

# Math 254: Introduction to Linear Algebra

## Notes #3.4 — Coordinates

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After this lecture you should:

- Know the relation between the basis  $\mathcal{B}$  (of a subspace  $V$ ), the  $\mathcal{B}$ -coordinates of a vector  $\vec{x} \in V$ , and the  $\mathcal{B}$ -coordinate vector  $[\vec{x}]_{\mathcal{B}}$ .
- Be able to identify the Matrix and  $\mathcal{B}$ -Matrix of a linear transformation.
- Know the basic definition of *Similarity of Matrices* (to be revisited in the context of Eigenvalues and Eigenvectors).



## Outline

- 1 Student Learning Objectives
  - SLOs: Coordinates
- 2 Subspaces of  $\mathbb{R}^n$  and Their Dimensions: Coordinates
  - Coordinates: Introduction, Definition, and Properties
  - Coordinates: Examples
  - The Matrix of a Linear Transformation: Change of Coordinates
- 3 Suggested Problems
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- 4 Supplemental Material
  - Metacognitive Reflection
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- 5 [FOCUS :: MATH] Beyond Vectors & Matrices — Linear Spaces
  - Definition and Examples
  - Span, Linear Independence, Basis, Coordinates
  - Theorems, and One More Definition



## Coordinates: Where are We?

### Example (Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates)

Consider the vectors (in  $\mathbb{R}^3$ )

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and define the plane  $V = \text{span}(\vec{v}_1, \vec{v}_2)$  in  $\mathbb{R}^3$ . Is the vector

$$\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

in the plane? (Visualize)



Coefficients in a Linear Combination  $\rightsquigarrow$  Coordinates

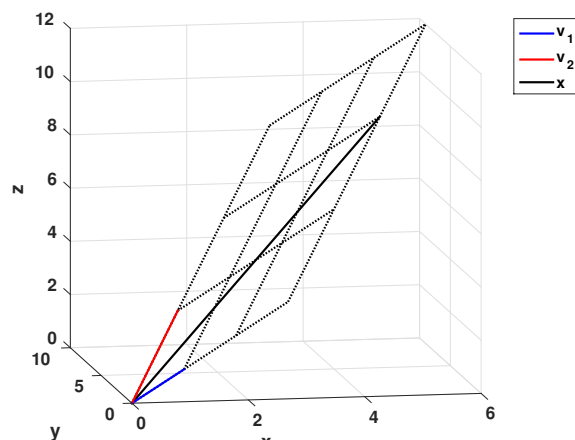
We are really asking “Can we find  $c_1$  and  $c_2$  so that:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 = \vec{x} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} ?”$$

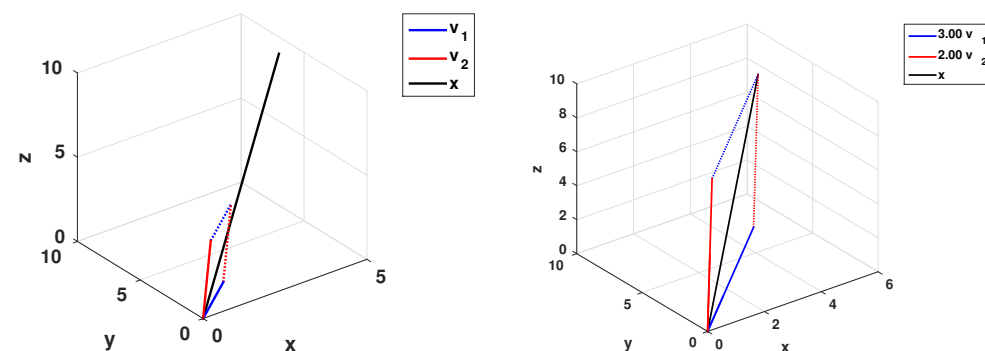
We look in our toolbox, and what do we find...

$$\text{rref} \left( \left[ \begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array} \right] \right) = \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

that is, the short answer is “Yes,” and the slightly longer  $c_1 = 3$ ,  $c_2 = 2$ .

Coefficients in a Linear Combination  $\rightsquigarrow$  Coordinates

**Figure:** We can think of  $\vec{c} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$  as the address (coordinate) of the vector  $\vec{x}$  in a grid where  $\vec{v}_1$  and  $\vec{v}_2$  are the coordinate-axes.

Coefficients in a Linear Combination  $\rightsquigarrow$  Coordinates

**Figure:** In the [LEFT] panel we see the vectors  $\vec{v}_1$ ,  $\vec{v}_2$ , and the linear combination  $\vec{v}_1 + \vec{v}_2$ . In the [RIGHT] panel we see the vectors  $3\vec{v}_1$ ,  $2\vec{v}_2$ , and the linear combination  $3\vec{v}_1 + 2\vec{v}_2$  which reaches the vector  $\vec{x}$ . ( $\exists$  Movie)

Coefficients in a Linear Combination  $\rightsquigarrow$  CoordinatesCoefficients in a Linear Combination  $\rightsquigarrow$  Coordinates

By introducing the  $c_1$ - $c_2$  coordinates along the vectors  $\vec{v}_1$  and  $\vec{v}_2$ , we transform the plane  $V$  to  $\mathbb{R}^2$ .

It is natural to have a brief panic attack when you realize that the coordinate axes  $\vec{v}_1$  and  $\vec{v}_2$  are not perpendicular; but, really, it is not a problem... each point in the plane does get its own unique (coordinate) address.

**Notation** (Basis,  $\mathfrak{B}$ ; coordinate vector  $[\vec{x}]_{\mathfrak{B}}$ )

Let  $\mathfrak{B}$  denote the basis  $\vec{v}_1$ - $\vec{v}_2$  of  $V$ , and let the coordinate vector of  $\vec{x}$  with respect to  $\mathfrak{B}$  be denoted by  $[\vec{x}]_{\mathfrak{B}}$ .

In our example we had  $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ , with  $\mathfrak{B} = \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$



Coordinates in a Subspace of  $\mathbb{R}^n$ Definition (Coordinates in a Subspace of  $\mathbb{R}^n$ )

Consider a basis  $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$  of a subspace  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$  of  $\mathbb{R}^n$ ;  $\dim(V) = m \leq n$ . Any vector  $\vec{x} \in V$  can be written uniquely as

$$\vec{x} = c_1 \vec{v}_1 + \dots + c_m \vec{v}_m.$$

The scalars  $c_1, \dots, c_m$  are called  **$\mathcal{B}$ -coordinates** of  $\vec{x}$ , and the vector

$$\vec{c} = [c_1 \quad \dots \quad c_m]^T$$

is the  **$\mathcal{B}$ -coordinate vector** of  $\vec{x}$ , denoted by  $[\vec{x}]_{\mathcal{B}}$ . Thus,

$$[\vec{x}]_{\mathcal{B}} = [c_1 \quad \dots \quad c_m]^T \quad \text{means that} \quad \vec{x} = c_1 \vec{v}_1 + \dots + c_m \vec{v}_m.$$

Note that

$$\vec{x} = S[\vec{x}]_{\mathcal{B}}, \text{ where } S = [\vec{v}_1 \quad \dots \quad \vec{v}_m], \text{ an } (n \times m) \text{ matrix.}$$



## Checking Our Example

We had:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}, \quad \rightsquigarrow S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

We computed:

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We can reconstruct  $\vec{x}$  from  $S$  and  $[\vec{x}]_{\mathcal{B}}$ :

$$S[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \vec{x}. \quad \checkmark$$



## Property: Linearity of Coordinates

## Theorem (Linearity of Coordinates)

If  $\mathcal{B}$  is a basis of a subspace  $V$  of  $\mathbb{R}^n$ , then

- $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}} \quad \forall \vec{x}, \vec{y} \in V, \text{ and}$
- $[k\vec{x}]_{\mathcal{B}} = k[\vec{x}]_{\mathcal{B}} \quad \forall \vec{x} \in V, \forall k \in \mathbb{R}$

## Building Blocks for the Proof

With  $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$  keep in mind

$$\left. \begin{aligned} \vec{x} &= c_1 \vec{v}_1 + \dots + c_m \vec{v}_m \\ \vec{y} &= d_1 \vec{v}_1 + \dots + d_m \vec{v}_m \end{aligned} \right\} \Leftrightarrow [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}, [\vec{y}]_{\mathcal{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix},$$

and use the definitions of vector-addition and scaling.



## Example: Coordinates / Basis / Vectors

## Example (Basis-Vector-Coordinate Transformations)

Consider the basis  $\mathcal{B}$  of  $\mathbb{R}^2$  consisting of vectors

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \rightsquigarrow S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

- (a) If  $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ , find  $[\vec{x}]_{\mathcal{B}}$ ;      (b) if  $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ , find  $\vec{y}$ .

For (a) we need to solve (for  $[\vec{x}]_{\mathcal{B}}$ )

$$S[\vec{x}]_{\mathcal{B}} = \vec{x}, \text{ rref} \left( \left[ \begin{array}{cc|cc} 3 & -1 & 10 & 0 \\ 1 & 3 & 10 & 0 \end{array} \right] \right) = \left( \left[ \begin{array}{cc|cc} 1 & 0 & 4 & 2 \\ 0 & 1 & 2 & 2 \end{array} \right] \right), \quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$



## Example: Coordinates / Basis / Vectors

## Example (Basis-Vector-Coordinate Transformations)

Consider the basis  $\mathfrak{B}$  of  $\mathbb{R}^2$  consisting of vectors

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \rightsquigarrow S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

(a) If  $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ , find  $[\vec{x}]_{\mathfrak{B}}$ ; (b) if  $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ , find  $\vec{y}$ .

For (b) we need to compute  $(\vec{y})$

$$\vec{y} = S[\vec{y}]_{\mathfrak{B}}, \quad \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$



## The Matrix of a Linear Transformation

Theorem (The  $\mathfrak{B}$ -Matrix of a Linear Transformation)

Consider a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$  of  $\mathbb{R}^n$ . There there exists a unique “ $\exists!$ ”  $B \in \mathbb{R}^{n \times n}$  matrix that transforms  $[\vec{x}]_{\mathfrak{B}}$  into  $[T(\vec{x})]_{\mathfrak{B}}$ :

$$[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}},$$

$\forall \vec{x} \in \mathbb{R}^n$ . This matrix  $B$  is called the  $\mathfrak{B}$ -matrix of  $T$ . We can construct  $B$  column-by-column, as follows:

$$B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathfrak{B}} & \dots & [T(\vec{v}_n)]_{\mathfrak{B}} \end{bmatrix}.$$

**Note:** In [MATH 524] we use the notation  $\mathcal{M}(T, \mathfrak{B})$  — “The matrix of  $T$  with respect to the basis  $\mathfrak{B}$ .”



## Example: Projection

## Example (Projection)

Let  $L$  be the line in  $\mathbb{R}^2$  spanned by  $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . Let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear transformation that projects any vector  $\vec{x}$  orthogonally onto  $L$ . It is quite useful to think of this in a coordinate system where one axis is  $L$  and the other is  $L^\perp$ ...

Let  $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  and  $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$  (clearly  $\vec{v}_1 \cdot \vec{v}_2 = 0$ , so they are perpendicular.) Now, if we have a vector  $\vec{x} = c_1 \vec{v}_1 + c_2 \vec{v}_2$ , then  $T(\vec{x}) = c_1 \vec{v}_1$ , or

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \rightsquigarrow [T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix};$$

which means that the projection matrix is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ in } \mathfrak{B}\text{-coordinates; compare with } \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \text{ in standard coordinates.}$$



## The Matrix of a Linear Transformation

Proof (The  $\mathfrak{B}$ -Matrix of a Linear Transformation))

$\mathfrak{B}$ -coordinates:  $\vec{x} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$ ; then

$$\begin{aligned} [T(\vec{x})]_{\mathfrak{B}} &= [T(c_1 \vec{v}_1 + \dots + c_n \vec{v}_n)]_{\mathfrak{B}} && [\vec{x} \text{ as lin. comb.}] \\ &= [c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n)]_{\mathfrak{B}} && [T \text{ is lin. trans.}] \\ &= c_1 [T(\vec{v}_1)]_{\mathfrak{B}} + \dots + c_n [T(\vec{v}_n)]_{\mathfrak{B}} && [[\cdot]_{\mathfrak{B}} \text{ is linear.}] \\ &= \begin{bmatrix} [T(\vec{v}_1)]_{\mathfrak{B}} & \dots & [T(\vec{v}_n)]_{\mathfrak{B}} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} && [\text{book-keeping}] \\ &= B[\vec{x}]_{\mathfrak{B}} && [\text{identify}] \end{aligned}$$

(Sequence of Definitions / Properties)



Standard Matrix vs.  $\mathfrak{B}$ -matrixTheorem (Standard Matrix vs.  $\mathfrak{B}$ -Matrix)

Consider a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$  of  $\mathbb{R}^n$ . Let  $B$  be the  $\mathfrak{B}$ -matrix of  $T$ , and let  $A$  be the standard matrix of  $T$  — so that  $T(\vec{x}) = A\vec{x} \forall \vec{x} \in \mathbb{R}^n$ ; then

$$AS = SB, \quad B = S^{-1}AS, \quad A = SBS^{-1}, \quad \text{where } S = [\vec{v}_1 \ \dots \ \vec{v}_n]$$

This follows from the linear transform relations:

$$T(\vec{x}) = A\vec{x}, \quad [T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}},$$

and

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}}, \quad T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}}$$

We formalize the matrix relations (in 2 slides)...



## Similar Matrices — Definition

## Definition (Similar Matrices)

Consider two matrices  $A, B \in \mathbb{R}^{n \times n}$ . We say that  $A$  is *similar* to  $B$  if there exists an invertible matrix  $S$  such that

$$AS = SB, \quad B = S^{-1}AS, \quad A = SBS^{-1}$$

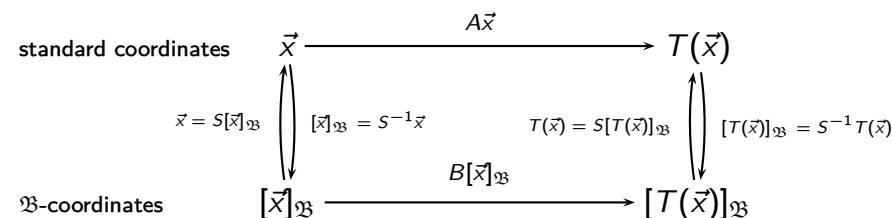
At this point we do not have an efficient way of finding out whether two given matrices are similar.

(We can set up a matrix  $S$  and leave its entries as variables, create  $AS$  and  $SB$ , and then set the two results equal and solve for the  $S$ -entries... However, better methods will be developed in the near future.)

Standard Matrix vs.  $\mathfrak{B}$ -matrix

((Change of Basis))

## Visualizing the Theorem:



$$S = [\vec{v}_1 \ \dots \ \vec{v}_n], \quad \mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$$

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}}, \quad S^{-1}\vec{x} = [\vec{x}]_{\mathfrak{B}}; \quad T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}}, \quad S^{-1}T(\vec{x}) = [T(\vec{x})]_{\mathfrak{B}}$$

Therefore

$$A\vec{x} = T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}} = SB[\vec{x}]_{\mathfrak{B}} = SBS^{-1}\vec{x}$$



## [FOCUS :: MATH] Similar Matrices — Properties

## Theorem (Matrix Similarity is an Equivalence Relation\*)

**reflexivity**  $A \in \mathbb{R}^{n \times n}$  is similar to itself.

**symmetry** If  $A$  is similar to  $B$ , then  $B$  is similar to  $A$

**transitivity** If  $A$  is similar to  $B$ , and  $B$  is similar to  $C$ , then  $A$  is similar to  $C$

**Reflexivity** let  $S = I_n$ .

**Symmetry** given  $AS_A = S_A B$ , let  $S_B = S_A^{-1}$ ; then  $S_B A S_A S_B = S_B A$ , and  $S_B S_A B S_B = B S_B$ , so that  $B S_B = S_B A$ .

**Transitivity**, we have  $AS_1 = S_1 B$ , and  $BS_2 = S_2 C$ ; now  $AS_1 S_2 = S_1 B S_2 = S_1 S_2 C$ ; so with  $S_3 = S_1 S_2$  we have  $AS_3 = S_3 C$ .



Diagonal  $\mathcal{B}$ -matrix

## Example

Given  $T(\vec{x}) = A\vec{x}$  ( $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ), we often want the basis  $\mathcal{B}$  be such that the  $\mathcal{B}$ -matrix of  $T$  is *diagonal*, that is

$$B = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}$$

The big question is how to pick the basis  $(\vec{v}_1, \vec{v}_2)$  so that happens?!

Recall that each column in the  $\mathcal{B}$ -matrix is of the form

$$[T(\vec{v}_k)]_{\mathcal{B}}$$

and the components of the column vectors are the coordinates expressed in the basis. We want *only* the  $k^{\text{th}}$  component to be non-zero, which means we must have  $T(\vec{v}_k) = b_{kk}\vec{v}_k$ .



## Suggested Problems 3.4

## Available on Learning Glass videos:

3.4 — 1, 3, 4, 7, 9, 17, 19, 23, 27, 29, 37

Diagonal  $\mathcal{B}$ -matrixTheorem (When is the  $\mathcal{B}$ -matrix Diagonal?)

Consider a linear transformation  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a basis  $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$  of  $\mathbb{R}^n$ .

- The  $\mathcal{B}$ -matrix  $B$  of  $T$  is diagonal **if and only if**  $T(\vec{v}_k) = c_k \vec{v}_k$   $\forall k \in \{1, \dots, n\}$ , for some scalars  $c_1, \dots, c_n \in \mathbb{R}$ .
- From a geometric point of view, this means that  $T(\vec{v}_k)$  is parallel to  $\vec{v}_k$   $\forall k \in \{1, \dots, n\}$ .

In general it is hard (we don't have the tools yet) to find a basis which makes the  $\mathcal{B}$ -matrix diagonal... We will return to this topic **[EIGENVECTORS and EIGENVALUES]** in the future... Simple examples with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are given by the vectors parallel and orthogonal to a line  $L$  we are orthogonally projecting onto, or reflecting across.



## Lecture – Book Roadmap

Lecture	Book, [GS5–]
3.1	§3.1, §3.2, §3.3
3.2	§3.1, §3.2, §3.3, §3.4
3.3	§3.1, §3.2, §3.3, §3.4, §3.5
3.4	§8.2, (§8.3)

§8.2 “Change of Basis” (p.412), “Choosing the Best Basis” (p.415–416)

§8.3 Extension of our discussion (we will revisit this)



## Metacognitive Exercise — Thinking About Thinking &amp; Learning

I know / learned

Almost there

Huh?!?

Right After Lecture

After Thinking / Office Hours / SI-session

After Reviewing for Quiz/Midterm/Final



## (3.4.4), (3.4.7)

**(3.4.4)** Determine whether the vector  $\vec{x}$  is in  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ . If  $\vec{x} \in V$ , find the coordinates of  $\vec{x}$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$  of  $V$ , and write the coordinate vector  $[\vec{x}]_{\mathfrak{B}}$ :

$$\vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

**(3.4.7)** Determine whether the vector  $\vec{x}$  is in  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ . If  $\vec{x} \in V$ , find the coordinates of  $\vec{x}$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$  of  $V$ , and write the coordinate vector  $[\vec{x}]_{\mathfrak{B}}$ :

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$



## (3.4.1), (3.4.3)

**(3.4.1)** Determine whether the vector  $\vec{x}$  is in  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ . If  $\vec{x} \in V$ , find the coordinates of  $\vec{x}$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$  of  $V$ , and write the coordinate vector  $[\vec{x}]_{\mathfrak{B}}$ :

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

**(3.4.3)** Determine whether the vector  $\vec{x}$  is in  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ . If  $\vec{x} \in V$ , find the coordinates of  $\vec{x}$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$  of  $V$ , and write the coordinate vector  $[\vec{x}]_{\mathfrak{B}}$ :

$$\vec{x} = \begin{bmatrix} 31 \\ 37 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 23 \\ 29 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 31 \\ 37 \end{bmatrix}.$$



## (3.4.9), (3.4.17)

**(3.4.9)** Determine whether the vector  $\vec{x}$  is in  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ . If  $\vec{x} \in V$ , find the coordinates of  $\vec{x}$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$  of  $V$ , and write the coordinate vector  $[\vec{x}]_{\mathfrak{B}}$ :

$$\vec{x} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

**(3.4.17)** Determine whether the vector  $\vec{x}$  is in  $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$ . If  $\vec{x} \in V$ , find the coordinates of  $\vec{x}$  with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$  of  $V$ , and write the coordinate vector  $[\vec{x}]_{\mathfrak{B}}$ :

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$





(3.4.19), (3.4.23)

**(3.4.19)** Find the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$ , with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ . Solve in three ways: (a) Use the formula  $B = S^{-1}AS$ , (b) Use a commutative diagram, and (c) construct  $B$  column-by-column.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

**(3.4.23)** Find the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$ , with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ . Solve in three ways: (a) Use the formula  $B = S^{-1}AS$ , (b) Use a commutative diagram, and (c) construct  $B$  column-by-column.

$$A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$



(3.4.37)

**(3.4.37)** Find a basis  $\mathfrak{B}$  of  $\mathbb{R}^n$  such that the  $\mathfrak{B}$ -matrix of the given linear transformation is diagonal.

$$T(\vec{x}) = [\text{Orthogonal Projection onto the line}] L = k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



(3.4.27), (3.4.29)

**(3.4.27)** Find the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$ , with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$ .

$$A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & 2 & 4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

**(3.4.29)** Find the matrix  $B$  of the linear transformation  $T(\vec{x}) = A\vec{x}$ , with respect to the basis  $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$ .

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$



## Motivation

So far, we have talked about vectors in  $\mathbb{R}^n$ , and matrix operations from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ ; expressed as linear transformations, via matrix-vector operations.

Some of the key concepts we have covered are: linear combination, linear transformation, kernel, image, subspace, span, linear independence, basis, dimension, and coordinates.

It turns out that this language (really, think of it as a *language*) can be applied to mathematical objects other than matrices and vectors; e.g. functions, equations, or infinite sequences.

The “language” of Linear Algebra is used throughout mathematics and other sciences.

Here, we “free” ourselves from the constraint of “living in  $\mathbb{R}^n$ ,” and re-state some of our result in a way that is useful in many settings.





## Linear Spaces

## Definition

## Definition (Linear Spaces)

A Linear Space  $V$  is a set with a definition (rule) for addition “+”, and a definition (rule) for scalar multiplication; and the following must hold ( $\forall u, v, w \in V, \forall c, k \in \mathbb{R}$ )

- a.  $v + w \in V$ .
- b.  $kv \in V$ .
- c.  $(u + v) + w = u + (v + w)$ .
- d.  $u + v = v + u$ .
- e.  $\exists n \in V: u + n = u$ , [NEUTRAL ELEMENT, denoted by 0]
- f.  $\exists \hat{u}: u + \hat{u} = 0; \hat{u}$  unique, and denoted by  $-u$ .
- g.  $k(u + v) = ku + kv$ .
- h.  $(c + k)u = cu + ku$ .
- i.  $c(ku) = (ck)u$ .
- j.  $1u = u$ .



## Examples: Linear Spaces

Example ( $\mathbb{R}^{n \times m}$ )

Given our previous definitions [NOTES#1.3] of matrix addition and scalar multiplication of a matrix, then  $\mathbb{R}^{n \times m}$ , the set of all  $n \times m$  matrices, is a linear space. The zero-matrix is the neutral element.

## Example (Infinite Sequences)

The set of all infinite sequences  $\bar{x} = (x_1, x_2, \dots, x_\infty)$  is a linear space; addition is defined  $\bar{x} + \bar{y} = (x_1 + y_1, x_2 + y_2, \dots, x_\infty + y_\infty)$ ; scalar multiplication  $k\bar{x} = (kx_1, kx_2, \dots, kx_\infty)$ . The zero-sequence  $(0, 0, \dots)$  is the neutral element.



## Examples: Linear Spaces

We have already seen the “prototype” linear spaces:

Example (Linear Space(s)  $\mathbb{R}^n$ )

Here, the natural element is the zero vector  $\vec{0} \in \mathbb{R}^n$ .

We give a few other examples

## Example

Let  $F(\mathbb{R}, \mathbb{R})$  set the set of all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$ , with the operations

$$(f + g)(x) = f(x) + g(x), \quad \text{and} \quad (kf)(x) = kf(x)$$

then  $F(\mathbb{R}, \mathbb{R})$  is a linear space; the function  $f(x) = 0$  is the neutral element.



## Examples: Linear Spaces

## Example (Linear Equations)

The linear equations in 3 unknowns

$$ax + by + cz = d$$

where  $a, b, c$ , and  $d$  are constants, form a linear space. The neutral element is  $0 = 0$ , i.e.  $a = b = c = d = 0$ .

## Example (Complex Numbers)

Let  $\mathbb{C}$  be the set of complex numbers  $z = a + bi$ ; with addition defined by  $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$ , and scalar multiplication by  $kz = ka + (kb)i$  ( $\forall k \in \mathbb{R}$ ).  $\mathbb{C}$  with these two operations form a linear space, with neutral element  $0 + 0i$ .



## Definitions

### Definition (Linear Combination)

We say that an element  $u$  of a linear space is a *linear combination* of the elements  $v_1, \dots, v_n$  if  $u = c_1 v_1 + \dots + c_n v_n$ .

Since the basic notation for Linear Algebra (on  $\mathbb{R}^n$ ) are defined in terms of linear combinations, we can generalize those concepts to all Linear Spaces without generalizations:

### Definition (Subspaces)

A subset  $W$  of a linear space  $V$  is called a subspace of  $V$  if

- a.  $W$  contains the neutral element, 0, of  $V$
- b.  $W$  is closed under addition
- c.  $W$  is closed under scalar multiplication
- b+c.**  $\Rightarrow W$  is closed under linear combinations



## Examples: Subspaces of $F(\mathbb{R}, \mathbb{R})$

### Example (More subspaces of $F(\mathbb{R}, \mathbb{R})$ )

- $C^n$ ,  $n \in \{1, 2, \dots, \infty\}$  — the functions with  $n$  (possibly infinitely) many continuous derivatives form subspaces of  $F(\mathbb{R}, \mathbb{R})$ .
- $\mathcal{P}$ , the set of polynomials forms a subspace of  $F(\mathbb{R}, \mathbb{R})$ .
- $\mathcal{P}_n$ , the set of all polynomials of degree  $\leq n$  forms a subspace of  $F(\mathbb{R}, \mathbb{R})$ .



## Examples: Subspaces of $F(\mathbb{R}, \mathbb{R})$

### Example

The polynomials of degree 2 —  $\mathcal{P}_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ , form a subspace of  $F(\mathbb{R}, \mathbb{R})$ .

- $f(x) = 0 = 0x^2 + 0x + 0$
- $kp_1(x) + p_2(x) = (ka_1 + a_2)x^2 + (kb_1 + b_2)x + (kc_1 + c_2)$

### Example

The differentiable functions,  $C^0$  form a subspace of  $F(\mathbb{R}, \mathbb{R})$ .

- $f(x) = 0$ , with  $f'(x) = 0$
- Calculus tell us that  $(kf(x) + g(x))' = kf'(x) + g'(x)$ .



## Span, Linear Independence, Basis, Coordinates

### Example (Span, Linear Independence, Basis, Coordinates)

Consider the elements  $u_1, \dots, u_n$  in a linear space  $V$ .

- a.  $u_1, \dots, u_n$  *span*  $V$  if every  $v \in V$  can be expressed as a linear combination of  $u_1, \dots, u_n$
- b-i.  $u_i$  is *linearly dependent* if it is a linear combination of  $u_1, \dots, u_{i-1}$ .
- b-ii. The elements  $u_1, \dots, u_n$  are *linearly independent* if none of them is linearly dependent. This is the case if the equation

$$c_1 u_1 + \dots + c_n u_n = 0$$

only has the trivial solution  $c_1 = \dots = c_n = 0$ .



## Span, Linear Independence, Basis, Coordinates

## Example (Span, Linear Independence, Basis, Coordinates)

Consider the elements  $u_1, \dots, u_n$  in a linear space  $V$ .

**c-i.**  $u_1, \dots, u_n$  are a *basis* of  $V$  if they span  $V$  and are linearly independent. This means every  $v \in V$  can be written as a unique linear combination  $v = c_1 u_1 + \dots + c_n u_n$ ,

**c-ii.** The coefficients  $c_1, \dots, c_n$  are called the *coefficients* of  $v$  with respect to the basis  $\mathfrak{B} = (u_1, \dots, u_n)$ . The vector

$$\vec{c}^T = [c_1 \quad \dots \quad c_n]^T$$

in  $\mathbb{R}^n$  is called the  $\mathfrak{B}$ -coordinate vector of  $v$ , denoted by  $[v]_{\mathfrak{B}}$

**c-iii.** The transformation  $L(v) = [v]_{\mathfrak{B}} = [c_1 \quad \dots \quad c_n]^T$  is called the  $\mathfrak{B}$ -coordinate transformation, sometimes denoted by  $L_{\mathfrak{B}}$



## Linear (Ordinary) Differential Equations — ODEs

## Important for the Future!

## Theorem (Linear Differential Equations)

The solutions of the differential equation ( $a, b \in \mathbb{R}$  are constants)

$$u''(x) + au'(x) + bu(x) = 0$$

form a two-dimensional subspace of the space  $C^\infty$  of smooth functions; more generally, the solutions of the differential equation

$$v^{(n)}(x) + a_{n-1}v^{(n-1)}(x) + \dots + a_1v'(x) + a_0v(x) = 0$$

(where the coefficients  $a_0, \dots, a_{n-1}$  are constants) form an  $n$ -dimensional subspace of  $C^\infty$ . A differential equation of this form is called an  $n^{\text{th}}$ -order linear differential equation with constant coefficients.

The connection between linear algebra and ODEs (both in terms of theory and applications) is VERY STRONG. In many places the topics are taught together in a joint (sequence of) class(es).



## Linear Spaces: Theorems

## Properties

Theorem (Linearity of the  $\mathfrak{B}$ -coordinate transformation,  $L_{\mathfrak{B}}$ )

If  $\mathfrak{B}$  is a basis of a linear space, then  $\forall u, v \in V, \forall k \in \mathbb{R}$ :

a.  $[u + v]_{\mathfrak{B}} = [u]_{\mathfrak{B}} + [v]_{\mathfrak{B}}$

b.  $[ku]_{\mathfrak{B}} = k[u]_{\mathfrak{B}}$

(The proof is pretty much a copy of the  $\mathbb{R}^n$  version from [NOTES#3.4]).

## Theorem (Dimension(!!!))

If a linear space  $V$  has a basis with  $n$  elements, then all other bases of  $V$  consist of  $n$  elements as well, and we say

$$\dim(V) = n$$



## Finite Dimensional Subspaces

## Definition (Finite Dimensional Subspaces)

A linear space  $V$  is called finite dimensional if it has a (finite) basis  $v_1, \dots, v_n$ , so that  $\dim(V) = n$ . Otherwise the space is called *infinite dimensional*.

The space of polynomials,  $\mathcal{P}$ , is infinite dimensional.

The study of infinite dimensional linear spaces — e.g. Hilbert-, Banach-, and Sobolev spaces, belong in a course on functional analysis; somewhere beyond the horizon of ADVANCED CALCULUS... really, it's fun stuff!

