) in [Math 524 (Notes#3.1)].

Orthogonal Projections in the Context of Rank-Nullity

Consider the linear transformation describing the projection onto a plane — $T : \mathbb{R}^3 \to V$, where V is a plane in $\mathbb{R}^3$.

- A plane is spanned by two vectors; $\dim(\mathrm{im}(T)) = 2$
- $\ker(T) = V^\perp = \{$the line thru the origin perpendicular to $V\}$; $\dim(\ker(T)) = 1$
- We can think of a projection like a "collapse" along the perpendicular direction(s).
- Here we get: $m - \dim(\ker(T)) = \dim(\mathrm{im}(T))$; $(3 - 1 = 2)$.

Note: $V^\perp$ is the collection of all vectors that are orthogonal (perpendicular, $\perp$) to all vectors in V. In [Notes#5.1] we will formally define this as the *orthogonal complement* of V.

Another Example

1 of 3

Example

Find the bases of the image and kernel of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}$$

We can eye-ball and realize that columns #2, #3, and #5 can be written as linear combinations of #1, and #4. But let's pretend we don't see that, and compute

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Another Example

We have

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}, \quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, yeah columns #1 and #4 do indeed form a basis for $\mathrm{im}(A)$, and $\dim(\mathrm{im}(A)) = 2$.

Further, we have

Dependent Vectors	Relation
$\vec{v}_2 = 2\vec{v}_1$	$-2\vec{v}_1 + \vec{v}_2 = \vec{0}$
$\vec{v}_3 = \vec{0}$	$\vec{v}_3 = \vec{0}$
$\vec{v}_5 = \vec{v}_1 + \vec{v}_4$	$-\vec{v}_1 - \vec{v}_4 + \vec{v}_5 = \vec{0}$

Another Example

The three relations define the three vectors spanning the kernel:

Dependent Vectors	Relation
$\vec{v}_2 = 2\vec{v}_1$	$-2\vec{v}_1 + \vec{v}_2 = \vec{0}$
$\vec{v}_3 = \vec{0}$	$\vec{v}_3 = \vec{0}$
$\vec{v}_5 = \vec{v}_1 + \vec{v}_4$	$-\vec{v}_1 - \vec{v}_4 + \vec{v}_5 = \vec{0}$

Kernel vectors:

$$\vec{w}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{w}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

so, we have our basis for the kernel; and $\dim(\ker(A)) = 3$; $\dim(\mathrm{im}(A)) + \dim(\ker(A)) = 5$

Summarizing the Procedure

Theorem (Finding Bases of the Image and Kernel)

Identify the dependent columns of A (maybe with the "help" of $\mathrm{rref}(A)$), then:

1. *Identify null-space vectors —*
   - a-i *Express each dependent column as a linear combination of preceding columns*
     $$\vec{v}_i = c_1\vec{v}_1 + \cdots + c_{i-1}\vec{v}_{i-1}$$
   - a-ii *Write the corresponding relation*
     $$-c_1\vec{v}_1 - \cdots - c_{i-1}\vec{v}_{i-1} + \vec{v}_i = \vec{0}$$
   - a-iii *Identify the null-space vector*
     $$\begin{bmatrix} -c_1 & \dots & -c_{i-1} & 1 & 0 & \dots & 0 \end{bmatrix}^T$$
   - b *(Alternative) Parameterize as usual to get null-space vectors*
2. *Collect all such vectors and you have the basis for $\ker(A)$.*
3. *The other (independent) columns of A form a basis of $\mathrm{im}(A)$.*

Bases of $\mathbb{R}^n$

Theorem (Bases of $\mathbb{R}^n$)

The vectors $\vec{v}_1, \dots, \vec{v}_n$ form a basis of $\mathbb{R}^n$ if and only if the matrix

$$A = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$$

is invertible.

Note: We have n vectors $\in \mathbb{R}^n$, which means $A \in \mathbb{R}^{n \times n}$.

Characteristics of Invertible Matrices

IMPORTANT!

Equivalent Statements: Invertible Matrices

For an $n \times n$ matrix A, the following statements are equivalent; that is for a given A, they are either all `true` or all `false`:

- **i.** A is invertible
- **ii.** The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, $\forall \vec{b} \in \mathbb{R}^n$
- **iii.** $\mathrm{rref}(A) = I_n$
- **iv.** $\mathrm{rank}(A) = n$
- **v.** $\mathrm{im}(A) = \mathbb{R}^n$
- **vi.** $\ker(A) = \{\vec{0}\}$
- **vii.** The column vectors of A form a basis of $\mathbb{R}^n$
- **viii.** The column vectors of A span $\mathbb{R}^n$
- **ix.** The column vectors of A are linearly independent

Summary introduced in [NOTES#2.4], added to in [NOTES#3.1]; and will be re-visited again in [NOTES#7.1].

Suggested Problems 3.3

Available on Learning Glass videos:

3.3 — 1, 3, 19, 23, 25, 27, 29, 30, 31, 32

Lecture – Book Roadmap

Lecture	Book, [GS5–]
3.1	§3.1, §3.2, §3.3
3.2	§3.1, §3.2, §3.3, §3.4
3.3	§3.1, §3.2, §3.3, §3.4, §3.5
3.4	

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(3.3.1), (3.3.3)

(3.3.1) Find the linearly dependent (redundant) column vectors; then find a basis for the image of A, and a basis for the kernel of A, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

(3.3.3) Find the linearly dependent (redundant) column vectors; then find a basis for the image of A, and a basis for the kernel of A, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

(3.3.19), (3.3.23)

(3.3.19) Find the linearly dependent (redundant) column vectors; then find a basis for the image of A, and a basis for the kernel of A, where

$$A = \begin{bmatrix} 1 & 0 & 5 & 3 & 0 \\ 0 & 1 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(3.3.23) Find the reduced row echelon form of A; then find a basis for the image of A, and a basis for the kernel of A, where

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}.$$

(3.3.25), (3.3.27)

(3.3.25) Find the reduced row echelon form of A; then find a basis for the image of A, and a basis for the kernel of A, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 2 \end{bmatrix}.$$

(3.3.27) Determine whether the following vectors form a basis of $\mathbb{R}^4$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix}.$$

(3.3.29), (3.3.30)

(3.3.29) Find a basis of the subspace of $\mathbb{R}^3$ defined by the equation

$$2x_1 + 3x_2 + x_3 = 0.$$

(3.3.30) Find a basis of the subspace of $\mathbb{R}^4$ defined by the equation

$$2x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

(3.3.31), (3.3.32)

(3.3.31) Let V be the subspace of $\mathbb{R}^4$ defined by the equation

$$x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that $\ker(T) = \{\vec{0}\}$, and $\mathrm{im}(T) = V$. Describe T by its matrix.

(3.3.32) Find a basis of the subspace of $\mathbb{R}^4$ that consists of all vectors perpendicular to both

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Bonus Questions from Reddit

(a) Show that the kernel of a linear transformation

$$T_A : \mathbb{R}^5 \to \mathbb{R}^3$$

must have dimension at least 2.

(b) Show that the image of a linear transformation

$$T_B : \mathbb{R}^3 \to \mathbb{R}^5$$

must have dimension at most 3.