Math 254: Introduction to Linear Algebra

Notes #5.2 — Gram-Schmidt Process and QR Factorization

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Outline

SLOs 5.2 Gram-Schmidt Process and QR Factorization

After this lecture you should know how:

- to perform *The Gram-Schmidt Orthogonalization Process* on a set of vectors, and
- it can be used to compute *The QR-factorization* of a matrix A: $A = QR$

  ⇒ This builds an orthonormal basis (the columns of Q) for the subspace $V = \mathrm{im}(A)$, which gives us the means to compute the orthogonal projection $\mathrm{proj}_V(\vec{x})$ onto V.
- to orthogonally project onto *any* subspace.

Orthogonal Projection onto a Subspace V

From [Notes#5.1] we have:

Theorem (Formula for the Orthogonal Projection)

If V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$, then

$$\mathrm{proj}_V(\vec{x}) = \vec{x}^{\|} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

$\forall x \in \mathbb{R}^n$.

How do you project onto a subspace if/when the given basis is not orthonormal?!?

It turns out that before we compute the projection, we have to find a new — *orthonormal* — basis...

THIS IS ALL WRONG!!!!!

Ponder what happens if we use the formula, but the given basis is **not** orthonormal...

Let's live in $\mathbb{R}^2$, let $V = \mathbb{R}^2$, with basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ defined by

$$\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix}2\\3\end{bmatrix}; \quad \|\vec{x}\| = \sqrt{13}.$$

Clearly $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, and $\vec{x} = 1\vec{v}_1 + 1\vec{v}_2$, but the projection formula goes haywire:

$$\text{proj}_V(\vec{x}) = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + (\vec{v}_2 \cdot \vec{x})\vec{v}_2 = 5\vec{v}_1 + 8\vec{v}_2 = \begin{bmatrix}13\\21\end{bmatrix}.$$

... even if we remember to correct for the non-unit length of $\vec{v}_{1,2}$:

$$\text{proj}_V(\vec{x}) = \frac{(\vec{v}_1 \cdot \vec{x})}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{(\vec{v}_2 \cdot \vec{x})}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{5}{2}\vec{v}_1 + \frac{8}{5}\vec{v}_2 = \begin{bmatrix}4.1\\5.7\end{bmatrix}.$$

Comments

There are other ways to realize the "projection" went awry:

- This is "life in $\mathbb{R}^2$," and since

$$\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix},$$

  are linearly independent $\rightsquigarrow$ they form a basis for $\mathbb{R}^2$ $\rightsquigarrow$ any projection of a vector $\vec{w} \in \mathbb{R}^2$ onto the subspace $V = \text{span}(\vec{v}_1, \vec{v}_2) \equiv \mathbb{R}^2$ must be the original vector $\vec{w}$.
- Even simpler, the famous *Method of the Eyeball* already showed that $\vec{v}_1 + \vec{v}_2 = \vec{x}$:

$$\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}.$$

Example: Doing it Right... Build an Orthonormal Basis

In this case, given a basis of $\mathbb{R}^2$, the answer is "obvious."

Next, we develop (still in $\mathbb{R}^2$ so we easily can visualize and use our intuition) a method for building an *orthonormal basis* given *any* starting basis.

Once we have the orthonormal basis, we can use the projection formula...

(!) The method will work in the general case: *Given* $\vec{x} \in \mathbb{R}^n$, *and* $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m) \subset \mathbb{R}^n$; *compute* $\text{proj}_V(\vec{x})$:

We find an orthonormal basis $\vec{q}_1, \dots, \vec{q}_m$, so that

$$V = \text{span}(\vec{v}_1, \dots, \vec{v}_m) = \text{span}(\vec{q}_1, \dots, \vec{q}_m);$$

and then use the projection formula.

Example: Doing it Right... Build an Orthonormal Basis

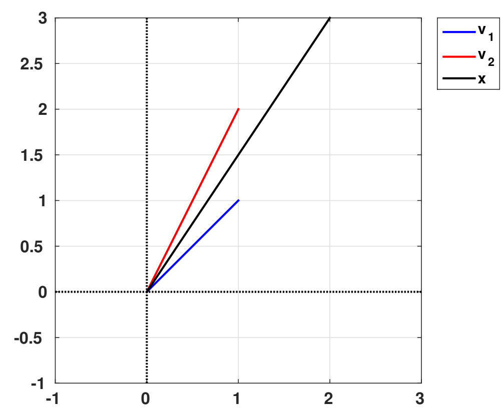


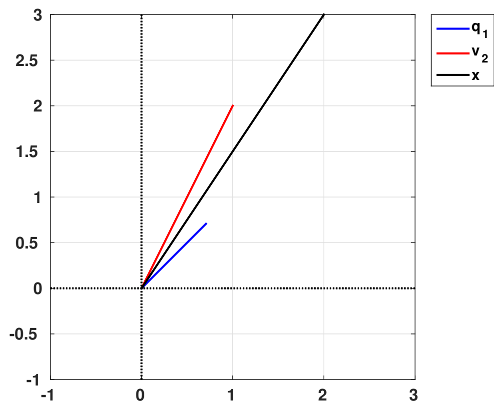
Figure: This is where we start.

$$\vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix}2\\3\end{bmatrix}.$$

Example: Doing it Right...

(i) Rescale the first vector

Figure: Rescale* $\vec{v}_1$ to be norm 1, and call it $\vec{q}_1$.



$$\vec{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix}2\\3\end{bmatrix}.$$

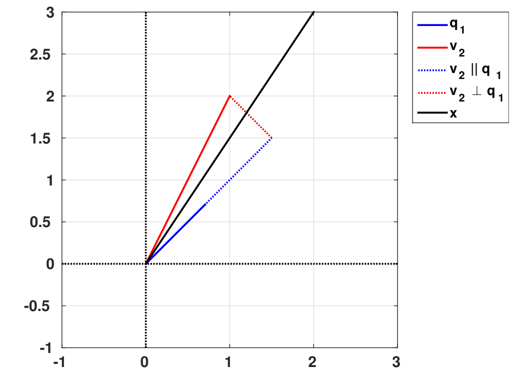
* Divide by the original norm, $\sqrt{2}$.

Gram-Schmidt Orthogonalization and QR Factorization
Suggested Problems
The Gram-Schmidt Orthogonalization Process
The QR Factorization
Observations

Example: Doing it Right...

(ii) Split the next vector into $\|$ and $\perp$ parts...

Figure: Next, project* $\vec{v}_2$ onto $\vec{q}_1$ and get $\vec{v}_2^{\,||\vec{q}_1}$, and $\vec{v}_2^{\,\perp\vec{q}_1}$.



$$\vec{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad \vec{v}_2^{\,||\vec{q}_1} = \begin{bmatrix}1.5\\1.5\end{bmatrix}; \quad \vec{v}_2^{\,\perp\vec{q}_1} = \begin{bmatrix}-0.5\\0.5\end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix}2\\3\end{bmatrix}.$$

* $\vec{v}_2^{\,||\vec{q}_1} = (\vec{q}_1 \cdot \vec{v}_2)\vec{q}_1 = \left(\frac{3}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}; \quad \vec{v}_2^{\,\perp\vec{q}_1} = \begin{bmatrix}1\\2\end{bmatrix} - \vec{v}_2^{\,||\vec{q}_1} = \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}.$

Gram-Schmidt Orthogonalization and QR Factorization
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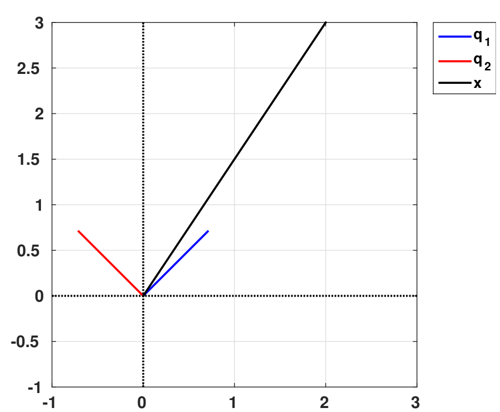
Example: Doing it Right...

(iii) Rescale the $\perp$ part, discard $\|$ part

Figure: Next, throw away $\vec{v}_2^{\,||\vec{q}_1}$; rescale $\vec{v}_2^{\,\perp\vec{q}_1}$ to norm 1, and name it $\vec{q}_2$.



$$\vec{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad \vec{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix}2\\3\end{bmatrix},$$

Now we have an orthonormal basis $\mathfrak{Q} = \langle \vec{q}_1, \vec{q}_2 \rangle$!

Gram-Schmidt Orthogonalization and QR Factorization
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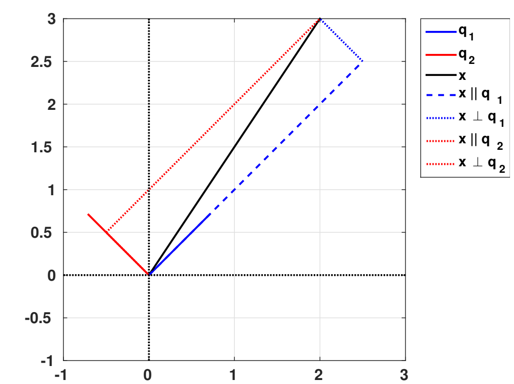
Example: Doing it Right...

(iv) Project using the new orthonormal basis!

Figure: Finally, we can use the projection formula.



$$\text{proj}_V(\vec{x}) = (\vec{q}_1 \cdot \vec{x})\vec{q}_1 + (\vec{q}_2 \cdot \vec{x})\vec{q}_2 = \frac{5}{\sqrt{2}}\vec{q}_1 + \frac{1}{\sqrt{2}}\vec{q}_2$$

$$\frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$$

Example: Doing it Right... Coordinates

In the context of [COORDINATES (NOTES#3.4)], we have

$$\text{BASIS: } \mathfrak{Q} = \langle \vec{q}_1, \vec{q}_2 \rangle$$

$$\text{COORDINATES: } [\vec{x}]_{\mathfrak{Q}} = \begin{bmatrix} \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Milking the Example for More Details...

We have performed a *Change of Basis*, in this case for the purpose of making the projection onto the subspace easily (after the change of basis, that is) computable.

It is "easy" to see that

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A=\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{Q=\begin{bmatrix}\vec{q}_1 & \vec{q}_2\end{bmatrix}} \underbrace{\begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R},$$

we have $A = QR$, where Q is the new orthonormal basis, and R is an upper triangular matrix.

The entries in the R matrix are — $\sqrt{2}$: the original norm of $\vec{v}_1$; $\frac{3}{\sqrt{2}}$: the dot product $(\vec{q}_1 \cdot \vec{v}_2)$; $\frac{1}{\sqrt{2}}$: the norm of $\vec{v}_2^{\perp \vec{q}_1}$. Not likely a coincidence...

Let's Ponder Higher Dimensions

When you have more basis vectors $\vec{v}_1, \dots, \vec{v}_n$ needing orthogonalization (to make an orthonormal basis):

Theorem (Gram-Schmidt Process (annotated))

- *Start like we did:*
  - $\vec{q}_1 = \vec{v}_1 / \|\vec{v}_1\|$
  - $\vec{w}_2 = \vec{v}_2 - (\vec{q}_1 \cdot \vec{v}_2)\vec{q}_1$, *note that this is a vector in the orthogonal complement of* $\text{span}(\vec{q}_1) = \text{span}(\vec{v}_1)$.
  - $\vec{q}_2 = \vec{w}_2 / \|\vec{w}_2\|$
- *Each time we grab a new vector* ($\vec{v}_k$), *find a "help vector"* $\vec{w}_k$ *in the orthogonal complement of the space spanned by the previously computed* $\vec{q}$*-vectors:*
  - $\vec{w}_k = \vec{v}_k - (\vec{q}_1 \cdot \vec{v}_k)\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_k)\vec{q}_2 - \cdots - (\vec{q}_{k-1} \cdot \vec{v}_k)\vec{q}_{k-1}$
- *Then* $\vec{q}_k = \vec{w}_k / \|\vec{w}_k\|$.

The QR Factorization

The Gram-Schmidt process computed a change of basis from the old basis (funky-script-A)

$$\mathfrak{A} = (\vec{v}_1, \dots, \vec{v}_n)$$

to a new *orthonormal* basis (funky-script-Q)

$$\mathfrak{Q} = (\vec{q}_1, \dots, \vec{q}_n).$$

We describe the result using the change-of-basis-Matrix R from $\mathfrak{A}$ to $\mathfrak{Q}$, writing

$$\underbrace{\begin{pmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{pmatrix}}_{A} = \underbrace{\begin{pmatrix}\vec{q}_1 & \cdots & \vec{q}_n\end{pmatrix}}_{Q} R$$

Interpretations and Relations

With $A = QR$, we have to following relations:

- $[\vec{x}]_{\mathfrak{Q}} = R[\vec{x}]_{\mathfrak{A}}$
  - Multiplication by R moves us from A-coordinates to Q-coordinates.
- $\vec{x} = Q[\vec{x}]_{\mathfrak{Q}} = QR[\vec{x}]_{\mathfrak{A}}$
  - Multiplying the Q-coordinate vector by Q "builds" the vector $\vec{x}$.
- $\vec{x} = A[\vec{x}]_{\mathfrak{A}}$
  - Multiplying the A-coordinate vector by A "builds" the (same) vector $\vec{x}$.

The "burning" question is *how do we construct R?* It turn out we already have all the pieces, we just need some book-keeping.

What's in R? 1 of 3

If we think back to the k^{th} step, we compute

$$\underbrace{\vec{w}_k}_{\vec{v}_k^{\perp}} = \vec{v}_k - \underbrace{(\vec{q}_1 \cdot \vec{v}_k)\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_k)\vec{q}_2 - \cdots - (\vec{q}_{k-1} \cdot \vec{v}_k)\vec{q}_{k-1}}_{\vec{v}_k^{\parallel}}$$

$\vec{v}_k^{\perp}$ is orthogonal to $V_{k-1} = \mathrm{span}(\vec{q}_1, \dots, \vec{q}_{k-1}) = \mathrm{span}(\vec{v}_1, \dots, \vec{v}_{k-1})$, and $\vec{v}_k^{\parallel} \in \mathrm{span}(\vec{q}_1, \dots, \vec{q}_{k-1})$.

Note: Subspaces, Orthogonal Complements, and Bases

We are constructing a sequence of subspace-pairs

$$V_k \oplus V_k^{\perp} = \mathbb{R}^n; \quad \dim(V_k) = k, \ \dim(V_k^{\perp}) = (n-k); \quad k = 1, \dots, n$$

and orthonormal bases $\mathfrak{Q}_k = (\vec{q}_1, \dots, \vec{q}_k)$ for each of the V_k-spaces; and we have $V_{k-1} \subset V_k$ and $V_k^{\perp} \subset V_{k-1}^{\perp}$.

We are explicitly constructing V_k and $\mathfrak{Q}_k$; whereas we're only concerned with a specific vector $\vec{v}_k^{\perp} \in V_k^{\perp}$.

What's in R? 2 of 3

OK, let's rearrange the previous expression:

$$\vec{v}_k = \underbrace{(\vec{q}_1 \cdot \vec{v}_k)\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_k)\vec{q}_2 - \cdots - (\vec{q}_{k-1} \cdot \vec{v}_k)\vec{q}_{k-1}}_{\vec{v}_k^{\parallel}} + \underbrace{\vec{w}_k}_{\vec{v}_k^{\perp}}$$

The next thing we do is normalize $\vec{v}_k^{\perp}$ to be norm 1, and name it $\vec{q}_k$; which means we can write the relation above:

$$\vec{v}_k = \underbrace{(\vec{q}_1 \cdot \vec{v}_k)\vec{q}_1 + (\vec{q}_2 \cdot \vec{v}_k)\vec{q}_2 + \cdots + (\vec{q}_{k-1} \cdot \vec{v}_k)\vec{q}_{k-1}}_{\vec{v}_k^{\parallel}} + \underbrace{\|\vec{v}_k^{\perp}\|\vec{q}_k}_{\vec{v}_k^{\perp}}$$

This is the "recipe" for rebuilding the k^{th} column of A using the first k columns of Q. The entries in R are given by

- $r_{\ell,k} = (\vec{q}_\ell \cdot \vec{v}_k)$, $\ell < k$; $(r_{\ell,k} = 0,\ \ell > k)$, and
- $r_{k,k} = \|\vec{v}_k^{\perp}\|$.

What's in R? 3 of 3

$$\vec{v}_k = \underbrace{\overbrace{(\vec{q}_1 \cdot \vec{v}_k)}^{r_{1,k}}\vec{q}_1 + \overbrace{(\vec{q}_2 \cdot \vec{v}_k)}^{r_{2,k}}\vec{q}_2 + \cdots + \overbrace{(\vec{q}_{k-1} \cdot \vec{v}_k)}^{r_{k-1,k}}\vec{q}_{k-1}}_{\vec{v}_k^{\parallel}} + \underbrace{\|\vec{v}_k^{\perp}\|\vec{q}_k}_{\vec{v}_k^{\perp}}$$

$$\underbrace{\begin{bmatrix} | & & | & \\ \vec{v}_1 & \cdots & \vec{v}_k & \cdots \\ | & & | & \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} | & & | & \\ \vec{q}_1 & \cdots & \vec{q}_k & \cdots \\ | & & | & \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \|\vec{v}_1\| & & & r_{1,k} & \\ 0 & \ddots & & \vdots & \\ \vdots & \ddots & \|\vec{v}_{k-1}^{\perp}\| & r_{k-1,k} & \\ \vdots & & 0 & \|\vec{v}_k^{\perp}\| & \cdots \\ \vdots & & & 0 & \\ \vdots & & & \vdots & \\ 0 & \cdots & \cdots & 0 & \end{bmatrix}}_{R}$$

Summarizing ⇝ The QR-factorization

Theorem (QR-Factorization)

Consider an $(n \times m)$ matrix A, with linearly independent columns, $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Then there exists an $(n \times m)$ matrix Q whose columns $\vec{q}_1, \ldots, \vec{q}_m \in \mathbb{R}^n$ are orthonormal, and an upper triangular matrix R with positive diagonal entries such that $A = QR$. This representation is unique.

Further

- $r_{11} = \|\vec{v}_1\|$,
- $r_{kk} = \|\vec{v}_k^{\perp \operatorname{span}(\vec{q}_1, \cdots, \vec{q}_{k-1})}\|$, $k \in \{2, \ldots, m\}$, *and*
- $r_{\ell,k} = (\vec{q}_\ell \cdot \vec{v}_k)$, $\ell \in \{1, \ldots, k-1\}$.

Note that

$$[QR\text{-factorization}] = [\text{Gram-Schmidt}] + [\text{Bookkeeping}].$$

Observations $A = [\vec{v}_1 \cdots \vec{v}_m] = QR$, $A \in \mathbb{R}^{n \times m}$

- Note that $\operatorname{span}(\vec{q}_1, \ldots, \vec{q}_k) = \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_k)$, $k = 1, \ldots, m$ (that's the point — we are building an orthonormal set of vectors, describing the same subspaces spanned the columns of the matrix A)
- Let $V_k = \operatorname{span}(\vec{q}_1, \ldots, \vec{q}_k) \equiv \operatorname{span}(\vec{v}_1, \ldots, \vec{v}_k)$; these subspaces are "nested":
$$V_0 \subset V_1 \subset \cdots \subset V_k,$$
$$\dim(V_0) \leq \dim(V_1) \leq \cdots \leq \dim(V_k),$$
(the maximal dimension is limited by the number of linearly independent vectors in $\{\vec{v}_1, \ldots, \vec{v}_k\}$)
- **#ProjectionFestival**
$$\operatorname{proj}_{V_k}(\vec{x}) = (\vec{x} \cdot \vec{q}_1)\vec{q}_1 + \cdots + (\vec{x} \cdot \vec{q}_k)\vec{q}_k$$

Suggested Problems 5.2

Available on Learning Glass videos:
5.2 — 3, 7, 13, 31, 32, 33, <u>35</u>, 39

Lecture – Book Roadmap

Lecture	Book, [GS5–]
5.1	§4.1, §4.2, §**4.4**
5.2	§4.1, §4.2, §**4.4**
5.3	§4.1, §4.2, §**4.4**

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(5.2.3), (5.2.7)

(5.2.3) Perform the Gram-Schmidt process on the sequence of vectors given:

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}.$$

(5.2.7) Perform the Gram-Schmidt process on the sequence of vectors given:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}.$$

(5.2.13), (5.2.31)

(5.2.13) Perform the Gram-Schmidt process on the sequence of vectors given:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}.$$

(5.2.31) Perform the Gram-Schmidt process on the following basis of $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

(5.2.33), (5.2.35)

(5.2.33) Find an orthonormal basis for the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(5.2.35) Find an orthonormal basis for the image of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}.$$

(5.2.39)

(5.2.39) Find an orthonormal basis $\langle \vec{u}_1, \vec{u}_2, \vec{u}_3 \rangle$ of $\mathbb{R}^3$, such that

$$\text{span}\,(\vec{u}_1) = \text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}\right),$$

and

$$\text{span}\,(\vec{u}_1, \vec{u}_2) = \text{span}\left(\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right),$$

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Experience shows that at this point, most students tend to be a bit lost...

Known — We need orthogonal bases to perform (correct) orthogonal projections to higher dimensional ($n \geq 2$) subspaces.

But — The previous example (projecting from $\mathbb{R}^2 \to \mathbb{R}^2$) was not very satisfying...

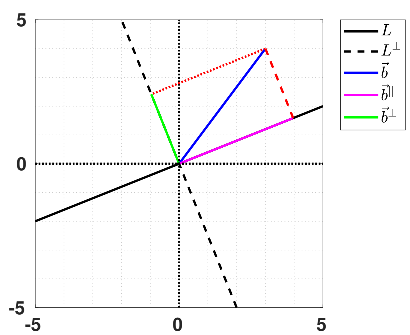
Mystery — Why are orthogonal projections such a big deal? (Bad reasons include:)

- "The professor said so." (multiple times)
- "It'll be on the test."

The goal of the next example is to give some idea as to why orthogonal projections can be useful... while re-visiting and connecting several "old" ideas.

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Recall our old cartoon of orthogonal projections:



where $\vec{w} \in \mathbb{R}^n$, $L = \{\, k\vec{w},\ k \in \mathbb{R}\,\}$ is the (line) subspace of $\mathbb{R}^n$.

Important Note:
$\vec{b}^{\parallel}$ is the point (in the subspace L) which is closest to $\vec{b}$.

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Now, let

$$A = \begin{bmatrix} | \\ \vec{w} \\ | \end{bmatrix} \in \mathbb{R}^{n\times 1},$$

then we are interested in solving the linear system $A\vec{x} = \vec{b}$, where $\vec{x} \in \mathbb{R}^1$ (for now), and $\vec{b} \in \mathbb{R}^n$.

The system has a solution if and only if $\vec{b} \in \text{im}(A) = L$.

When $\vec{b} \notin \text{im}(A)$ we can either

- say " *you guys, I'm going home!*" , or
- extend the concept of a "solution" to the problem...

Why Orthogonal Projections Matter ⇝ Solving the "Unsolvable"

Since this is not a South Park episode, we decide to extend the concept of what it means to "solve" this problem:

We decide to look for a value $\vec{x}^*$ which makes the **residual***

$$r(\vec{x}) = \|A\vec{x} - \vec{b}\|$$

as small as possible.

In our example, that value is $\vec{x}^* = \left(\frac{\vec{b}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)$, which makes $A\vec{x}^* = \vec{b}^{\|}$, and $r(\vec{x}^*) = \|\vec{b}^{\|} - \vec{b}\| = \|-\vec{b}^{\perp}\| = \|\vec{b}^{\perp}\|$.

It is true in general that the shortest distance between $\vec{b}$ and a subspace L, is $\vec{b}^{\perp} = \vec{b} - \mathrm{proj}_L(\vec{b})$.

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* think of is as a measure of how far we are from solving the linear system in the "traditional" sense.

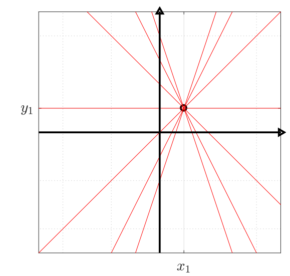
Supplemental Material
Solved Problems
Metacognitive Reflection
Problem Statements 5.2
Why Orthogonal Projections Matter

Why Orthogonal Projections Matter ⇝ Solving the "Unsolvable"

Next we consider a slightly different category of problems: fitting a straight line $y = a + bx$ to some number of given points in the x-y–plane, $\{(x_k, y_k)\}_{k=1}^{n}$.

Case ($n = 1$, a single point): In this case we have infinitely many solutions. In our notation the solutions are given by

$$\underbrace{\begin{bmatrix} 1 & x_1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \end{bmatrix}$$



which gives

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -x_1 \\ 1 \end{bmatrix}$$

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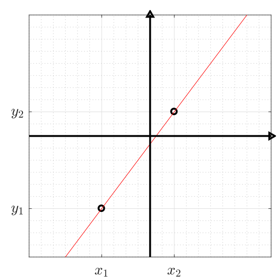
Why Orthogonal Projections Matter

Why Orthogonal Projections Matter ⇝ Solving the "Unsolvable"

Case ($n = 2$, two distinct points): In this case we have a unique solution. In our notation the solutions are given by

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$



which gives

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where the inverse is guaranteed to exist when $x_1 \neq x_2$.

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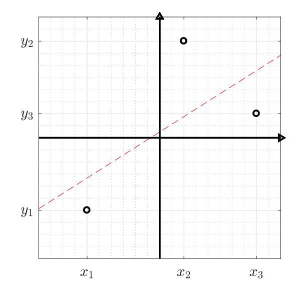
Why Orthogonal Projections Matter ⇝ Solving the "Unsolvable"

Case ($n = 3$, three distinct points): In this case we have no solution. In our notation the solutions would be given by

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$



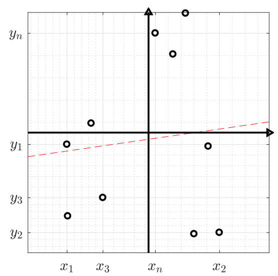
which gives

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \text{MAGIC} \\ \text{MATRIX} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad ???$$

There is no solution, unless the 3 points are on a common line...

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Case ($n = \text{large}$, many (distinct) points):
In this case we have no solution. In our notation the solutions would be given by



$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A \in \mathbb{R}^{n\times 2}} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y} \in \mathbb{R}^n}$$

which gives

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \textsc{Magic} \\ \textsc{Matrix} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \;???$$

There is no solution, unless the ALL points are on a common line...

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Staying in the general $n = \text{large}$ case, with

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A \in \mathbb{R}^{n\times 2}} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y} \in \mathbb{R}^n}$$

In our linear algebra language, we "know" that $P = \text{im}(A)$ is a 2-dimensional subspace of $\mathbb{R}^n$ (the two columns are different, unless all the x_k s coincide)...

and, of course, we only have a solution if/when $\vec{y}$ can be written as a linear combination of the columns of $A \Leftrightarrow$ "$\vec{y} \in \text{im}(A)$."

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Now, if we are looking for a best-extended-concept-of-solution candidate; we compute $\text{proj}_P(\vec{y}) \equiv \vec{y}^{\|}$, and the system

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A \in \mathbb{R}^{n\times 2}} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{c}} = \text{proj}_P(\vec{y})$$

does have a unique solution, call it $\vec{c}^*$; and the residual

$$r(\vec{c}^*) = \|A\vec{c}^* - \vec{y}\| = \|\vec{y}^{\|} - \vec{y}\| = \|\vec{y}^{\perp}\|$$

is minimized.

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

We have defined a new type of "solution" for inconsistent non-square (matrix) problems.

The way we have discussed it, the best name would be a

- "Minimum Residual Solution"

However, the most common mathematical name is the

- "Least Squares Solution"

In many applications (related to statistics), the most common name is the

- "Linear Regression Solution"

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$

What is your problem?!?
Find an orthonormal basis for the subspace

$$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4,$$

then project the vectors

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{y}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

onto V.

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$

- First, we need *a* basis for V; finding $\ker(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix})$ will do the trick.
- Since $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ already is in `rref`, we can identify the solutions to $A\vec{x} = 0$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so our basis is

$$B_V = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right); \quad A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

as an added bonus we will compute the QR-factorization of A.

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$

- $\|\vec{v}_1\| = \sqrt{(-1)^2 + 1^1 + 0^2 + 0^2} = \sqrt{2}$
- $\vec{q}_1 = \dfrac{1}{\|\vec{v}_1\|} \vec{v}_1$

$$Q = \begin{bmatrix} -1/\sqrt{2} & \times & \times \\ 1/\sqrt{2} & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

- We move on to $\vec{v}_2$...

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$

- $\vec{q}_1 \cdot \vec{v}_2 = \dfrac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \dfrac{1}{\sqrt{2}} \left((-1)^2 + 1 \times 0 + 0 \times 1 + 0 \times 0 \right) = \dfrac{1}{\sqrt{2}}$
- $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{q}_1 \cdot \vec{v}_2)\vec{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \left(\dfrac{1}{\sqrt{2}} \right) \left(\dfrac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) = \dfrac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$
- $\|\vec{v}_2^{\perp}\| = \dfrac{1}{2}\sqrt{1 + 1 + 4 + 0} = \dfrac{\sqrt{6}}{2}$
- $\vec{q}_2 = \dfrac{1}{\|\vec{v}_2^{\perp}\|} \vec{v}_2^{\perp} = \dfrac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$
- $$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & \times \\ 1/\sqrt{2} & -1/\sqrt{6} & \times \\ 0 & 2/\sqrt{6} & \times \\ 0 & 0 & \times \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \times \\ 0 & \sqrt{6}/2 & \times \\ 0 & 0 & \times \end{bmatrix}$$

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$ 5 of 8

- $\vec{q}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left((-1)^2 + 1 \times 0 + 0 \times 1 + 0 \times 0 \right) = \frac{1}{\sqrt{2}}$
- $\vec{q}_2 \cdot \vec{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \left((-1)^2 + (-1) \times 0 + 0 \times 1 + 0 \times 1 \right) = \frac{1}{\sqrt{6}}$
- $$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & \times \\ 1/\sqrt{2} & -1/\sqrt{6} & \times \\ 0 & 2/\sqrt{6} & \times \\ 0 & 0 & \times \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 & 1/\sqrt{6} \\ 0 & 0 & \times \end{bmatrix}$$

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$ 6 of 8

- $\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{q}_1 \cdot \vec{v}_3)\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_3)\vec{q}_2$:
$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) - \left(\frac{1}{\sqrt{6}} \right) \left(\frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right) = \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$
- $\|\vec{v}_3^{\perp}\| = \frac{1}{3}\sqrt{1+1+1+9} = \frac{\sqrt{12}}{3}$
- $\vec{q}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|} \vec{v}_3^{\perp} = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$
- $$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & -1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 & 1/\sqrt{6} \\ 0 & 0 & \sqrt{12}/3 \end{bmatrix}$$

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$ Projections! 7 of 8

- $\vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
- $\vec{q}_1 \cdot \vec{y}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (-1+1+0+0) = 0$
- $\vec{q}_2 \cdot \vec{y}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} (-1-1+2+0) = 0$
- $\vec{q}_3 \cdot \vec{y}_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{12}} (-1-1-1+3) = 0$
- $\mathrm{proj}_V(\vec{y}_1) = \vec{0}$
- Of course! We constructed $B_V = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ by finding all vectors orthogonal to $\vec{y}_1$ ((Solving $[1\ 1\ 1\ 1]\vec{x} = \vec{0}$))

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$ Projections! 8 of 8

- $\vec{y}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
- $\vec{q}_1 \cdot \vec{y}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}} (-1+2+0+0) = \frac{1}{\sqrt{2}}$
- $\vec{q}_2 \cdot \vec{y}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{6}} (-1-2+6+0) = \frac{3}{\sqrt{6}}$
- $\vec{q}_3 \cdot \vec{y}_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{12}} (-1-2-3+12) = \frac{6}{\sqrt{12}}$
- $\mathrm{proj}_V(\vec{y}_2) = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} + \frac{6}{12} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

5.2.35 and Beyond

What is your problem?!?

Given A, find an orthonormal basis for $\mathrm{im}(A)$, and the QR-factorization $QR = A$:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} . & . & . \\ . & . & . \\ . & . & . \end{bmatrix}, \quad R = \begin{bmatrix} . & . & . \\ 0 & . & . \\ 0 & 0 & . \end{bmatrix}$$

$$\vec{v}_1 :: \quad \|\vec{v}_1\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3; \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/3 & . & . \\ 2/3 & . & . \\ 2/3 & . & . \end{bmatrix}, \quad R = \begin{bmatrix} 3 & . & . \\ 0 & . & . \\ 0 & 0 & . \end{bmatrix}$$

5.2.35 and Beyond

$$\vec{v}_2 :: \quad \vec{v}_2^{\perp} = \vec{v}_2 - (\vec{q}_1 \cdot \vec{v}_2)\,\vec{q}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} - \underbrace{\left(\frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right)}_{0} \left(\frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

$$\|\vec{v}_2^{\perp}\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3, \quad \vec{q}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/3 & 2/3 & . \\ 2/3 & 1/3 & . \\ 2/3 & -2/3 & . \end{bmatrix}, \quad R = \begin{bmatrix} 3 & 0 & . \\ 0 & 3 & . \\ 0 & 0 & . \end{bmatrix}$$

5.2.35 and Beyond

$$\begin{aligned} \vec{v}_3^{\perp} &= \vec{v}_3 - (\vec{q}_1 \cdot \vec{v}_3)\,\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_3)\,\vec{q}_2 \\ &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \underbrace{\left(\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)}_{1} \left(\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right) - \underbrace{\left(\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)}_{1} \left(\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right) \\ &= \frac{1}{3}\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\|\vec{v}_3^{\perp}\| = 0, \quad \vec{q}_q = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/3 & 2/3 & ? \\ 2/3 & 1/3 & ? \\ 2/3 & -2/3 & ? \end{bmatrix}, \quad R = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

5.2.35 and Beyond

- $\vec{v}_3^{\perp} = 0$ means that $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.
- Therefore $\mathrm{im}(A) = \mathrm{span}(\vec{v}_1, \vec{v}_2) = \mathrm{span}(\vec{q}_1, \vec{q}_2)$
- We have 2 options for the QR-factorization:

$$A = \underbrace{\begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}}_{\text{"Economy Size" } QR\text{-factorization}}, \text{ or } \underbrace{\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{"Full" } QR\text{-factorization}}.$$

Note that in the second version, we have added a third orthonormal vector to the Q-matrix, and a row of zeros to the R-matrix.