# Math 254: Introduction to Linear Algebra

## Notes #3.4 — Coordinates

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# Outline

## SLOs 3.4 Coordinates

After this lecture you should:

- Know the relation between the basis $\mathfrak{B}$ (of a subspace $V$), the $\mathfrak{B}$-coordinates of a vector $\vec{x} \in V$, and the $\mathfrak{B}$-coordinate vector $[\vec{x}]_{\mathfrak{B}}$.
- Be able to identify the Matrix and $\mathfrak{B}$-Matrix of a linear transformation.
- Know the basic definition of *Similarity of Matrices* (to be revisited in the context of Eigenvalues and Eigenvectors).

# Coordinates: Where are We?

**Example (Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates)**

Consider the vectors (in $\mathbb{R}^3$)

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and define the plane $V = \mathrm{span}(\vec{v}_1, \vec{v}_2)$ in $\mathbb{R}^3$. Is the vector

$$\vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

in the plane? (Visualize)

## Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates

We are really asking *"Can we find $c_1$ and $c_2$ so that:*

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \text{?''}$$

We look in our toolbox, and what do we find...

$$\mathrm{rref}\left( \left[ \begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array} \right] \right) = \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

that is, the short answer is *"Yes,"* and the slightly longer $c_1 = 3$, $c_2 = 2$.

# Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates

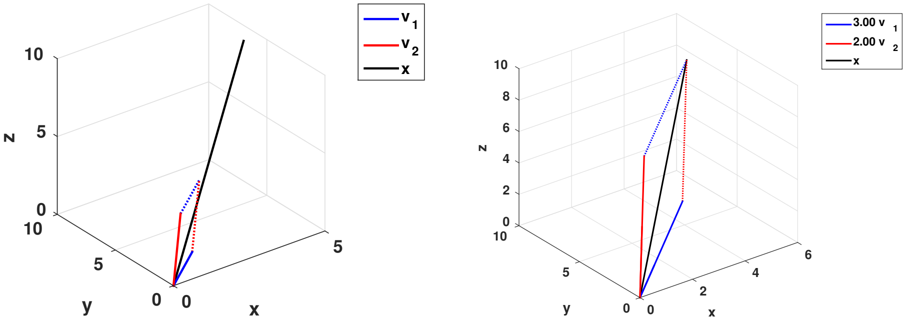


**Figure:** In the [LEFT] panel we see the vectors $\vec{v}_1$, $\vec{v}_2$, and the linear combination $\vec{v}_1 + \vec{v}_2$. In the [RIGHT] panel we see the vectors $3\vec{v}_1$, $2\vec{v}_2$, and the linear combination $3\vec{v}_1 + 2\vec{v}_2$ which reaches the vector $\vec{x}$. ($\exists$ Movie)

# Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates

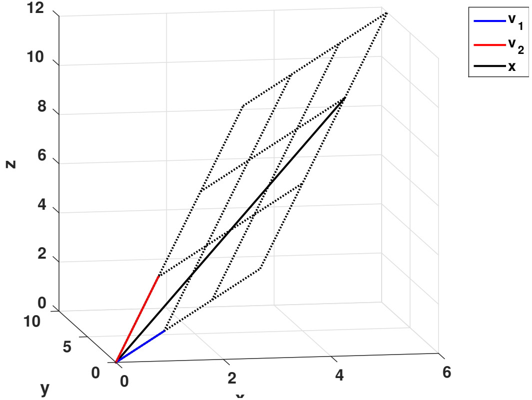


**Figure:** We can think of $\vec{c} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ as the address (coordinate) of the vector $\vec{x}$ in a grid where $\vec{v}_1$ and $\vec{v}_2$ are the coordinate-axes.

## Coefficients in a Linear Combination $\rightsquigarrow$ Coordinates

By introducing the $c_1$-$c_2$ coordinates along the vectors $\vec{v}_1$ and $\vec{v}_2$, we transform the plane $V$ to $\mathbb{R}^2$.

It is natural to have a brief panic attack when you realize that the coordinate axes $\vec{v}_1$ and $\vec{v}_2$ are not perpendicular; but, really, it is not a problem... each point in the plane does get its own unique (coordinate) address.

**Notation (Basis, $\mathfrak{B}$; coordinate vector $[\vec{x}]_{\mathfrak{B}}$)**

Let $\mathfrak{B}$ denote the basis $\vec{v}_1$-$\vec{v}_2$ of $V$, and let the coordinate vector of $\vec{x}$ with respect to $\mathfrak{B}$ be denoted by $[\vec{x}]_{\mathfrak{B}}$.

In our example we had $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, with $\mathfrak{B} = \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$

# Coordinates in a Subspace of $\mathbb{R}^n$

**Definition (Coordinates in a Subspace of $\mathbb{R}^n$)**

Consider a basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of a subspace $V = \mathrm{span}\,(\vec{v}_1, \dots, \vec{v}_m)$ of $\mathbb{R}^n$; $\dim(V) = m \leq n$. Any vector $\vec{x} \in V$ can be written uniquely as

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m.$$

The scalars $c_1, \dots, c_m$ are called *$\mathfrak{B}$-coordinates* of $\vec{x}$, and the vector

$$\vec{c} = \begin{bmatrix} c_1 & \dots & c_m \end{bmatrix}^T$$

is the *$\mathfrak{B}$-coordinate vector* of $\vec{x}$, denoted by $[\vec{x}]_{\mathfrak{B}}$. Thus,

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 & \cdots & c_m \end{bmatrix}^T \quad \text{means that } \vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m.$$

Note that

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}}, \text{ where } S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}, \text{ an } (n \times m) \text{ matrix.}$$

# Checking Our Example

We had:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}, \quad \rightsquigarrow S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

We computed:

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

We can reconstruct $\vec{x}$ from $S$ and $[\vec{x}]_{\mathfrak{B}}$:

$$S[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \vec{x}. \quad \checkmark$$

# Property: Linearity of Coordinates

**Theorem (Linearity of Coordinates)**

*If $\mathfrak{B}$ is a basis of a subspace $V$ of $\mathbb{R}^n$, then*

$$\begin{array}{lll} \textbf{a.} & [\vec{x}+\vec{y}]_{\mathfrak{B}} = [\vec{x}]_{\mathfrak{B}} + [\vec{y}]_{\mathfrak{B}} & \forall \vec{x}, \vec{y} \in V, \text{ and} \\ \textbf{b.} & [k\vec{x}]_{\mathfrak{B}} = k[\vec{x}]_{\mathfrak{B}} & \forall \vec{x} \in V, \forall k \in \mathbb{R} \end{array}$$

**Building Blocks for the Proof**

With $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ keep in mind

$$\left.\begin{array}{rcl} \vec{x} & = & c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \\ \vec{y} & = & d_1\vec{v}_1 + \cdots + d_m\vec{v}_m \end{array}\right\} \Leftrightarrow [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}, \; [\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix},$$

and use the definitions of vector-addition and scaling.

# Example: Coordinates / Basis / Vectors

### Example (Basis-Vector-Coordinate Transformations)

Consider the basis $\mathfrak{B}$ of $\mathbb{R}^2$ consisting of vectors

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \rightsquigarrow S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

(a) If $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, find $[\vec{x}]_{\mathfrak{B}}$; (b) if $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{y}$.

For (a) we need to solve (for $[\vec{x}]_{\mathfrak{B}}$)

$$S[\vec{x}]_{\mathfrak{B}} = \vec{x}, \ \mathrm{rref}\left(\left[\begin{array}{cc|c} 3 & -1 & 10 \\ 1 & 3 & 10 \end{array}\right]\right) = \left(\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right]\right), \ [\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

# Example: Coordinates / Basis / Vectors

**Example (Basis-Vector-Coordinate Transformations)**

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$$\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \rightsquigarrow S = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

(a) If $\vec{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$, find $[\vec{x}]_{\mathfrak{B}}$; (b) if $[\vec{y}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, find $\vec{y}$.

For (b) we need to compute ($\vec{y}$)

$$\vec{y} = S[\vec{y}]_{\mathfrak{B}}, \quad \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

# Example: Projection

## Example (Projection)

Let $L$ be the line in $\mathbb{R}^2$ spanned by $\vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects any vector $\vec{x}$ orthogonally onto $L$. It is quite useful to think of this in a coordinate system where one axis is $L$ and the other is $L^\perp$...

Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ (clearly $\vec{v}_1 \cdot \vec{v}_2 = 0$, so they are perpendicular.)
Now, if we have a vector $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$, then $T(\vec{x}) = c_1\vec{v}_1$, or

$$[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \rightsquigarrow [T(\vec{x})]_{\mathfrak{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix};$$

which means that the projection matrix is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ in } \mathfrak{B}\text{-coordinates; compare with } \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \text{ in standard coordinates.}$$

# The Matrix of a Linear Transformation

**Theorem (The $\mathfrak{B}$-Matrix of a Linear Transformation)**

*Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$ of $\mathbb{R}^n$. There there exits a unique "$\exists!$" $B \in \mathbb{R}^{n\times n}$ matrix that transforms $[\vec{x}]_{\mathfrak{B}}$ into $[T(\vec{x})]_{\mathfrak{B}}$:*

$$[T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}},$$

*$\forall \vec{x} \in \mathbb{R}^n$. This matrix $B$ is called the $\mathfrak{B}$-matrix of $T$. We can construct $B$ column-by-column, as follows:*

$$B = \begin{bmatrix} [T(\vec{v}_1)]_{\mathfrak{B}} & \dots & [T(\vec{v}_n)]_{\mathfrak{B}} \end{bmatrix}.$$

Proof

**Note:** In [MATH 524] we use the notation $\mathcal{M}(T, \mathfrak{B})$ — "The matrix of $T$ with respect to the basis $\mathfrak{B}$."

# The Matrix of a Linear Transformation

**Proof (The $\mathfrak{B}$-Matrix of a Linear Transformation))**

$\mathfrak{B}$-coordinates: $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$; then

$$
\begin{aligned}
[T(\vec{x})]_{\mathfrak{B}} &= [T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n)]_{\mathfrak{B}} && [\vec{x} \text{ as lin.comb.}] \\
&= [c_1 T(\vec{v}_1) + \cdots + c_n T(\vec{v}_n)]_{\mathfrak{B}} && [T \text{ is lin.trans.}] \\
&= c_1[T(\vec{v}_1)]_{\mathfrak{B}} + \cdots + c_n[T(\vec{v}_n)]_{\mathfrak{B}} && [[\circ]_{\mathfrak{B}} \text{ is linear.}] \\
&= \big[[T(\vec{v}_1)]_{\mathfrak{B}} \quad \ldots \quad [T(\vec{v}_n)]_{\mathfrak{B}}\big] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} && [\text{book-keeping}] \\
&= B[\vec{x}]_{\mathfrak{B}}. && [\text{identify}]
\end{aligned}
$$

(Sequence of Definitions / Properties)

## Standard Matrix vs. $\mathfrak{B}$-matrix

**Theorem (Standard Matrix vs. $\mathfrak{B}$-Matrix)**

*Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^n$. Let $B$ be the $\mathfrak{B}$-matrix of $T$, and let $A$ be the standard matrix of $T$ — so that $T(\vec{x}) = A\vec{x}\ \forall x \in \mathbb{R}^n$; then*

$$AS = SB,\ B = S^{-1}AS,\ A = SBS^{-1},\ \text{where } S = \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_n \end{bmatrix}$$

This follows from the linear transform relations:

$$T(\vec{x}) = A\vec{x}, \quad [T(\vec{x})]_{\mathfrak{B}} = B[\vec{x}]_{\mathfrak{B}},$$

and

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}}, \quad T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}}$$

We formalize the matrix relations (in 2 slides)...

# Standard Matrix vs. $\mathfrak{B}$-matrix

((Change of Basis))

## Visualizing the Theorem:

$$
\begin{tikzcd}
\text{standard coordinates} & \vec{x} \arrow[rr, "A\vec{x}"] \arrow[d, shift left, "{[\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x}}"] & & T(\vec{x}) \arrow[d, shift left, "{[T(\vec{x})]_{\mathfrak{B}} = S^{-1}T(\vec{x})}"] \\
\mathfrak{B}\text{-coordinates} & {[\vec{x}]_{\mathfrak{B}}} \arrow[rr, "B[\vec{x}]_{\mathfrak{B}}"] \arrow[u, shift left, "{\vec{x} = S[\vec{x}]_{\mathfrak{B}}}"] & & {[T(\vec{x})]_{\mathfrak{B}}} \arrow[u, shift left, "{T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}}}"]
\end{tikzcd}
$$

$$S = \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_n \end{bmatrix}, \quad \mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_n)$$

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}}, \;\; S^{-1}\vec{x} = [\vec{x}]_{\mathfrak{B}}; \;\; T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}}, \;\; S^{-1}T(\vec{x}) = [T(\vec{x})]_{\mathfrak{B}}$$

Therefore

$$A\vec{x} = T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}} = SB[\vec{x}]_{\mathfrak{B}} = SBS^{-1}\vec{x}$$

# Similar Matrices — Definition

**Definition (Similar Matrices)**

Consider two matrices $A, B \in \mathbb{R}^{n \times n}$. We say that $A$ is *similar* to $B$ if there exists an invertible matrix $S$ such that

$$AS = SB, \quad B = S^{-1}AS, \quad A = SBS^{-1}$$

At this point we do not have an efficient way of finding out whether two given matrices are similar.

(We can set up a matrix $S$ and leave its entries as variables, create $AS$ and $SB$, and then set the two results equal and solve for the $S$-entries... However, better methods will be developed in the near future.)

# [Focus :: Math] Similar Matrices — Properties

**Theorem (Matrix Similarity is an Equivalence Relation*)**

**reflexivity** *$A \in \mathbb{R}^{n \times n}$ is similar to itself.*

**symmetry** *If $A$ is similar to $B$, then $B$ is similar to $A$*

**transitivity** *If $A$ is similar to $B$, and $B$ is similar to $C$, then $A$ is similar to $C$*

**Reflexivity** let $S = I_n$.

**Symmetry** given $AS_A = S_A B$, let $S_B = S_A^{-1}$; then $S_B A S_A S_B = S_B A$, and $S_B S_A B S_B = B S_B$, so that $B S_B = S_B A$.

**Transitivity**, we have $AS_1 = S_1 B$, and $BS_2 = S_2 C$; now $AS_1 S_2 = S_1 B S_2 = S_1 S_2 C$; so with $S_3 = S_1 S_2$ we have $AS_3 = S_3 C$.

# Diagonal $\mathfrak{B}$-matrix

**Example**

Given $T(\vec{x}) = A\vec{x}$ ($T : \mathbb{R}^2 \to \mathbb{R}^2$), we often want the basis $\mathfrak{B}$ be such that the $\mathfrak{B}$-matrix of $T$ is *diagonal*, that is

$$B = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}$$

The big question is how to pick the basis $(\vec{v}_1, \vec{v}_2)$ so that happens?!

Recall that each column in the $\mathfrak{B}$-matrix is of the form

$$[T(\vec{v}_k)]_{\mathfrak{B}}$$

and the components of the column vectors are the coordinates expressed in the basis. We want *only* the $k^{\text{th}}$ component to be non-zero, which means we must have $T(\vec{v}_k) = b_{kk}\vec{v}_k$.

# Diagonal $\mathfrak{B}$-matrix

**Theorem (When is the $\mathfrak{B}$-matrix Diagonal?)**

*Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and a basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ of $\mathbb{R}^n$.*

- *The $\mathfrak{B}$-matrix $B$ of $T$ is diagonal if and only if $T(\vec{v}_k) = c_k \vec{v}_k$ $\forall k \in \{1, \ldots, n\}$, for some scalars $c_1, \ldots, c_n \in \mathbb{R}$.*
- *From a geometric point of view, this means that $T(\vec{v}_k)$ is parallel to $\vec{v}_k$ $\forall k \in \{1, \ldots, n\}$.*

In general it is hard (we don't have the tools yet) to find a basis which makes the $\mathfrak{B}$-matrix diagonal... We will return to this topic [EIGENVECTORS and EIGENVALUES] in the future... Simple examples with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are given by the vectors parallel and orthogonal to a line $L$ we are orthogonally projecting onto, or reflecting across.

# Suggested Problems 3.4

**Available on Learning Glass videos:**

3.4 — 1, 3, 4, 7, 9, 17, 19, 23, 27, 29, 37

# Lecture – Book Roadmap

| Lecture | Book, [GS5–] |
|---|---|
| 3.1 | §3.1, §3.2, §3.3 |
| 3.2 | §3.1, §3.2, §3.3, §3.4 |
| 3.3 | §3.1, §3.2, §3.3, §3.4, §3.5 |
| 3.4 | §8.2, (§8.3) |

§8.2 "Change of Basis" (p.412), "Choosing the Best Basis" (p.415–416)

§8.3 Extension of our discussion (we will revisit this)

# Metacognitive Exercise — Thinking About Thinking & Learning

| I know / learned | Almost there | Huh?!? |
|---|---|---|
| **Right After Lecture** | | |
| | | |
| **After Thinking / Office Hours / SI-session** | | |
| | | |
| **After Reviewing for Quiz/Midterm/Final** | | |
| | | |

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# (3.4.1), (3.4.3)

**(3.4.1)** Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of $V$, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

**(3.4.3)** Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of $V$, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 31 \\ 37 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 23 \\ 29 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 31 \\ 37 \end{bmatrix}.$$

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# (3.4.4), (3.4.7)

**(3.4.4)** Determine whether the vector $\vec{x}$ is in $V = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of $V$, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

**(3.4.7)** Determine whether the vector $\vec{x}$ is in $V = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of $V$, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

# (3.4.9), (3.4.17)

**(3.4.9)** Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of $V$, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

**(3.4.17)** Determine whether the vector $\vec{x}$ is in $V = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of $V$, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

# (3.4.19), (3.4.23)

**(3.4.19)** Find the matrix $B$ of the linear transformation $T(\vec{x}) = A\vec{x}$, with respect to the basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$. Solve in three ways: *(a)* Use the formula $B = S^{-1}AS$, *(b)* Use a commutative diagram, and *(c)* construct $B$ column-by-column.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

**(3.4.23)** Find the matrix $B$ of the linear transformation $T(\vec{x}) = A\vec{x}$, with respect to the basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$. Solve in three ways: *(a)* Use the formula $B = S^{-1}AS$, *(b)* Use a commutative diagram, and *(c)* construct $B$ column-by-column.

$$A = \begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

# (3.4.27), (3.4.29)

**(3.4.27)** Find the matrix $B$ of the linear transformation $T(\vec{x}) = A\vec{x}$, with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$.

$$A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & =2 & 4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

**(3.4.29)** Find the matrix $B$ of the linear transformation $T(\vec{x}) = A\vec{x}$, with respect to the basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$.

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 2 \\ 3 & -9 & 6 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

# (3.4.37)

**(3.4.37)** Find a basis $\mathfrak{B}$ of $\mathbb{R}^n$such that the $\mathfrak{B}$-matrix of the given linear transformation is diagonal.

$$T(\vec{x}) = [\text{Orthogonal Projection onto the line}]\ L = k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

# Motivation

So far, we have talked about vectors in $\mathbb{R}^n$, and matrix operations from $\mathbb{R}^m$ to $\mathbb{R}^n$; expressed as linear transformations, via matrix-vector operations.

Some of the key concepts we have covered are: linear combination, linear transformation, kernel, image, subspace, span, linear independence, basis, dimension, and coordinates.

It turns out that this language (really, think of it as a *language*) can be applied to mathematical objects other than matrices and vectors; *e.g.* functions, equations, or infinite sequences.

The "language" of Linear Algebra is used throughout mathematics and other sciences.

Here, we "free" ourselves from the constraint of "living in $\mathbb{R}^n$," and re-state some of our result in a way that is useful in many settings.

# Linear Spaces — Definition

Definition (Linear Spaces)

A Linear Space $V$ is a set with a definition (rule) for addition "+", and a definition (rule) for scalar multiplication; and the following must hold ($\forall u, v, w \in V$, $\forall c, k \in \mathbb{R}$)

- **a.** $v + w \in V$.
- **b.** $kv \in V$.
- **c.** $(u + v) + w = u + (v + w)$.
- **d.** $u + v = v + u$.
- **e.** $\exists n \in V$: $u + n = u$, [NEUTRAL ELEMENT, denoted by 0]
- **f.** $\exists \widehat{u}$: $u + \widehat{u} = 0$; $\widehat{u}$ unique, and denoted by $-u$.
- **g.** $k(u + v) = ku + kv$.
- **h.** $(c + k)u = cu + ku$.
- **i.** $c(ku) = (ck)u$.
- **j.** $1u = u$.

# Examples: Linear Spaces

We have already seen the "prototype" linear spaces:

**Example (Linear Space(s) $\mathbb{R}^n$)**

Here, the natural element is the zero vector $\vec{0} \in \mathbb{R}^n$.

We give a few other examples

**Example**

Let $F(\mathbb{R}, \mathbb{R})$ set the set of all functions $f : \mathbb{R} \to \mathbb{R}$, with the operations

$$(f+g)(x) = f(x) + g(x), \quad \text{and } (kf)(x) = kf(x)$$

then $F(\mathbb{R}, \mathbb{R})$ is a linear space; the function $f(x) = 0$ is the neutral element.

# Examples: Linear Spaces

**Example ($\mathbb{R}^{n\times m}$)**

Given our previous definitions [NOTES#1.3] of matrix addition and scalar multiplication of a matrix, then $\mathbb{R}^{n\times m}$, the set of all $n \times m$ matrices, is a linear space. The zero-matrix is the neutral element.

**Example (Infinite Sequences)**

The set of all infinite sequences $\overline{x} = (x_1, x_2, \ldots, x_\infty)$ is a linear space; addition is defined $\overline{x} + \overline{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_\infty + y_\infty)$; scalar multiplication $k\overline{x} = (kx_1, kx_2, \ldots, kx_\infty)$. The zero-sequence $(0, 0, \ldots)$ is the neutral element.

# Examples: Linear Spaces

## Example (Linear Equations)

The linear equations in 3 unknowns

$$ax + by + cz = d$$

where $a, b, c$, and $d$ are constants, form a linear space. The neutral element is $\mathbf{0} = 0$, *i.e.* $a = b = c = d = 0$.

## Example (Complex Numbers)

Let $\mathbb{C}$ be the set of complex numbers $z = a + bi$; with addition defined by $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$, and scalar multiplication by $kz = ka + (kb)i$ $(\forall k \in \mathbb{R})$. $\mathbb{C}$ with these two operations form a linear space, with neutral element $0 + 0i$.

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# Definitions

Definition (Linear Combination)

We say that an element $u$ of a linear space is a *linear combination* of the elements $v_1, \ldots, v_n$ if $u = c_1 v_1 + \cdots + c_n v_n$.

Since the basic notation for Linear Algebra (on $\mathbb{R}^n$) are defined in terms of linear combinations, we can generalize those concepts to all Linear Spaces without generalizations:

Definition (Subspaces)

A subset $W$ of a linear space $V$ is called a subspace of $V$ if

- **a.** $W$ contains the neutral element, 0, of $V$
- **b.** $W$ is closed under addition
- **c.** $W$ is closed under scalar multiplication

**b+c**. $\Rightarrow$ $W$ is closed under linear combinations

# Examples: Subspaces of $F(\mathbb{R}, \mathbb{R})$

## Example

The polynomials of degree 2 — $\mathcal{P}_2 = \{\, ax^2 + bx + c \;:\; a, b, c \in \mathbb{R} \,\}$, form a subspace of $F(\mathbb{R}, \mathbb{R})$.

- $f(x) = 0 = 0x^2 + 0x + 0$
- $kp_1(x) + p_2(x) = (ka_1 + a_2)x^2 + (kb_1 + b_2)x + (kc_1 + c_2)$

## Example

The differentiable functions, $C^0$ form a subspace of $F(\mathbb{R}, \mathbb{R})$.

- $f(x) = 0$, with $f'(x) = 0$
- Calculus tell us that $(kf(x) + g(x))' = kf'(x) + g'(x)$.

# Examples: Subspaces of $F(\mathbb{R}, \mathbb{R})$

## Example (More subspaces of $F(\mathbb{R}, \mathbb{R})$)

- $C^n$, $n \in \{1, 2, \ldots, \infty\}$ — the functions with $n$ (possibly infinitely) many continuous derivatives form subspaces of $F(\mathbb{R}, \mathbb{R})$.
- $\mathcal{P}$, the set of polynomials forms a subspace of $F(\mathbb{R}, \mathbb{R})$.
- $\mathcal{P}_n$, the set of all polynomials of degree $\leq n$ forms a subspace of $F(\mathbb{R}, \mathbb{R})$.

# Span, Linear Independence, Basis, Coordinates

## Example (Span, Linear Independence, Basis, Coordinates)

Consider the elements $u_1, \ldots, u_n$ in a linear space $V$.

**a.** $u_1, \ldots, u_n$ *span* $V$ if every $v \in V$ can be expressed as a linear combination of $u_1, \ldots, u_n$

**b-*i*.** $u_i$ is *linearly dependent* if it is a linear combination of $u_1, \ldots, u_{i-1}$.

**b-*ii*.** The elements $u_1, \ldots, u_n$ are *linearly independent* if none of them is linearly dependent. This is the case if the equation

$$c_1 u_1 + \cdots + c_n u_n = 0$$

only has the trivial solution $c_1 = \cdots = c_n = 0$.

# Span, Linear Independence, Basis, Coordinates

## Example (Span, Linear Independence, Basis, Coordinates)

Consider the elements $u_1, \ldots, u_n$ in a linear space $V$.

**c-*i*.** $u_1, \ldots, u_n$ are a *basis* of $V$ is they span $V$ and are linearly independent. This means every $v \in V$ can be written as a unique linear combination $v = c_1 u_1 + \cdots + c_n u_n$,

**c-*ii*.** The coefficients $c_1, \ldots, c_n$ are called the *coefficients* of $v$ with respect to the basis $\mathfrak{B} = (u_1, \ldots, u_n)$. The vector

$$\vec{c}^T = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^T$$

in $\mathbb{R}^n$ is called the $\mathfrak{B}$-*coordinate vector* of $v$, denoted by $[v]_{\mathfrak{B}}$

**c-*iii*.** The transformation $L(v) = [v]_{\mathfrak{B}} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^T$ is called the $\mathfrak{B}$-*coordinate transformation*, sometimes denoted by $L_{\mathfrak{B}}$

## Linear Spaces: Theorems — Properties

**Theorem** (Linearity of the $\mathfrak{B}$-*coordinate transformation*, $L_{\mathfrak{B}}$)

*If $\mathfrak{B}$ is a basis of a linear space, then $\forall u, v \in V$, $\forall k \in \mathbb{R}$:*

**a.** $[u+v]_{\mathfrak{B}} = [u]_{\mathfrak{B}} + [v]_{\mathfrak{B}}$

**b.** $[ku]_{\mathfrak{B}} = k[u]_{\mathfrak{B}}$

(The proof is pretty much a copy of the $\mathbb{R}^n$ version from [NOTES#3.4]).

**Theorem** (Dimension(!!!))

*If a linear space $V$ has a basis with $n$ elements, then all other bases of $V$ consist of $n$ elements as well, and we say*

$$\dim(V) = n$$

## Linear (Ordinary) Differential Equations — ODEs Important for the Future!

**Theorem (Linear Differential Equations)**

*The solutions of the differential equation ($a, b \in \mathbb{R}$ are constants)*

$$u''(x) + au'(x) + bu(x) = 0$$

*form a two-dimensional subspace of the space $C^\infty$ of smooth functions; more generally, the solutions of the differential equation*

$$v^{(n)}(x) + a_{n-1}v^{(n-1)}(x) + \cdots + a_1 v'(x) + a_0 u(x) = 0$$

*(where the coefficients $a_0, \ldots, a_{n-1}$ are constants) form an $n$-dimensional subspace of $C^\infty$. A differential equation of this form is called an $n^{th}$-order linear differential equation with constant coefficients.*

The connection between linear algebra and ODEs (both in terms of theory and applications) is VERY STRONG. In many places the topics are taught together in a joint (sequence of) class(es).

## Finite Dimensional Subspaces

**Definition (Finite Dimensional Subspaces)**

A linear space $V$ is called finite dimensional if it has a (finite) basis $v_1, \ldots, v_n$, so that $\dim(V) = n$. Otherwise the space is called *infinite dimensional*.

The space of polynomials, $\mathcal{P}$, is infinite dimensional.

The study of infinite dimensional linear spaces — *e.g.* Hilbert-, Banach-, and Sobolev spaces, belong in a course on functional analysis; somewhere beyond the horizon of ADVANCED CALCULUS... really, it's fun stuff!