Math 254: Introduction to Linear Algebra

Notes #3.4 — Coordinates

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Spring 2022
(Revised: March 8, 2022)

Outline

SLOs 3.4 Coordinates

After this lecture you should:

- Know the relation between the basis $\mathfrak{B}$ (of a subspace V), the $\mathfrak{B}$-coordinates of a vector $\vec{x} \in V$, and the $\mathfrak{B}$-coordinate vector $[\vec{x}]_{\mathfrak{B}}$.
- Be able to identify the Matrix and $\mathfrak{B}$-Matrix of a linear transformation.
- Know the basic definition of *Similarity of Matrices* (to be revisited in the context of Eigenvalues and Eigenvectors).

Coordinates: Where are We?

Example (Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates)

Consider the vectors (in $\mathbb{R}^3$)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and define the plane $V = \mathrm{span}(\vec{v}_1, \vec{v}_2)$ in $\mathbb{R}^3$. Is the vector

$$\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

in the plane? (Visualize)

Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates

We are really asking *"Can we find c_1 and c_2 so that:*

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \text{ ?"}$$

We look in our toolbox, and what do we find...

$$\text{rref}\left(\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array} \right] \right) = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

that is, the short answer is *"Yes,"* and the slightly longer $c_1 = 3$, $c_2 = 2$.

Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates

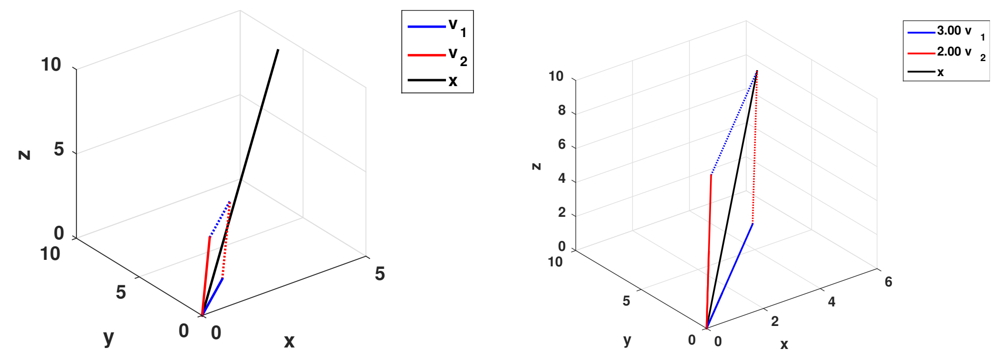

Figure: In the [LEFT] panel we see the vectors $\vec{v}_1$, $\vec{v}_2$, and the linear combination $\vec{v}_1 + \vec{v}_2$. In the [RIGHT] panel we see the vectors $3\vec{v}_1$, $2\vec{v}_2$, and the linear combination $3\vec{v}_1 + 2\vec{v}_2$ which reaches the vector $\vec{x}$. ($\exists$ Movie)

Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates

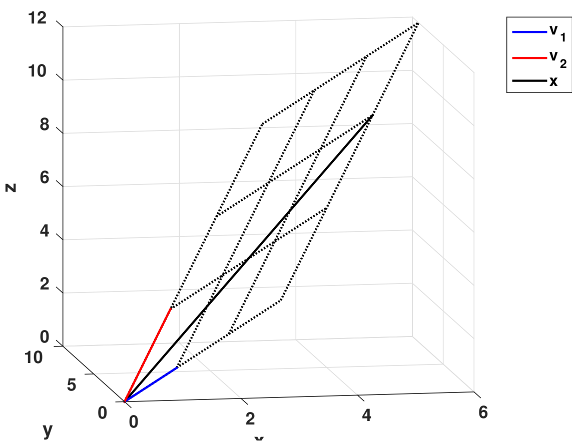

Figure: We can think of $\vec{c} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as the address (coordinate) of the vector $\vec{x}$ in a grid where $\vec{v}_1$ and $\vec{v}_2$ are the coordinate-axes.

Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates

By introducing the c_1-c_2 coordinates along the vectors $\vec{v}_1$ and $\vec{v}_2$, we transform the plane V to $\mathbb{R}^2$.

It is natural to have a brief panic attack when you realize that the coordinate axes $\vec{v}_1$ and $\vec{v}_2$ are not perpendicular; but, really, it is not a problem... each point in the plane does get its own unique (coordinate) address.

Notation (Basis, $\mathfrak{B}$; coordinate vector $[\vec{x}]_{\mathfrak{B}}$)

Let $\mathfrak{B}$ denote the basis $\vec{v}_1$-$\vec{v}_2$ of V, and let the coordinate vector of $\vec{x}$ with respect to $\mathfrak{B}$ be denoted by $[\vec{x}]_{\mathfrak{B}}$.

In our example we had $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, with $\mathfrak{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$

Coordinates in a Subspace of $\mathbb{R}^n$

Definition (Coordinates in a Subspace of $\mathbb{R}^n$)

Consider a basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of a subspace $V = \mathrm{span}(\vec{v}_1, \dots, \vec{v}_m)$ of $\mathbb{R}^n$; $\dim(V) = m \le n$. Any vector $\vec{x} \in V$ can be written uniquely as

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m.$$

The scalars $c_1, \dots, c_m$ are called *$\mathfrak{B}$-coordinates* of $\vec{x}$, and the vector

$$\vec{c} = \begin{bmatrix} c_1 & \dots & c_m \end{bmatrix}^T$$

is the *$\mathfrak{B}$-coordinate vector* of $\vec{x}$, denoted by $[\vec{x}]_{\mathfrak{B}}$. Thus,

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix}^T \quad \text{means that } \vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m.$$

Note that

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}}, \text{ where } S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}, \text{ an } (n \times m) \text{ matrix.}$$

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Checking Our Example

We had:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}, \quad \rightsquigarrow S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

We computed:

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We can reconstruct $\vec{x}$ from S and $[\vec{x}]_{\mathfrak{B}}$:

$$S[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \vec{x}. \quad \surd$$

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Property: Linearity of Coordinates

Theorem (Linearity of Coordinates)

If $\mathfrak{B}$ is a basis of a subspace V of $\mathbb{R}^n$, then

a. $[\vec{x} + \vec{y}]_{\mathfrak{B}} = [\vec{x}]_{\mathfrak{B}} + [\vec{y}]_{\mathfrak{B}} \quad \forall \vec{x}, \vec{y} \in V$, *and*

b. $[k\vec{x}]_{\mathfrak{B}} = k[\vec{x}]_{\mathfrak{B}} \quad \forall \vec{x} \in V, \forall k \in \mathbb{R}$

Building Blocks for the Proof

With $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$ keep in mind

$$\left.\begin{array}{rcl} \vec{x} & = & c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \\ \vec{y} & = & d_1\vec{v}_1 + \cdots + d_m\vec{v}_m \end{array}\right\} \Leftrightarrow [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}, \ [\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix},$$

and use the definitions of vector-addition and scaling.

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Example: Coordinates / Basis / Vectors

Example (Basis-Vector-Coordinate Transformations)

Consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of vectors

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \rightsquigarrow S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

(a) If $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, find $[\vec{x}]_{\mathfrak{B}}$; (b) if $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{y}$.

For (a) we need to solve (for $[\vec{x}]_{\mathfrak{B}}$)

$$S[\vec{x}]_{\mathfrak{B}} = \vec{x}, \ \mathrm{rref}\left(\left[\begin{array}{cc|c} 3 & -1 & 10 \\ 1 & 3 & 10 \end{array}\right]\right) = \left(\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]\right), \ [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

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Example: Coordinates / Basis / Vectors

Example (Basis-Vector-Coordinate Transformations)

Consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of vectors

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \rightsquigarrow S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

(a) If $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, find $[\vec{x}]_{\mathfrak{B}}$; (b) if $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{y}$.

For (b) we need to compute $(\vec{y})$

$$\vec{y} = S[\vec{y}]_{\mathfrak{B}}, \quad \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

Example: Projection

Example (Projection)

Let L be the line in $\mathbb{R}^2$ spanned by $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects any vector $\vec{x}$ orthogonally onto L. It is quite useful to think of this in a coordinate system where one axis is L and the other is $L^\perp$...

Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ (clearly $\vec{v}_1 \cdot \vec{v}_2 = 0$, so they are perpendicular.)
Now, if we have a vector $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$, then $T(\vec{x}) = c_1\vec{v}_1$, or

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \rightsquigarrow [T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix};$$

which means that the projection matrix is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ in } \mathfrak{B}\text{-coordinates; compare with } \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \text{ in standard coordinates.}$$

The Matrix of a Linear Transformation

Theorem (The $\mathfrak{B}$-Matrix of a Linear Transformation)

Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^n$. There there exits a unique "$\exists!$" $B \in \mathbb{R}^{n\times n}$ matrix that transforms $[\vec{x}]_{\mathfrak{B}}$ into $[T(\vec{x})]_{\mathfrak{B}}$:

$$[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}},$$

$\forall \vec{x} \in \mathbb{R}^n$. This matrix B is called the $\mathfrak{B}$-matrix of T. We can construct B column-by-column, as follows:

$$B = \big[[T(\vec{v}_1)]_{\mathfrak{B}} \quad \ldots \quad [T(\vec{v}_n)]_{\mathfrak{B}} \big].$$

Proof

Note: In [MATH 524] we use the notation $\mathcal{M}(T, \mathfrak{B})$ — "The matrix of T with respect to the basis $\mathfrak{B}$."

The Matrix of a Linear Transformation

Proof (The $\mathfrak{B}$-Matrix of a Linear Transformation))

$\mathfrak{B}$-coordinates: $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$; then

$$\begin{aligned}
[T(\vec{x})]_{\mathfrak{B}} &= [T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)]_{\mathfrak{B}} && [\vec{x} \text{ as lin.comb.}] \\
&= [c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n)]_{\mathfrak{B}} && [T \text{ is lin.trans.}] \\
&= c_1[T(\vec{v}_1)]_{\mathfrak{B}} + \cdots + c_n[T(\vec{v}_n)]_{\mathfrak{B}} && [[\circ]_{\mathfrak{B}} \text{ is linear.}] \\
&= \big[[T(\vec{v}_1)]_{\mathfrak{B}} \quad \ldots \quad [T(\vec{v}_n)]_{\mathfrak{B}} \big] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} && [\text{book-keeping}] \\
&= B[\vec{x}]_{\mathfrak{B}}. && [\text{identify}]
\end{aligned}$$

(Sequence of Definitions / Properties)

Standard Matrix vs. $\mathfrak{B}$-matrix

Theorem (Standard Matrix vs. $\mathfrak{B}$-Matrix)

Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$ of $\mathbb{R}^n$. Let B be the $\mathfrak{B}$-matrix of T, and let A be the standard matrix of T — so that $T(\vec{x}) = A\vec{x}\ \forall x \in \mathbb{R}^n$; then

$$AS = SB,\ B = S^{-1}AS,\ A = SBS^{-1},\ \textit{where } S = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}$$

This follows from the linear transform relations:

$$T(\vec{x}) = A\vec{x}, \quad [T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}},$$

and

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}}, \quad T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}}$$

We formalize the matrix relations (in 2 slides)...

Standard Matrix vs. $\mathfrak{B}$-matrix ((Change of Basis))

Visualizing the Theorem:

$$\begin{tikzcd} \text{standard coordinates} & \vec{x} \arrow[rr, "A\vec{x}"] \arrow[d, shift right, "{\vec{x}=S[\vec{x}]_{\mathfrak{B}}}"'] & & T(\vec{x}) \arrow[d, shift right, "{T(\vec{x})=S[T(\vec{x})]_{\mathfrak{B}}}"'] \\ \mathfrak{B}\text{-coordinates} & {[\vec{x}]_{\mathfrak{B}}} \arrow[rr, "{B[\vec{x}]_{\mathfrak{B}}}"] \arrow[u, shift right, "{[\vec{x}]_{\mathfrak{B}}=S^{-1}\vec{x}}"'] & & {[T(\vec{x})]_{\mathfrak{B}}} \arrow[u, shift right, "{[T(\vec{x})]_{\mathfrak{B}}=S^{-1}T(\vec{x})}"'] \end{tikzcd}$$

$$S = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}, \quad \mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$$

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}},\ S^{-1}\vec{x} = [\vec{x}]_{\mathfrak{B}};\ T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}},\ S^{-1}T(\vec{x}) = [T(\vec{x})]_{\mathfrak{B}}$$

Therefore

$$A\vec{x} = T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}} = SB[\vec{x}]_{\mathfrak{B}} = SBS^{-1}\vec{x}$$

Similar Matrices — Definition

Definition (Similar Matrices)

Consider two matrices $A, B \in \mathbb{R}^{n \times n}$. We say that A is *similar* to B if there exists an invertible matrix S such that

$$AS = SB, \quad B = S^{-1}AS, \quad A = SBS^{-1}$$

At this point we do not have an efficient way of finding out whether two given matrices are similar.

(We can set up a matrix S and leave its entries as variables, create AS and SB, and then set the two results equal and solve for the S-entries... However, better methods will be developed in the near future.)

[FOCUS :: MATH] Similar Matrices — Properties

Theorem (Matrix Similarity is an Equivalence Relation*)

reflexivity *$A \in \mathbb{R}^{n \times n}$ is similar to itself.*

symmetry *If A is similar to B, then B is similar to A*

transitivity *If A is similar to B, and B is similar to C, then A is similar to C*

Reflexivity let $S = I_n$.

Symmetry given $AS_A = S_AB$, let $S_B = S_A^{-1}$; then $S_BAS_AS_B = S_BA$, and $S_BS_ABS_B = BS_B$, so that $BS_B = S_BA$.

Transitivity, we have $AS_1 = S_1B$, and $BS_2 = S_2C$; now $AS_1S_2 = S_1BS_2 = S_1S_2C$; so with $S_3 = S_1S_2$ we have $AS_3 = S_3C$.

Diagonal $\mathfrak{B}$-matrix

Example

Given $T(\vec{x}) = A\vec{x}$ ($T : \mathbb{R}^2 \to \mathbb{R}^2$), we often want the basis $\mathfrak{B}$ be such that the $\mathfrak{B}$-matrix of T is *diagonal*, that is

$$B = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}$$

The big question is how to pick the basis $(\vec{v}_1, \vec{v}_2)$ so that happens?!

Recall that each column in the $\mathfrak{B}$-matrix is of the form

$$[T(\vec{v}_k)]_{\mathfrak{B}}$$

and the components of the column vectors are the coordinates expressed in the basis. We want *only* the k^{th} component to be non-zero, which means we must have $T(\vec{v}_k) = b_{kk}\vec{v}_k$.

Diagonal $\mathfrak{B}$-matrix

Theorem (When is the $\mathfrak{B}$-matrix Diagonal?)

Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$ of $\mathbb{R}^n$.

- *The $\mathfrak{B}$-matrix B of T is diagonal if and only if $T(\vec{v}_k) = c_k\vec{v}_k$ $\forall k \in \{1, \dots, n\}$, for some scalars $c_1, \dots, c_n \in \mathbb{R}$.*
- *From a geometric point of view, this means that $T(\vec{v}_k)$ is parallel to $\vec{v}_k$ $\forall k \in \{1, \dots, n\}$.*

In general it is hard (we don't have the tools yet) to find a basis which makes the $\mathfrak{B}$-matrix diagonal... We will return to this topic [EIGENVECTORS and EIGENVALUES] in the future... Simple examples with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are given by the vectors parallel and orthogonal to a line L we are orthogonally projecting onto, or reflecting across.

Suggested Problems 3.4

Available on Learning Glass videos:

3.4 — 1, 3, 4, 7, 9, 17, 19, 23, 27, 29, 37

Lecture – Book Roadmap

Lecture	Book, [GS5–]
3.1	§3.1, §3.2, §3.3
3.2	§3.1, §3.2, §3.3, §3.4
3.3	§3.1, §3.2, §3.3, §3.4, §3.5
3.4	§8.2, (§8.3)

§8.2 "Change of Basis" (p.412), "Choosing the Best Basis" (p.415–416)

§8.3 Extension of our discussion (we will revisit this)

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(3.4.1), (3.4.3)

(3.4.1) Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(3.4.3) Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 31 \\ 37 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 23 \\ 29 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 31 \\ 37 \end{bmatrix}.$$

(3.4.4), (3.4.7)

(3.4.4) Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(3.4.7) Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(3.4.9), (3.4.17)

(3.4.9) Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

(3.4.17) Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

(3.4.19), (3.4.23)

(3.4.19) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$, with respect to the basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$. Solve in three ways: *(a)* Use the formula $B = S^{-1}AS$, *(b)* Use a commutative diagram, and *(c)* construct B column-by-column.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

(3.4.23) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$, with respect to the basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$. Solve in three ways: *(a)* Use the formula $B = S^{-1}AS$, *(b)* Use a commutative diagram, and *(c)* construct B column-by-column.

$$A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(3.4.27), (3.4.29)

(3.4.27) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$, with respect to the basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$.

$$A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & =2 & 4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(3.4.29) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$, with respect to the basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

(3.4.37)

(3.4.37) Find a basis $\mathfrak{B}$ of $\mathbb{R}^n$such that the $\mathfrak{B}$-matrix of the given linear transformation is diagonal.

$$T(\vec{x}) = [\text{Orthogonal Projection onto the line}]\ L = k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Motivation

So far, we have talked about vectors in $\mathbb{R}^n$, and matrix operations from $\mathbb{R}^m$ to $\mathbb{R}^n$; expressed as linear transformations, via matrix-vector operations.

Some of the key concepts we have covered are: linear combination, linear transformation, kernel, image, subspace, span, linear independence, basis, dimension, and coordinates.

It turns out that this language (really, think of it as a *language*) can be applied to mathematical objects other than matrices and vectors; *e.g.* functions, equations, or infinite sequences.

The “language” of Linear Algebra is used throughout mathematics and other sciences.

Here, we “free” ourselves from the constraint of “living in $\mathbb{R}^n$,” and re-state some of our result in a way that is useful in many settings.

Linear Spaces — Definition

Definition (Linear Spaces)

A Linear Space V is a set with a definition (rule) for addition "+", and a definition (rule) for scalar multiplication; and the following must hold $(\forall u, v, w \in V, \forall c, k \in \mathbb{R})$

- **a.** $v + w \in V$.
- **b.** $kv \in V$.
- **c.** $(u + v) + w = u + (v + w)$.
- **d.** $u + v = v + u$.
- **e.** $\exists n \in V \colon\ u + n = u$, [Neutral Element, denoted by 0]
- **f.** $\exists \widehat{u} \colon\ u + \widehat{u} = 0$; $\widehat{u}$ unique, and denoted by $-u$.
- **g.** $k(u + v) = ku + kv$.
- **h.** $(c + k)u = cu + ku$.
- **i.** $c(ku) = (ck)u$.
- **j.** $1u = u$.

Examples: Linear Spaces

We have already seen the "prototype" linear spaces:

Example (Linear Space(s) $\mathbb{R}^n$)

Here, the natural element is the zero vector $\vec{0} \in \mathbb{R}^n$.

We give a few other examples

Example

Let $F(\mathbb{R}, \mathbb{R})$ set the set of all functions $f : \mathbb{R} \to \mathbb{R}$, with the operations

$$(f + g)(x) = f(x) + g(x), \quad \text{and } (kf)(x) = kf(x)$$

then $F(\mathbb{R}, \mathbb{R})$ is a linear space; the function $f(x) = 0$ is the neutral element.

Examples: Linear Spaces

Example ($\mathbb{R}^{n \times m}$)

Given our previous definitions [Notes#1.3] of matrix addition and scalar multiplication of a matrix, then $\mathbb{R}^{n \times m}$, the set of all $n \times m$ matrices, is a linear space. The zero-matrix is the neutral element.

Example (Infinite Sequences)

The set of all infinite sequences $\overline{x} = (x_1, x_2, \ldots, x_\infty)$ is a linear space; addition is defined $\overline{x} + \overline{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_\infty + y_\infty)$; scalar multiplication $k\overline{x} = (kx_1, kx_2, \ldots, kx_\infty)$. The zero-sequence $(0, 0, \ldots)$ is the neutral element.

Examples: Linear Spaces

Example (Linear Equations)

The linear equations in 3 unknowns

$$ax + by + cz = d$$

where a, b, c, and d are constants, form a linear space. The neutral element is $0 = 0$, *i.e.* $a = b = c = d = 0$.

Example (Complex Numbers)

Let $\mathbb{C}$ be the set of complex numbers $z = a + bi$; with addition defined by $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$, and scalar multiplication by $kz = ka + (kb)i$ $(\forall k \in \mathbb{R})$. $\mathbb{C}$ with these two operations form a linear space, with neutral element $0 + 0i$.

Definitions

Definition (Linear Combination)

We say that an element u of a linear space is a *linear combination* of the elements $v_1, \ldots, v_n$ if $u = c_1 v_1 + \cdots + c_n v_n$.

Since the basic notation for Linear Algebra (on $\mathbb{R}^n$) are defined in terms of linear combinations, we can generalize those concepts to all Linear Spaces without generalizations:

Definition (Subspaces)

A subset W of a linear space V is called a subspace of V if

- **a.** W contains the neutral element, 0, of V
- **b.** W is closed under addition
- **c.** W is closed under scalar multiplication
- **b+c.** $\Rightarrow$ W is closed under linear combinations

Examples: Subspaces of $F(\mathbb{R}, \mathbb{R})$

Example

The polynomials of degree 2 — $\mathcal{P}_2 = \{\, ax^2 + bx + c \ :\ a, b, c \in \mathbb{R} \,\}$, form a subspace of $F(\mathbb{R}, \mathbb{R})$.

- $f(x) = 0 = 0x^2 + 0x + 0$
- $kp_1(x) + p_2(x) = (ka_1 + a_2)x^2 + (kb_1 + b_2)x + (kc_1 + c_2)$

Example

The differentiable functions, C^0 form a subspace of $F(\mathbb{R}, \mathbb{R})$.

- $f(x) = 0$, with $f'(x) = 0$
- Calculus tell us that $(kf(x) + g(x))' = kf'(x) + g'(x)$.

Examples: Subspaces of $F(\mathbb{R}, \mathbb{R})$

Example (More subspaces of $F(\mathbb{R}, \mathbb{R})$)

- C^n, $n \in \{1, 2, \ldots, \infty\}$ — the functions with n (possibly infinitely) many continuous derivatives form subspaces of $F(\mathbb{R}, \mathbb{R})$.
- $\mathcal{P}$, the set of polynomials forms a subspace of $F(\mathbb{R}, \mathbb{R})$.
- $\mathcal{P}_n$, the set of all polynomials of degree $\leq n$ forms a subspace of $F(\mathbb{R}, \mathbb{R})$.

Span, Linear Independence, Basis, Coordinates

Example (Span, Linear Independence, Basis, Coordinates)

Consider the elements $u_1, \ldots, u_n$ in a linear space V.

- **a.** $u_1, \ldots, u_n$ *span* V if every $v \in V$ can be expressed as a linear combination of $u_1, \ldots, u_n$
- **b-i.** u_i is *linearly dependent* if it is a linear combination of $u_1, \ldots, u_{i-1}$.
- **b-ii.** The elements $u_1, \ldots, u_n$ are *linearly independent* if none of them is linearly dependent. This is the case if the equation

$$c_1 u_1 + \cdots + c_n u_n = 0$$

only has the trivial solution $c_1 = \cdots = c_n = 0$.

Span, Linear Independence, Basis, Coordinates

Example (Span, Linear Independence, Basis, Coordinates)

Consider the elements $u_1, \ldots, u_n$ in a linear space V.

c-*i*. $u_1, \ldots, u_n$ are a *basis* of V is they span V and are linearly independent. This means every $v \in V$ can be written as a unique linear combination $v = c_1 u_1 + \cdots + c_n u_n$,

c-*ii*. The coefficients $c_1, \ldots, c_n$ are called the *coefficients* of v with respect to the basis $\mathfrak{B} = (u_1, \ldots, u_n)$. The vector

$$\vec{c}^T = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^T$$

in $\mathbb{R}^n$ is called the $\mathfrak{B}$-*coordinate vector* of v, denoted by $[v]_{\mathfrak{B}}$

c-*iii*. The transformation $L(v) = [v]_{\mathfrak{B}} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^T$ is called the $\mathfrak{B}$-*coordinate transformation*, sometimes denoted by $L_{\mathfrak{B}}$

Supplemental Material
[Focus :: Math] Beyond Vectors & Matrices — Linear Spaces
Definition and Examples
Span, Linear Independence, Basis, Coordinates
Theorems, and One More Definition

Linear Spaces: Theorems — Properties

Theorem (Linearity of the $\mathfrak{B}$-*coordinate transformation*, $L_{\mathfrak{B}}$)

If $\mathfrak{B}$ is a basis of a linear space, then $\forall u, v \in V$, $\forall k \in \mathbb{R}$:

a. $[u+v]_{\mathfrak{B}} = [u]_{\mathfrak{B}} + [v]_{\mathfrak{B}}$

b. $[ku]_{\mathfrak{B}} = k[u]_{\mathfrak{B}}$

(The proof is pretty much a copy of the $\mathbb{R}^n$ version from [NOTES#3.4]).

Theorem (Dimension(!!!))

If a linear space V has a basis with n elements, then all other bases of V consist of n elements as well, and we say

$$\dim(V) = n$$

Supplemental Material
[Focus :: Math] Beyond Vectors & Matrices — Linear Spaces
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Linear (Ordinary) Differential Equations — ODEs — Important for the Future!

Theorem (Linear Differential Equations)

The solutions of the differential equation ($a, b \in \mathbb{R}$ are constants)

$$u''(x) + au'(x) + bu(x) = 0$$

form a two-dimensional subspace of the space C^∞ of smooth functions; more generally, the solutions of the differential equation

$$v^{(n)}(x) + a_{n-1}v^{(n-1)}(x) + \cdots + a_1 v'(x) + a_0 u(x) = 0$$

(where the coefficients $a_0, \ldots, a_{n-1}$ are constants) form an n-dimensional subspace of C^∞. A differential equation of this form is called an n^{th}-order linear differential equation with constant coefficients.

The connection between linear algebra and ODEs (both in terms of theory and applications) is VERY STRONG. In many places the topics are taught together in a joint (sequence of) class(es).

Supplemental Material
[Focus :: Math] Beyond Vectors & Matrices — Linear Spaces
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Theorems, and One More Definition

Finite Dimensional Subspaces

Definition (Finite Dimensional Subspaces)

A linear space V is called finite dimensional if it has a (finite) basis $v_1, \ldots, v_n$, so that $\dim(V) = n$. Otherwise the space is called *infinite dimensional*.

The space of polynomials, $\mathcal{P}$, is infinite dimensional.

The study of infinite dimensional linear spaces — *e.g.* Hilbert-, Banach-, and Sobolev spaces, belong in a course on functional analysis; somewhere beyond the horizon of ADVANCED CALCULUS... really, it's fun stuff!