Math 254: Introduction to Linear Algebra

Notes #5.1 — Orthogonal Projections and Orthonormal Bases

Peter Blomgren
⟨blomgren@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

Spring 2022
(Revised: March 21, 2022)

Outline

SLOs 5.1 Orthogonal Projections and Orthonormal Bases

After this lecture you should:

- Understand the concept of *Orthonormality*
- Be able to compute the Projection onto a subspace V (with $\dim(V) > 1$), using an *Orthonormal Basis.*
- Be comfortable with the use of the *Orthogonal Complement*, $V^\perp$ of a subspace V.

Something Old + Something New...

Rewind (Orthogonality, Length, Unit Vectors)

a. Two vectors $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$ are called *orthogonal* (or *perpendicular*) if $\vec{v} \cdot \vec{w} = 0$.

b. The *length* (or *norm*, or *magnitude*) of a vector $\vec{v} \in \mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.

c. A vector $\vec{u} \in \mathbb{R}^n$ is called a *unit vector* if $\|\vec{u}\| = 1$.

Definition (Orthonormal Vectors)

The vectors $\vec{u}_1, \ldots, \vec{u}_m \in \mathbb{R}^n$ are called *orthonormal* if they are all unit vectors and orthogonal to one another:

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Orthonormal Vectors

Example (The "Standard Basis Vectors")

The vectors $\vec{e}_1, \ldots, \vec{e}_n \in \mathbb{R}^n$ are orthonormal. — They form an *orthonormal basis* for $\mathbb{R}^n$.

Example (Rotated Standard Vectors in $\mathbb{R}^2$)

Consider $\vec{e}_1$, and $\vec{e}_2$ in $\mathbb{R}^2$; and their rotated versions:

$$\vec{r}_1(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix},$$

$$\vec{r}_2(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix},$$

Orthonormal Vectors

Example (Orthonormal Vectors in $\mathbb{R}^4$)

The vectors

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad \vec{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

in $\mathbb{R}^4$ are orthonormal.

UNIT LENGTH: $\sqrt{4 \times \frac{1}{2^2}} = 1.$

ORTHOGONALITY: For each pair of vectors, two of the products $\vec{u}_{i,k}\vec{u}_{j,k}$ will be positive and two negative; hence the sum $\sum_{k=1}^{4} \vec{u}_{i,k}\vec{u}_{j,k}$ will be zero.

Properties

Theorem (Properties of Orthonormal Vectors)

a. *Orthonormal Vectors are linearly independent.*

b. *Orthonormal Vectors $\vec{u}_1, \ldots, \vec{u}_n \in \mathbb{R}^n$ form a basis of $\mathbb{R}^n$.*

a. Clearly, there is no way to (linearly) combine perpendicular vectors to describe each other (*for example*, think of $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ in $\mathbb{R}^3$.)

b. By previous theorems, n linearly independent vectors in $\mathbb{R}^n$ necessarily form a basis of $\mathbb{R}^n$. (Think of the standard basis $\vec{e}_1, \ldots, \vec{e}_n \in \mathbb{R}^n$; and rotations / reflections of it...)

Comment: In some sense, Orthonormal vectors are "maximally linearly independent."

Properties

Rewind (Orthogonal Projection)

Consider a vector $\vec{x} \in \mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$. Then we can write

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

where $\vec{x}^{\parallel} \in V$, and $\vec{x}^{\perp} \perp V$. **This representation is unique.** The vector $\vec{x}^{\parallel}$ is called the *orthogonal projection* of $\vec{x}$ onto V, sometimes denoted $\mathrm{proj}_V(\vec{x})$; the transformation $T(\vec{x}) = \mathrm{proj}_V(\vec{x}) = \vec{x}^{\parallel}$ from $\mathbb{R}^n \mapsto \mathbb{R}^n$ is linear.

We have discussed this previously, but only in the context of describing the *image* and *kernel* of the projection... We are now ready to start discussing HOW we can compute the projection in any dimension space (onto any subspace).

Orthogonal Projection: Formula

Theorem (Formula for the Orthogonal Projection)

If V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$ — that is $V = \operatorname{span}(\vec{u}_1, \dots, \vec{u}_m)$ then

$$\operatorname{proj}_V(\vec{x}) = \vec{x}^{\parallel} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

$\forall x \in \mathbb{R}^n$.

Having the orthonormal basis is the absolute key to this formula. Any non-orthonormal basis will produce strange (incorrect) results.

Orthogonal Projection: Example

Example (Orthogonal Projection (Part 1))

Consider the subspace $V = \operatorname{im}(A)$ of $\mathbb{R}^4$, find $\operatorname{proj}_V(\vec{x})$; where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}, \quad \left\| \begin{bmatrix} \pm 1 \\ \pm 1 \\ \pm 1 \\ \pm 1 \end{bmatrix} \right\| = \sqrt{4(\pm 1)^2} = \sqrt{4} = 2.$$

Since the columns $\vec{a}_1$ and $\vec{a}_2$ are linearly independent, and orthogonal (zero dot-product), they form a[n orthogonal] basis of V. Dividing each vector by its length gives us an orthonormal basis for $V = \operatorname{span}(\vec{u}_1, \vec{u}_2)$, where

$$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

Continued on the next slide...

Orthogonal Projection: Example

Example (Orthogonal Projection (Part 2))

Now we use the projection formula,

$$\vec{x} \cdot \vec{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = 6, \quad \vec{x} \cdot \vec{u}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = 2,$$

therefore

$$\operatorname{proj}_V(\vec{x}) = 6\vec{u}_1 + 2\vec{u}_2 = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}.$$

Orthogonal Projection: Example

Example (Orthogonal Projection (part 3))

It is worth noting:

$$\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 7 \end{bmatrix}, \quad \operatorname{proj}_V(\vec{x}) = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 4 \end{bmatrix}.$$

$$\|\vec{x}\| = \sqrt{1 + 9 + 1 + 49} = \sqrt{60},$$

$$\|\operatorname{proj}_V(\vec{x})\| = \sqrt{16 + 4 + 4 + 16} = \sqrt{40}$$

so that $\|\operatorname{proj}_V(\vec{x})\| \leq \|\vec{x}\|$. **This is ALWAYS true!**

Orthogonal Projection onto a [Subspace Spanned by a] Basis 1/2

Theorem (Orthogonal Projection onto a Basis)

Consider an orthonormal basis $\vec{u}_1, \dots, \vec{u}_n$ of $\mathbb{R}^n$. Then

$$\vec{x} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n$$

$\forall \vec{x} \in \mathbb{R}^n$.

Since the basis spans $\mathbb{R}^n$, we can "rebuild" $\vec{x}$ completely by adding up all the projected pieces.

We have $\vec{x}$ as a **unique linear combination**

$$\vec{x} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$$

where $c_k = (\vec{u}_k \cdot \vec{x})$, $k = 1, \dots, n$.

Orthogonal Projection onto a [Subspace Spanned by a] Basis 2/2

Looking in the rear-view mirror [COORDINATES (NOTES#3.4)], we can let

$$\text{BASIS: } \mathfrak{U} = \langle \vec{u}_1, \vec{u}_2, \dots, \vec{u}_n \rangle$$

$$\text{COORDINATES: } [\vec{x}]_{\mathfrak{U}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$c_k = (\vec{u}_k \cdot \vec{x}), \quad k = 1, \dots, n.$$

Orthogonality allows us to compute the coordinates one-at-a-time, *i.e.* they are independent from each other. This in itself is a useful property! $\rightsquigarrow$ Parallel Computing for speed!

Image, and Kernel...

Using our recently acquired vocabulary, we realize that

$$\mathrm{im}(\mathrm{proj}_V(\vec{x})) = V$$

Also, we know we can write

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$$

and $\mathrm{proj}_V(\vec{x})) = \vec{x}^{\parallel}$, so if we are looking for the kernel,

$$\ker(\mathrm{proj}_V(\vec{x})),$$

we want all $\vec{x}$ without a $\vec{x}^{\parallel}$ part, *i.e.* $\{\vec{x} \in \mathbb{R}^n \,:\, \vec{x} = \vec{x}^{\perp}\}$, *the collection of all vectors orthogonal to the subspace V.*

A formal definition follows on the next slide...

The Orthogonal Complement :: Definition and Related Expressions

Definition (Orthogonal Complement)

Consider a subspace V of $\mathbb{R}^m$. The *Orthogonal Complement* $V^{\perp}$ of V is the set of those vectors $\vec{x} \in \mathbb{R}^m$ that are orthogonal to all vectors in V:

$$V^{\perp} = \{\vec{x} \in \mathbb{R}^m \,:\, \vec{x} \cdot \vec{v} = 0,\ \forall \vec{v} \in V\}$$

Note* that $V^{\perp}$ is the kernel of $\mathrm{proj}_V(\vec{x})$.

* This means that if we have a description of V as the solution of a linear system ($A \in \mathbb{R}^{n \times m}$)

$$V = \{\vec{x} \in \mathbb{R}^m \,:\, A\vec{x} = \vec{0}\}$$

then (note that $\mathrm{proj}_V(\vec{x}) : \mathbb{R}^m \mapsto \mathbb{R}^m$)

$$\begin{array}{lllll} V & = \ker(A) & = \mathrm{im}(\mathrm{proj}_V(\vec{x})) & \subset & \mathbb{R}^m \\ V^{\perp} & = \mathrm{im}(A^T) & = \ker(\mathrm{proj}_V(\vec{x})) & \subset & \mathbb{R}^m \end{array}$$

The Orthogonal Complements of a Linear Transformation

Consider a linear transformation $T(\vec{x}) = A\vec{x}$, where $A \in \mathbb{R}^{n\times m}$:

"Input Space"		**"Output Space"**
$\vec{x} \in \mathbb{R}^m$	$\mapsto$	$\vec{y} = A\vec{x} \in \mathbb{R}^n$
$\ker(A)$	$\mapsto$	$\vec{0}$
$\ker(A)^\perp$	$\mapsto$	$\mathrm{im}(A)$
nothing	$\mapsto$	$\mathrm{im}(A)^\perp$
$\ker(A) \oplus \ker(A)^\perp = \mathbb{R}^m$		$\mathrm{im}(A) \oplus \mathrm{im}(A)^\perp = \mathbb{R}^n$

We use the symbol $\oplus$ to denote the "Direct Sum" of two subspaces (formal definition in a few slides).

We make a big deal of the direct sum in [MATH 524]...

The Orthogonal Complement :: Properties

Theorem (Properties of the Orthogonal Complement)

Consider a subspace V of $\mathbb{R}^n$.

- **a.** *The Orthogonal Complement $V^\perp$ of V is a subspace of $\mathbb{R}^n$.*
- **b.** *The intersection (common elements) of $V^\perp$ and V consists of the zero vector: $V^\perp \cap V = \{\vec{0}\}$.* [$\vec{x} \in V^\perp \cap V : \vec{x}\cdot\vec{x} = 0 \Rightarrow \vec{x} = \vec{0}$.]
- **c.** $\dim(V) + \dim(V^\perp) = n$. [BY RANK-NULLITY THEOREM; $T(\vec{x}) = \mathrm{proj}_V(\vec{x})$]
- **d.** $(V^\perp)^\perp = V$.
- **e.** *The "Direct Sum" $V \oplus V^\perp = \mathbb{R}^n$, where*

$$U = V \oplus V^\perp \stackrel{\text{def}}{=} \{\vec{u} = \vec{v} + \vec{w} \;:\; \vec{v} \in V, \vec{w} \in V^\perp\},$$

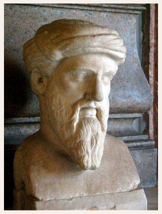
that is, V and $V^\perp$ "split" the space in two non-overlapping parts — in this context $\vec{0}$ does not "count" as an overlap.

From Pythagoras to Cauchy-(Bunyakovsky)-Schwarz

Example (Really Old Stuff in New Notation)

Consider a line L, and a vector $\vec{x}$ in $\mathbb{R}^n$. If we project $\vec{x}$ onto L and write $\vec{x} = \vec{x}^{\|} + \vec{x}^{\perp}$ good ole' Pythagoras says

$$\|\vec{x}\|^2 \equiv \|\vec{x}^{\|} + \vec{x}^{\perp}\|^2 = \|\vec{x}^{\|}\|^2 + \|\vec{x}^{\perp}\|^2.$$



Pythagoras, ~570–495 BC
© Public Domain; wikimedia.

Checking:

$$\begin{aligned}
\|\vec{x}\|^2 &= \|\vec{x}^{\|} + \vec{x}^{\perp}\|^2 \\
&= (\vec{x}^{\|} + \vec{x}^{\perp})\cdot(\vec{x}^{\|} + \vec{x}^{\perp}) \\
&= \vec{x}^{\|}\cdot\vec{x}^{\|} + \vec{x}^{\|}\cdot\vec{x}^{\perp} + \vec{x}^{\perp}\cdot\vec{x}^{\|} + \vec{x}^{\perp}\cdot\vec{x}^{\perp} \\
&= \|\vec{x}^{\|}\|^2 + 0 + 0 + \|\vec{x}^{\perp}\|^2 \quad = \quad \|\vec{x}^{\|}\|^2 + \|\vec{x}^{\perp}\|^2
\end{aligned}$$

... and there it is!

From Pythagoras to Cauchy-(Bunyakovsky)-Schwarz

Theorem (Pythagorean Theorem)

Consider two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$. The equation

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$$

holds if and only if $\vec{x} \perp \vec{y}$.

Theorem ($\mathrm{proj}_V(\vec{x})$ is no longer than $\vec{x}$)

Consider a subspace V of $\mathbb{R}^n$, and a vector $\vec{x} \in \mathbb{R}^n$. Then

$$\|\mathrm{proj}_V(\vec{x})\| \le \|\vec{x}\|$$

where equality is achieved if and only if $\vec{x} \in V$.

From Pythagoras to Cauchy-(Bunyakovsky)-Schwarz

Proof (by observation)

Since
$$\|\vec{x}^{\|}\|^2 + \|\vec{x}^{\perp}\|^2 = \|\vec{x}\|^2$$
(and all lengths are non-negative), we must have
$$\|\vec{x}^{\|}\|^2 = \|\vec{x}\|^2 - \|\vec{x}^{\perp}\|^2 \leq \|\vec{x}\|^2.$$

From Pythagoras to Cauchy-(Bunyakovsky)-Schwarz

Now, ponder a one-dimensional subspace (a line through the origin) V of $\mathbb{R}^n$, and let $\vec{y}$ be a vector in that subspace; let
$$\vec{u} = \frac{1}{\|\vec{y}\|}\vec{y}$$
be a unit vector spanning V.

We can now write the projection using $\vec{u}$:
$$\mathrm{proj}_V(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}$$

It follows that
$$\|\vec{x}\| \geq \|\mathrm{proj}_V(\vec{x})\| = \|(\vec{x}\cdot\vec{u})\vec{u}\| = |\vec{x}\cdot\vec{u}|\,\|\vec{u}\| = |\vec{x}\cdot\vec{u}| = \left|\vec{x}\cdot\frac{1}{\|\vec{y}\|}\vec{y}\right| = \frac{1}{\|\vec{y}\|}|\vec{x}\cdot\vec{y}|$$

multiply by $\|\vec{y}\|$ to get Cauchy-(Bunyakovsky)-Schwarz

so that
$$|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|$$

From Pythagoras to Cauchy-(Bunyakovsky)-Schwarz

What we showed on the previous slide is known as:

Theorem (The Cauchy-(Bunyakovsky)-Schwarz Inequality)

If $\vec{x}$ and $\vec{y} \in \mathbb{R}^n$, then
$$|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|.$$
The statement is an equality if and only if $\vec{x}$ and $\vec{y}$ are parallel.

Pythagoras ∼570 – 495 BC.
Augustin-Louis Cauchy, 21 August 1789 – 23 May 1857.
⇒ proof for sums (1821).
Viktor Yakovlevich Bunyakovsky, 16 December 1804 – 12 December 1889.
⇒ proof for integrals (1859).
Karl Hermann Amandus Schwarz, 25 January 1843 – 30 November 1921.
⇒ Modern proof (1888).

The Dot Product, $\cos(\theta)$, and Cauchy-Bunyakovsky-Schwarz

Consider two vectors $\vec{x}$, and $\vec{y} \in \mathbb{R}^n$. We have previously expressed the dot product in terms of the angle between the two vectors.
$$\vec{x}\cdot\vec{y} = \cos(\theta)\,\|\vec{x}\|\,\|\vec{y}\|$$

The real use of the formula is to find
$$\cos(\theta) = \frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\,\|\vec{y}\|}, \quad \theta = \arccos\left(\frac{\vec{x}\cdot\vec{y}}{\|\vec{x}\|\,\|\vec{y}\|}\right)$$

Cauchy-Bunyakovsky-Schwarz, $|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\,\|\vec{y}\|$, guarantees that the argument to arccos, and the value of $\cos(\theta)$ (as defined here) make sense.

Note that the angle θ is in the plane spanned by $\vec{x}$, and $\vec{y}$.

Example: Angle Between Vectors

Example (Angle Between Vectors)

Find the angle between

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We have

$$\|\vec{x}\| = 1, \quad \|\vec{y}\| = \sqrt{4} = 2, \quad \vec{x} \cdot \vec{y} = 1$$

so that

$$\cos(\theta) = \frac{1}{2}, \quad \theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

Suggested Problems 5.1

Available on Learning Glass videos:
5.1 — 7, 10, 11, 15, <u>17</u>, 27, 28

Lecture – Book Roadmap

Lecture	Book, [GS5–]
5.1	§4.1, §4.2, **§4.4**
5.2	§4.1, §4.2, **§4.4**
5.3	§4.1, §4.2, **§4.4**

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(5.1.7), (5.1.10)

(5.1.7) For vectors $\vec{u}$, $\vec{v}$, determine whether the angle is acute $(< \frac{\pi}{2})$, right $(= \frac{\pi}{2})$, or obtuse $(> \frac{\pi}{2})$.

$$\vec{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

(5.1.10) For which value(s) of $k \in \mathbb{R}$ are the vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ k \\ 1 \end{bmatrix}$$

perpendicular?

(5.1.11)

(5.1.11) Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

a. For $n = 2, 3, 4$, find the angle θ_n between $\vec{u}$ and $\vec{v}$.
b. Find the limit of θ_n as $n \to \infty$.

(5.1.15)

(5.1.15) Consider the vector

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathbb{R}^4.$$

Find a basis for the subspace of $\mathbb{R}^4$ consisting of all vectors perpendicular to $\vec{v}$.

(5.1.17), (5.1.27)

(5.1.17) Find a basis for $W^\perp$, where

$$W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \right).$$

(5.1.27) Find the orthogonal projection of $9\vec{e}_1$ onto the subspace W of $\mathbb{R}^4$, where

$$W = \text{span}\left(\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right), \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(5.1.28)

(5.1.28) Find the orthogonal projection of $\vec{e}_1$ onto the subspace W of $\mathbb{R}^4$, where

$$W = \mathrm{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} \right), \quad \vec{e}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}.$$

The Super-Slow, Slow, and Fast Fourier Transform

A lot of signal analysis and processing is frequency-based; meaning that it is highly useful to express a signal using basis functions that are determined by various frequencies.

Consider the functions:

$$\begin{cases} \Phi_0(x) &= \dfrac{1}{2} \\ \Phi_k(x) &= \cos(kx), \quad k = 1, \dots, n \\ \Phi_{n+k}(x) &= \sin(kx), \quad k = 1, \dots, n-1 \end{cases}$$

and let each one define a vector $\vec{v}_i \in \mathbb{R}^{2n}$, by evaluating the function in the points $x_j = -\pi + (j\pi/n),\ j = 0, 1, \dots, (2m-1)$.

The Super-Slow, Slow, and Fast Fourier Transform

- Let those vectors be the columns in a matrix, $M \in \mathbb{R}^{2n \times 2n}$.
  - It turns out that the vectors are *linearly independent*, which makes them a *basis*, $\mathfrak{B}$, for $\mathbb{R}^{2n}$ and the matrix M invertible;
  - further, the vectors are *orthogonal* (which will help us save some work).
- Now, if we have sampled a signal in $2n$ locations / timepoints; then we can collect those samples in $\vec{f} \in \mathbb{R}^{2n}$.
- If we can to express the signal as a linear combination of the cos/sin-vectors, all we have to do is solve the linear system

$$M[\vec{f}]_{\mathfrak{B}} = \vec{f},$$

which in general requires roughly $\frac{8}{3}n^3$ operations $(\times/+)$. This is the **Super-Slow Fourier Transform**.

The Super-Slow, Slow, and Fast Fourier Transform

- Since the vectors are orthogonal, solving the system is not necessary; we can get each coefficient by computing a length $2n$ dot-product:

$$a_k = \frac{1}{n}\sum_{j=0}^{2n-1} f_j \cos(kx_j) \quad b_k = \frac{1}{n}\sum_{j=0}^{2n-1} f_j \sin(kx_j).$$

where a_k, $k = 0, \dots, n$ are the first $(n+1)$ coefficients of $[\vec{f}]_{\mathfrak{B}}$, and b_k, $k = 1, \dots, (n-1)$ are the remaining coefficients.
- This approach, the **Slow Fourier Transform** requires roughly $4n^2$ operations.
- [Full Disclosure] We have omitted a few (3) factors of 2 which are necessary to make the vectors ortho*normal*.

The Super-Slow, Slow, and Fast Fourier Transform

Much of the analysis was done by **Jean Baptiste Joseph Fourier** in the early 1800s, but the use of the Fourier series representation was not practical until...

1965, when **Cooley and Tukey*** published a 4-page paper describing an algorithm which computes the coefficients using only $\mathcal{O}(n \log_2 n)$ operations.

It is hard to overstate the importance of this paper!!!

The algorithm is now known as the **"Fast Fourier Transform"** or just the **"FFT"**. We sweep the details of the FFT under the rug; it comes down to some clever complex analysis, and the facts that $1+1=2$, and $1-1=0$.

---

* James W. Cooley and John W. Tukey, *"An Algorithm for the Machine Calculation of Complex Fourier Series,"* Mathematics of Computation, Vol. 19, No. 90, April 1965, pp. 297-301, DOI: 10.2307/2003354, URL: http://www.jstor.org/stable/2003354

Comparing Operation Counts

2,500,000 s ≈ 1 month

n	$4n^2$	$3n + n\log_2 n$	Speedup
16	1,024	112	9
64	16,384	576	28
256	262,144	2,816	93
1,024	4,194,304	13,312	315
4,096	67,108,864	61,440	1,092
16,384	1,073,741,824	278,528	3,855
65,536	17,179,869,184	1,245,184	13,797
262,144	274,877,906,944	5,505,024	49,932
1,048,576	4,398,046,511,104	24,117,248	182,361
4,194,304	70,368,744,177,664	104,857,600	671,088
8,388,608	281,474,976,710,656	218,103,808	1,290,555
16,777,216	1,125,899,906,842,624	452,984,832	2,485,513
33,554,432	4,503,599,627,370,496	939,524,096	4,793,490

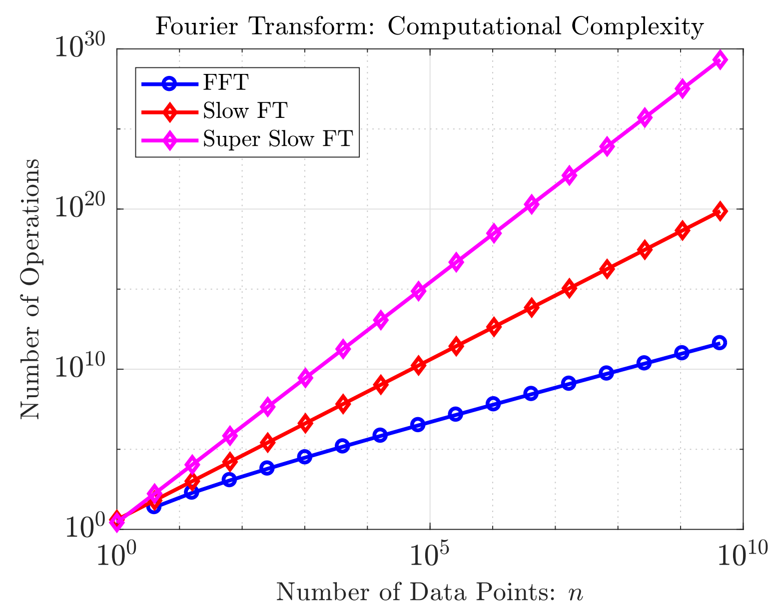
$$33,554,432 = 2^{13} \times 2^{12} = 8,196 \times 4,096^{\dagger}$$

† The "8k Digital Video Format" has a resolution of $8,192 \times 4,320$ pixels; the tentative Ultra High Definition Television (UHDTV) specification calls for $7,680 \times 4,320$ pixels for 16:9 aspect ratio (120 fps, 12 bits/channel (at least 3 – RGB) $\rightsquigarrow 4.0 \times 10^9$ 36-bit pixels/sec). IMAX shot on 70 mm *film* has a theoretical pixel resolution of $12,000 \times 8,700$ (at 24 fps, for a total of 2.5×10^9 pixels/sec).

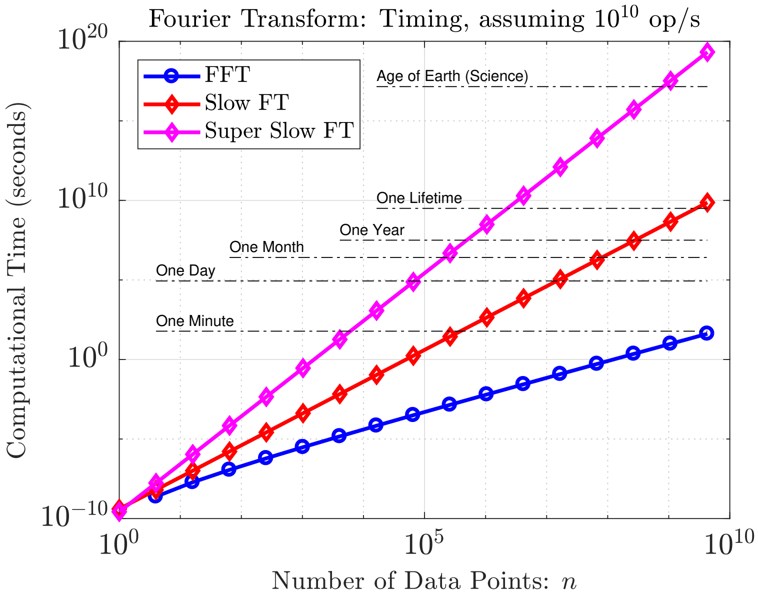
Comparing Operation Counts

n	SLOW-FT time	FFT time	Speedup
16	9 s	1 s	9
64	29 s	1 s	28
256	1 m 33 s	1 s	93
1,024	5 m 15 s	1 s	315
4,096	18 m 12 s	1 s	1,092
16,384	1:04:26	1 s	3,855
65,536	3:49:57	1 s	13,797
262,144	13:52:12	1 s	49,932
1,048,576	2d + 02:39:21	1 s	182,361
4,194,304	7d + 18:24:48	1 s	671,088
8,388,608	14d + 22:29:15	1 s	1,290,555
16,777,216	28d + 18:25:13	1 s	2,485,513
33,554,432	55d + 11:31:30	1 s	4,793,490

Comparing Operation Counts



Comparing Operation Counts



MPEG-4 Compression :: Main Ideas

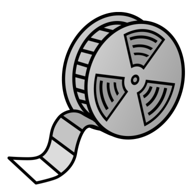
We used the idea of video compression to motivate *why* we should care about orthogonal basis... The discussion was somewhat hand-wavy

This is an attempt at describing the key ideas of video compression, using only concepts from calculus and half a semester of linear algebra. (Good luck to me!)

In order to communicate the main ideas, lots of "minor" details have been swept under the rug; several oversimplifications have been committed, and a few convenient lies have been told.

MPEG-4 Compression :: Description of the Problem — 1080p "HD" Movies

Imagine a 2-hour 1080p24 Movie; where we are showing 24 frames/second, and each frame is 1920×1080 pixels, each pixel has a bit depth of 8-bits per color (whether that's Red-Green-Blue, or Y-Cb-Cr, is a discussion for someplace else); but the bottom line is that we have 3 bytes/pixel, so we end of with a raw datastream with



	3	bytes/pixel
×	24	frames/second
×	7200	seconds
×	(1920×1080)	pixels/frame
=	1,074,954,240,000	bytes.

Image License: User:Bromskloss [CC BY-SA 3.0 (http://creativecommons.org/licenses/by-sa/3.0/)], via Wikimedia Commons.

MPEG-4 Compression :: Blu-ray Disc Capacity

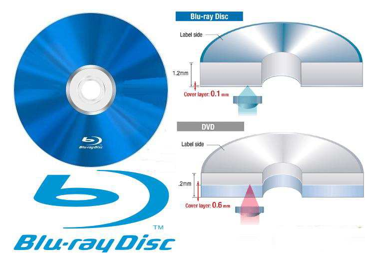


Image License: "Bluray," "Blu-ray Disc," logos, specifications, etc. are trademarks owned by the *Blu-ray Disc Association* (Hitachi, Ltd., LG Electronics Inc., Matsushita Electric Industrial Co., Ltd., PIONEER CORPORATION, Royal Philips Electronics, SAMSUNG ELECTRONICS CO., LTD., SHARP CORPORATION, Sony Corporation, THOMSON multimedia).

Now, keeping in mind that a standard dual-layer Blu-ray disc holds a measly 50,050,629,632 bytes of data, we need a compression ratio of 1 : 21.5 in order to fit the movie onto a disk. This means we can only store slightly less than 4.7% of the datastream.

OK, OK, OK, the extended version of *Lord of the Rings* is 208 minutes; so really we can only fit 2.7% of the datastream...

If you are streaming the movie, even LESS data is getting transmitted.

MPEG-4 Compression :: Movie $\rightsquigarrow$ A Single Frame $\rightsquigarrow$ Gray-scale

Let's for a moment restict our discussion to a single 1920×1080 pixel frame; and for simplicity, let's make it gray-scale.

Gandalf was vanquished by the "Copyright spell," so you're stuck with my ugly face...

MPEG-4 Compression :: Compressing the Single Frame

Note: Simplifying to a single gray-scale frame sweeps *a lot* of important details under the rug, but the example still gives the "right flavor" of how compression works...

Next we are going to discuss how we can compress this single snapshot to use only 4.0% storage. This is going to require a little bit of mathematics...

Looking at a Single Horizontal/Vertical Line of the Image: First, we can consider the image to be constructed out of 1080 lines, each with 1920 pixels; which means we have a collection of 1080 vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_{1080}$, each "living it up" in $\mathbb{R}^{1920}$; we can also (simultaneously) think of the as being constructed out of 1920 columns, each with 1080 (vertical) pixels; giving us vectors $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{1920}$, each "living it up" in $\mathbb{R}^{1080}$.

MPEG-4 Compression :: The Need for Orthonormality

What we need are some good (orthonormal) bases for $\mathbb{R}^{1080}$ and $\mathbb{R}^{1920}$. It turns out that if we are given an even number, $2n$ points, then we can use the $2n$ vectors generated by the functions

$$\begin{cases} \Phi_0(x) &= \dfrac{1}{2} \\ \Phi_k(x) &= \cos(kx), \quad k = 1, \ldots, n \\ \Phi_{n+k}(x) &= \sin(kx), \quad k = 1, \ldots, n-1 \end{cases}$$

evaluated in the interval $[-\pi, \pi]$, at the equally spaced points $x_j = -\pi + (j\pi/n)$, $j = 0, 1, \ldots, (2n-1)$. The generated set of vectors are orthonormal!

MPEG-4 Compression :: Enter the *Fourier Transform*

Now, align the points x_j with the pixels, numbered from $j = 0$ to $j = (2n-1)$, horizontally or vertically. Let p_j denote the pixel value (gray-scale intensity). Now, if we let

$$a_k = \frac{1}{n} \sum_{j=0}^{2n-1} f_j \cos(kx_j) \quad b_k = \frac{1}{n} \sum_{j=0}^{2n-1} f_j \sin(kx_j),$$

be the values of the [pixel-vector]–[cos/sin-vector] dot-products. In our language the a_k and b_k coefficients are coordinates in the cos/sin-vector basis for $\mathbb{R}^{2n}$; and given the coordinates, we can fully reconstruct the pixel values:

$$p_j \equiv S(x_j) = \frac{a_0}{2} + \frac{a_n}{2}\cos(nx_j) + \sum_{k=1}^{n-1} \left[a_k \cos(kx_j) + b_k \sin(kx_j)\right].$$

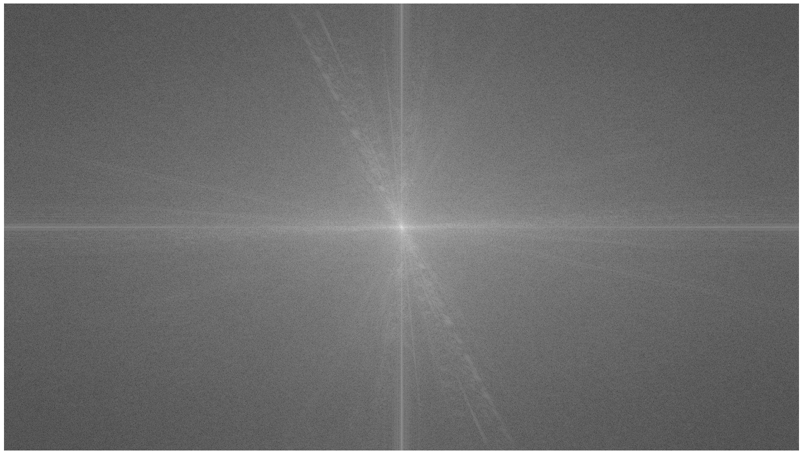
MPEG-4 Compression :: 1D ⇝ 2D Fourier Transform

We now have a set-up where we can go from "image coordinates" to [cos/sin-vector] coordinates (and back) using only dot products. What we have defined is known as the (one dimensional) *Fourier transform.*

Back to 2D
Even though the previos discussion gave us a nice way to build orthonormal bases in one dimension, it is far from clear *why* this is desirable.

Now, consider the Office-Scene-image we had; and lets perform the above procedure first in the horizontal direction (which transforms the image into 1080 lines of [cos/sin-vector] coordinates. Next, transform *that* "image" in the vertical direction. This now gives us an "image" of vertial [cos/sin-vector] coordinates of (horizonal [cos/sin-vector] of imagef). This is known as the two dimensional *Fourier transform.*

MPEG-4 Compression :: The Office Scene through "Fourier Goggles"



The two dimensional Fourier transform of the original Office Scene.

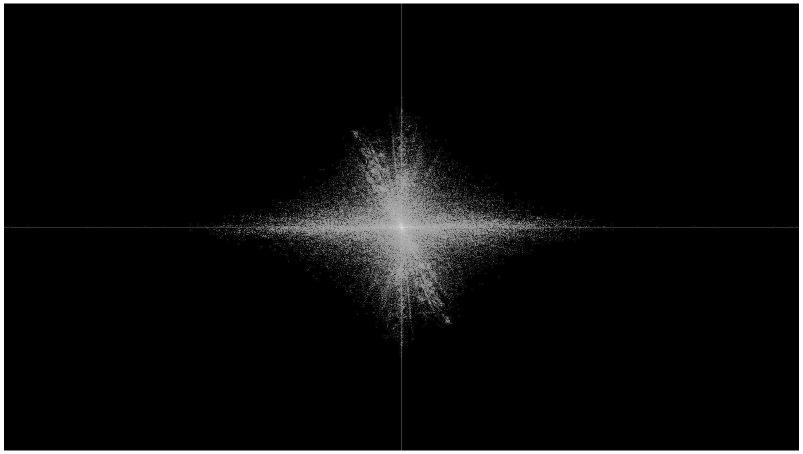
MPEG-4 Compression :: Compressing

Next, we throw away some Fourier Coefficients.

Here, we take the time to figure out what (in magnitude) 4.0% of coefficients are the largest. — We keep those, and discard the rest.

Then we reconstruct the Office Scene using only the leading 4.0% of the coefficients.

MPEG-4 Compression :: The Leading 4.0% of Fourier Coefficients



Fourier transformed, and filtered; only the largest 4.0% of coefficients have been kept.

MPEG-4 Compression :: Office Scene — Compressed Reconstruction

Office-Scene: Fourier transformed, filtered and reconstructed: only the largest 4.0% of coefficients have been kept.

MPEG-4 Compression :: Wrap-up

For more details on how video compression actually is implemented, check out the following

References:

- *What is H.264* — http://www.h264info.com/h264.html
- Wikipedia — H.264/MPEG-4 AVC — http://en.wikipedia.org/wiki/H.264/MPEG-4_AVC
- Whitepaper: *H.264 video compression standard: New possibilities within video surveillance* — http://www.axis.com/files/whitepaper/wp_h264_31669_en_0803_lo.pdf