Math 254: Introduction to Linear Algebra

Notes #5.2 — Gram-Schmidt Process and QR Factorization

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Outline

SLOs 5.2 — Gram-Schmidt Process and QR Factorization

After this lecture you should know how:

- to perform *The Gram-Schmidt Orthogonalization Process* on a set of vectors, and
- it can be used to compute *The QR-factorization* of a matrix A: $A = QR$
  - $\Rightarrow$ This builds an orthonormal basis (the columns of Q) for the subspace $V = \mathrm{im}(A)$, which gives us the means to compute the orthogonal projection $\mathrm{proj}_V(\vec{x})$ onto V.
- to orthogonally project onto *any* subspace.

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Orthogonal Projection onto a Subspace V

From [Notes#5.1] we have:

Theorem (Formula for the Orthogonal Projection)

If V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$, then

$$\mathrm{proj}_V(\vec{x}) = \vec{x}^{\|} = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_m \cdot \vec{x})\vec{u}_m$$

$\forall x \in \mathbb{R}^n$.

How do you project onto a subspace if/when the given basis is not orthonormal?!?

It turns out that before we compute the projection, we have to find a new — *orthonormal* — basis...

WARNING WARNING WARNING THIS IS ALL WRONG!!!!! WARNING WARNING WARNING

Ponder what happens if we use the formula, but the given basis is **not** orthonormal...

Let's live in $\mathbb{R}^2$, let $V = \mathbb{R}^2$, with basis $\mathfrak{B} = (\vec{v}_1, \vec{v}_2)$ defined by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}; \quad \|\vec{x}\| = \sqrt{13}.$$

Clearly $\vec{v}_1$ and $\vec{v}_2$ are linearly independent, and $\vec{x} = 1\vec{v}_1 + 1\vec{v}_2$, but the projection formula goes haywire:

$$\operatorname{proj}_V(\vec{x}) = (\vec{v}_1 \cdot \vec{x})\vec{v}_1 + (\vec{v}_2 \cdot \vec{x})\vec{v}_2 = 5\vec{v}_1 + 8\vec{v}_2 = \begin{bmatrix} 13 \\ 21 \end{bmatrix}.$$

... even if we remember to correct for the non-unit length of $\vec{v}_{1,2}$:

$$\operatorname{proj}_V(\vec{x}) = \frac{(\vec{v}_1 \cdot \vec{x})}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{(\vec{v}_2 \cdot \vec{x})}{\|\vec{v}_2\|^2}\vec{v}_2 = \frac{5}{2}\vec{v}_1 + \frac{8}{5}\vec{v}_2 = \begin{bmatrix} 4.1 \\ 5.7 \end{bmatrix}.$$

Comments

There are other ways to realize the "projection" went awry:

- This is "life in $\mathbb{R}^2$," and since

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

  are linearly independent $\rightsquigarrow$ they form a basis for $\mathbb{R}^2$ $\rightsquigarrow$ any projection of a vector $\vec{w} \in \mathbb{R}^2$ onto the subspace $V = \mathrm{span}(\vec{v}_1, \vec{v}_2) \equiv \mathbb{R}^2$ must be the original vector $\vec{w}$.

- Even simpler, the famous *Method of the Eyeball* already showed that $\vec{v}_1 + \vec{v}_2 = \vec{x}$:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Example: Doing it Right...

Build an Orthonormal Basis

In this case, given a basis of $\mathbb{R}^2$, the answer is "obvious."

Next, we develop (still in $\mathbb{R}^2$ so we easily can visualize and use our intuition) a method for building an *orthonormal basis* given *any* starting basis.

Once we have the orthonormal basis, we can use the projection formula...

! The method will work in the general case: *Given* $\vec{x} \in \mathbb{R}^n$, *and* $V = \mathrm{span}\,(\vec{v}_1, \ldots, \vec{v}_m) \subset \mathbb{R}^n$; *compute* $\mathrm{proj}_V(\vec{x})$:

We find an orthonormal basis $\vec{q}_1, \ldots, \vec{q}_m$, so that

$$V = \mathrm{span}\,(\vec{v}_1, \ldots, \vec{v}_m) = \mathrm{span}\,(\vec{q}_1, \ldots, \vec{q}_m)\,;$$

and then use the projection formula.

Example: Doing it Right...

Build an Orthonormal Basis

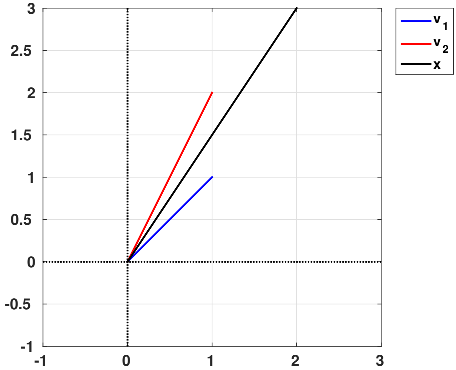


Figure: This is where we start.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Example: Doing it Right...

(i) Rescale the first vector

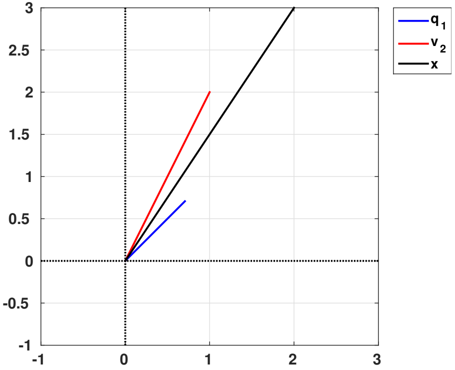


Figure: Rescale* $\vec{v}_1$ to be norm 1, and call it $\vec{q}_1$.

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

* Divide by the original norm, $\sqrt{2}$.

Example: Doing it Right...

(ii) Split the next vector into $\|$ and $\perp$ parts...

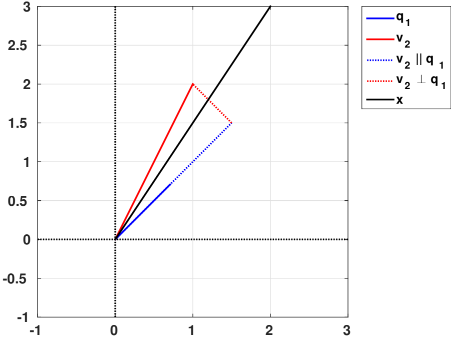


Figure: Next, project* $\vec{v}_2$ onto $\vec{q}_1$ and get $\vec{v}_2^{||\vec{q}_1}$, and $\vec{v}_2^{\perp\vec{q}_1}$.

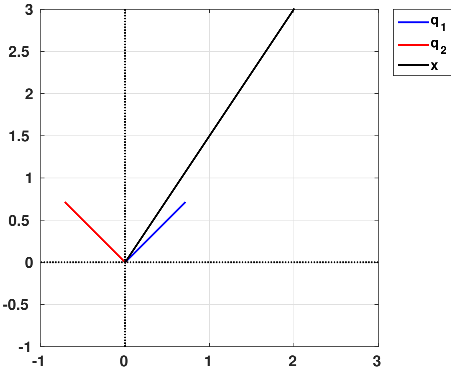
$$\vec{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad \vec{v}_2^{||\vec{q}_1} = \begin{bmatrix}1.5\\1.5\end{bmatrix}; \quad \vec{v}_2^{\perp\vec{q}_1} = \begin{bmatrix}-0.5\\0.5\end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix}2\\3\end{bmatrix}.$$

$*$ $\vec{v}_2^{||\vec{q}_1} = (\vec{q}_1 \cdot \vec{v}_2)\vec{q}_1 = \left(\frac{3}{\sqrt{2}}\right) \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix}; \quad \vec{v}_2^{\perp\vec{q}_1} = \begin{bmatrix}1\\2\end{bmatrix} - \vec{v}_2^{||\vec{q}_1} = \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}.$

Example: Doing it Right...

(iii) Rescale the $\perp$ part, discard $\|$ part

Figure: Next, throw away $\vec{v}_2^{\|\vec{q}_1}$; rescale $\vec{v}_2^{\perp\vec{q}_1}$ to norm 1, and name it $\vec{q}_2$.



$$\vec{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad \vec{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}; \quad \text{and } \vec{x} = \begin{bmatrix}2\\3\end{bmatrix},$$

Now we have an orthonormal basis $\mathfrak{Q} = \langle \vec{q}_1, \vec{q}_2 \rangle$!

Example: Doing it Right...

(iv) Project using the new orthonormal basis!

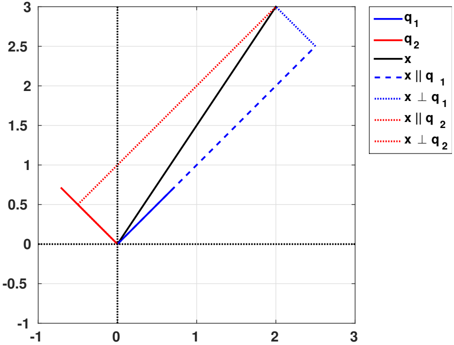


Figure: Finally, we can use the projection formula.

$$\mathrm{proj}_V(\vec{x}) = (\vec{q}_1 \cdot \vec{x})\vec{q}_1 + (\vec{q}_2 \cdot \vec{x})\vec{q}_2 = \frac{5}{\sqrt{2}}\vec{q}_1 + \frac{1}{\sqrt{2}}\vec{q}_2$$

$$\frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$$

Example: Doing it Right... Coordinates

In the context of [Coordinates (Notes#3.4)], we have

$$\text{Basis: } \mathfrak{Q} = \langle \vec{q}_1, \vec{q}_2 \rangle$$

$$\text{Coordinates: } [\vec{x}]_{\mathfrak{Q}} = \begin{bmatrix} \dfrac{5}{\sqrt{2}} \\ \dfrac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

Milking the Example for More Details...

We have performed a *Change of Basis*, in this case for the purpose of making the projection onto the subspace easily (after the change of basis, that is) computable.

It is "easy" to see that

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A=\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}} = \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{Q=\begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}} \underbrace{\begin{bmatrix} \sqrt{2} & \frac{3}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R},$$

we have $A = QR$, where Q is the new orthonormal basis, and R is an upper triangular matrix.

The entries in the R matrix are — $\sqrt{2}$: the original norm of $\vec{v}_1$; $\frac{3}{\sqrt{2}}$: the dot product $(\vec{q}_1 \cdot \vec{v}_2)$; $\frac{1}{\sqrt{2}}$: the norm of $\vec{v}_2^{\perp \vec{q}_1}$. Not likely a coincidence...

Let's Ponder Higher Dimensions

When you have more basis vectors $\vec{v}_1, \dots, \vec{v}_n$ needing orthogonalization (to make an orthonormal basis):

Theorem (Gram-Schmidt Process (annotated))

- *Start like we did:*
  - $\vec{q}_1 = \vec{v}_1 / \|\vec{v}_1\|$
  - $\vec{w}_2 = \vec{v}_2 - (\vec{q}_1 \cdot \vec{v}_2)\vec{q}_1$, *note that this is a vector in the orthogonal complement of* $\mathrm{span}(\vec{q}_1) = \mathrm{span}(\vec{v}_1)$.
  - $\vec{q}_2 = \vec{w}_2 / \|\vec{w}_2\|$
- *Each time we grab a new vector ($\vec{v}_k$), find a "help vector" $\vec{w}_k$ in the orthogonal complement of the space spanned by the previously computed $\vec{q}$-vectors:*
  - $\vec{w}_k = \vec{v}_k - (\vec{q}_1 \cdot \vec{v}_k)\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_k)\vec{q}_2 - \cdots - (\vec{q}_{k-1} \cdot \vec{v}_k)\vec{q}_{k-1}$
- *Then* $\vec{q}_k = \vec{w}_k / \|\vec{w}_k\|$.

The QR Factorization

The Gram-Schmidt process computed a change of basis from the old basis (funky-script-A)

$$\mathfrak{A} = (\vec{v}_1, \ldots, \vec{v}_n)$$

to a new *orthonormal* basis (funky-script-Q)

$$\mathfrak{Q} = (\vec{q}_1, \ldots, \vec{q}_n).$$

We describe the result using the change-of-basis-Matrix R from $\mathfrak{A}$ to $\mathfrak{Q}$, writing

$$\underbrace{\begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{pmatrix}}_{Q} R$$

Interpretations and Relations

With $A = QR$, we have to following relations:

- $[\vec{x}]_{\mathfrak{Q}} = R[\vec{x}]_{\mathfrak{A}}$
  - Multiplication by R moves us from A-coordinates to Q-coordinates.
- $\vec{x} = Q[\vec{x}]_{\mathfrak{Q}} = QR[\vec{x}]_{\mathfrak{A}}$
  - Multiplying the Q-coordinate vector by Q "builds" the vector $\vec{x}$.
- $\vec{x} = A[\vec{x}]_{\mathfrak{A}}$
  - Multiplying the A-coordinate vector by A "builds" the (same) vector $\vec{x}$.

The "burning" question is *how do we construct R?* It turn out we already have all the pieces, we just need some book-keeping.

What's in R?

If we think back to the k^{th} step, we compute

$$\underbrace{\vec{w}_k}_{\vec{v}_k^{\perp}} = \vec{v}_k - \underbrace{(\vec{q}_1 \cdot \vec{v}_k)\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_k)\vec{q}_2 - \cdots - (\vec{q}_{k-1} \cdot \vec{v}_k)\vec{q}_{k-1}}_{\vec{v}_k^{\|}}$$

$\vec{v}_k^{\perp}$ is orthogonal to $V_{k-1} = \mathrm{span}(\vec{q}_1, \dots, \vec{q}_{k-1}) = \mathrm{span}(\vec{v}_1, \dots, \vec{v}_{k-1})$, and $\vec{v}_k^{\|} \in \mathrm{span}(\vec{q}_1, \dots, \vec{q}_{k-1})$.

Note: Subspaces, Orthogonal Complements, and Bases

We are constructing a sequence of subspace-pairs

$$V_k \oplus V_k^{\perp} = \mathbb{R}^n; \quad \dim(V_k) = k,\ \dim(V_k^{\perp}) = (n-k); \quad k = 1, \dots, n$$

and orthonormal bases $\mathfrak{Q}_k = (\vec{q}_1, \dots, \vec{q}_k)$ for each of the V_k-spaces; and we have $V_{k-1} \subset V_k$ and $V_k^{\perp} \subset V_{k-1}^{\perp}$.

We are explicitly constructing V_k and $\mathfrak{Q}_k$; whereas we're only concerned with a specific vector $\vec{v}_k^{\perp} \in V_k^{\perp}$.

What's in R?

OK, let's rearrange the previous expression:

$$\vec{v}_k = \underbrace{(\vec{q}_1 \cdot \vec{v}_k)\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_k)\vec{q}_2 - \cdots - (\vec{q}_{k-1} \cdot \vec{v}_k)\vec{q}_{k-1}}_{\vec{v}_k^{||}} + \underbrace{\vec{w}_k}_{\vec{v}_k^{\perp}}$$

The next thing we do is normalize $\vec{v}_k^{\perp}$ to be norm 1, and name it $\vec{q}_k$; which means we can write the relation above:

$$\vec{v}_k = \underbrace{(\vec{q}_1 \cdot \vec{v}_k)\vec{q}_1 + (\vec{q}_2 \cdot \vec{v}_k)\vec{q}_2 + \cdots + (\vec{q}_{k-1} \cdot \vec{v}_k)\vec{q}_{k-1}}_{\vec{v}_k^{||}} + \underbrace{\|\vec{v}_k^{\perp}\|\vec{q}_k}_{\vec{v}_k^{\perp}}$$

This is the "recipe" for rebuilding the k^{th} column of A using the first k columns of Q. The entries in R are given by

- $r_{\ell,k} = (\vec{q}_\ell \cdot \vec{v}_k)$, $\ell < k$; $(r_{\ell,k} = 0,\ \ell > k)$, and
- $r_{k,k} = \|\vec{v}_k^{\perp}\|$.

What's in R?

$$\vec{v}_k = \underbrace{\overbrace{(\vec{q}_1 \cdot \vec{v}_k)}^{r_{1,k}}\vec{q}_1 + \overbrace{(\vec{q}_2 \cdot \vec{v}_k)}^{r_{2,k}}\vec{q}_2 + \cdots + \overbrace{(\vec{q}_{k-1} \cdot \vec{v}_k)}^{r_{k-1,k}}\vec{q}_{k-1}}_{\vec{v}_k^{\parallel}} + \underbrace{\|\vec{v}_k^{\perp}\|\vec{q}_k}_{\vec{v}_k^{\perp}}$$

$$\underbrace{\begin{bmatrix} | & & | & \\ \vec{v}_1 & \cdots & \vec{v}_k & \cdots \\ | & & | & \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} | & & | & \\ \vec{q}_1 & \cdots & \vec{q}_k & \cdots \\ | & & | & \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \|\vec{v}_1\| & & & r_{1,k} & \\ 0 & \ddots & & \vdots & \\ \vdots & \ddots & \|\vec{v}_{k-1}^{\perp}\| & r_{k-1,k} & \\ \vdots & & 0 & \|\vec{v}_k^{\perp}\| & \cdots \\ \vdots & & & 0 & \\ \vdots & & & \vdots & \\ 0 & \cdots & \cdots & 0 & \end{bmatrix}}_{R}$$

Summarizing $\rightsquigarrow$ The QR-factorization

Theorem (QR-Factorization)

Consider an $(n \times m)$ matrix A, with linearly independent columns, $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Then there exists an $(n \times m)$ matrix Q whose columns $\vec{q}_1, \ldots, \vec{q}_m \in \mathbb{R}^n$ are orthonormal, and an upper triangular matrix R with positive diagonal entries such that $A = QR$. This representation is unique.

Further

- $r_{11} = \|\vec{v}_1\|$,
- $r_{kk} = \|\vec{v}_k^{\perp \operatorname{span}(\vec{q}_1, \cdots, \vec{q}_{k-1})}\|$, $k \in \{2, \ldots, m\}$, *and*
- $r_{\ell,k} = (\vec{q}_\ell \cdot \vec{v}_k)$, $\ell \in \{1, \ldots, k-1\}$.

Note that

$$[QR\text{-factorization}] = [\text{Gram-Schmidt}] + [\text{Bookkeeping}].$$

Observations $A = [\vec{v}_1 \cdots \vec{v}_m] = QR$, $A \in \mathbb{R}^{n \times m}$

- Note that $\mathrm{span}(\vec{q}_1, \ldots, \vec{q}_k) = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k)$, $k = 1, \ldots, m$ (that's the point — we are building an orthonormal set of vectors, describing the same subspaces spanned the columns of the matrix A)
- Let $V_k = \mathrm{span}(\vec{q}_1, \ldots, \vec{q}_k) \equiv \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_k)$; these subspaces are "nested":
$$V_0 \subset V_1 \subset \cdots \subset V_k,$$
$$\dim(V_0) \leq \dim(V_1) \leq \cdots \leq \dim(V_k),$$
(the maximal dimension is limited by the number of linearly independent vectors in $\{\vec{v}_1, \ldots, \vec{v}_k\}$)
- **#ProjectionFestival**
$$\mathrm{proj}_{V_k}(\vec{x}) = (\vec{x} \cdot \vec{q}_1)\vec{q}_1 + \cdots + (\vec{x} \cdot \vec{q}_k)\vec{q}_k$$

Suggested Problems 5.2

Available on Learning Glass videos:

5.2 — 3, 7, 13, 31, 32, 33, <u>35</u>, 39

Lecture – Book Roadmap

Lecture	Book, [GS5–]
5.1	§4.1, §4.2, **§4.4**
5.2	§4.1, §4.2, **§4.4**
5.3	§4.1, §4.2, **§4.4**

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(5.2.3), (5.2.7)

(5.2.3) Perform the Gram-Schmidt process on the sequence of vectors given:

$$\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}.$$

(5.2.7) Perform the Gram-Schmidt process on the sequence of vectors given:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}.$$

(5.2.13), (5.2.31)

(5.2.13) Perform the Gram-Schmidt process on the sequence of vectors given:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}.$$

(5.2.31) Perform the Gram-Schmidt process on the following basis of $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} b \\ c \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}.$$

(5.2.33), (5.2.35)

(5.2.33) Find an orthonormal basis for the kernel of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(5.2.35) Find an orthonormal basis for the image of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}.$$

(5.2.39)

(5.2.39) Find an orthonormal basis $\langle \vec{u}_1, \vec{u}_2, \vec{u}_3 \rangle$ of $\mathbb{R}^3$, such that

$$\mathrm{span}\,(\vec{u}_1) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right),$$

and

$$\mathrm{span}\,(\vec{u}_1, \vec{u}_2) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right),$$

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Experience shows that at this point, most students tend to be a bit lost...

Known We need orthogonal bases to perform (correct) orthogonal projections to higher dimensional ($n \geq 2$) subspaces.

But The previous example (projecting from $\mathbb{R}^2 \to \mathbb{R}^2$) was not very satisfying...

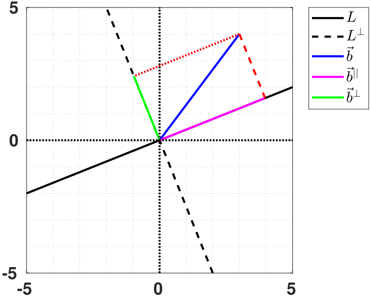
Mystery Why are orthogonal projections such a big deal? (Bad reasons include:)

- "The professor said so." (multiple times)
- "It'll be on the test."

The goal of the next example is to give some idea as to why orthogonal projections can be useful... while re-visiting and connecting several "old" ideas.

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Recall our old cartoon of orthogonal projections:



where $\vec{w} \in \mathbb{R}^n$, $L = \{\, k\vec{w},\ k \in \mathbb{R} \,\}$ is the (line) subspace of $\mathbb{R}^n$.

Important Note:
$\vec{b}^{\parallel}$ is the point (in the subspace L) which is closest to $\vec{b}$.

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Now, let

$$A = \begin{bmatrix} | \\ \vec{w} \\ | \end{bmatrix} \in \mathbb{R}^{n\times 1},$$

then we are interested in solving the linear system $A\vec{x} = \vec{b}$, where $\vec{x} \in \mathbb{R}^1$ (for now), and $\vec{b} \in \mathbb{R}^n$.

The system has a solution if and only if $\vec{b} \in \mathrm{im}(A) = L$.

When $\vec{b} \notin \mathrm{im}(A)$ we can either

- say " *you guys, I'm going home!*" , or
- extend the concept of a "solution" to the problem...

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Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Since this is not a South Park episode, we decide to extend the concept of what it means to "solve" this problem:

We decide to look for a value $\vec{x}^*$ which makes the **residual***

$$r(\vec{x}) = \|A\vec{x} - \vec{b}\|$$

as small as possible.

In our example, that value is $\vec{x}^* = \left(\frac{\vec{b}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)$, which makes $A\vec{x}^* = \vec{b}^{\|}$, and $r(\vec{x}^*) = \|\vec{b}^{\|} - \vec{b}\| = \|-\vec{b}^{\perp}\| = \|\vec{b}^{\perp}\|$.

It is true in general that the shortest distance between $\vec{b}$ and a subspace L, is $\vec{b}^{\perp} = \vec{b} - \mathrm{proj}_L(\vec{b})$.

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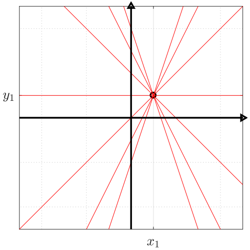
* think of is as a measure of how far we are from solving the linear system in the "traditional" sense.

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Next we consider a slightly different category of problems: fitting a straight line $y = a + bx$ to some number of given points in the x-y–plane, $\{(x_k, y_k)\}_{k=1}^{n}$.

Case ($n = 1$, a single point): In this case we have infinitely many solutions. In our notation the solutions are given by

$$\underbrace{\begin{bmatrix} 1 & x_1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \end{bmatrix}$$



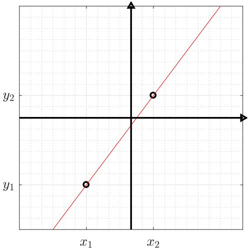
which gives

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -x_1 \\ 1 \end{bmatrix}$$

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Case ($n = 2$, two distinct points): In this case we have a unique solution. In our notation the solutions are given by

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$



which gives

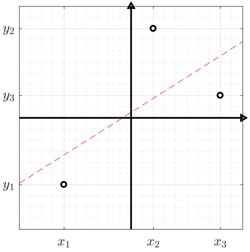
$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where the inverse is guaranteed to exist when $x_1 \neq x_2$.

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Case ($n = 3$, three distinct points): In this case we have no solution. In our notation the solutions would be given by

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$



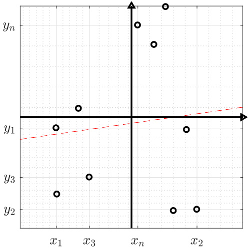
which gives

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \textsc{Magic} \\ \textsc{Matrix} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad ???$$

There is no solution, unless the 3 points are on a common line...

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Case ($n = \text{large}$, many (distinct) points): In this case we have no solution. In our notation the solutions would be given by



$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A \in \mathbb{R}^{n\times 2}} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y} \in \mathbb{R}^n}$$

which gives

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \textsc{Magic} \\ \textsc{Matrix} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad ???$$

There is no solution, unless the ALL points are on a common line...

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Staying in the general $n = \mathrm{large}$ case, with

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A \in \mathbb{R}^{n \times 2}} \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y} \in \mathbb{R}^n}$$

In our linear algebra language, we "know" that $P = \mathrm{im}(A)$ is a 2-dimensional subspace of $\mathbb{R}^n$ (the two columns are different, unless all the x_k s coincide)...

and, of course, we only have a solution if/when $\vec{y}$ can be written as a linear combination of the columns of $A \Leftrightarrow$ "$\vec{y} \in \mathrm{im}(A)$."

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

Now, if we are looking for a best-extended-concept-of-solution candidate; we compute $\mathrm{proj}_P(\vec{y}) \equiv \vec{y}^{\|}$, and the system

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A \in \mathbb{R}^{n\times 2}} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{c}} = \mathrm{proj}_P(\vec{y})$$

does have a unique solution, call it $\vec{c}^*$; and the residual

$$r(\vec{c}^*) = \|A\vec{c}^* - \vec{y}\| = \|\vec{y}^{\|} - \vec{y}\| = \|\vec{y}^{\perp}\|$$

is minimized.

Why Orthogonal Projections Matter $\rightsquigarrow$ Solving the "Unsolvable"

We have defined a new type of "solution" for inconsistent non-square (matrix) problems.

The way we have discussed it, the best name would be a

- "Minimum Residual Solution"

However, the most common mathematical name is the

- "Least Squares Solution"

In many applications (related to statistics), the most common name is the

- "Linear Regression Solution"

$V = \{\, x_1 + x_2 + x_3 + x_4 = 0 \,\} \subset \mathbb{R}^4$

What is your problem?!?

Find an orthonormal basis for the subspace

$$V = \{\, x_1 + x_2 + x_3 + x_4 = 0 \,\} \subset \mathbb{R}^4,$$

then project the vectors

$$\vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{y}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

onto V.

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$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$

- First, we need *a* basis for V; finding $\ker(\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix})$ will do the trick.

- Since $A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$ already is in `rref`, we can identify the solutions to $\vec{A}\vec{x} = 0$:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

so our basis is

$$B_V = (\vec{v}_1, \vec{v}_2, \vec{v}_3) = \left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right); \quad A = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

as an added bonus we will compute the QR-factorization of A.

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$

- $\|\vec{v}_1\| = \sqrt{(-1)^2 + 1^1 + 0^2 + 0^2} = \sqrt{2}$

- $\vec{q}_1 = \dfrac{1}{\|\vec{v}_1\|}\vec{v}_1$

$$Q = \begin{bmatrix} -1/\sqrt{2} & \times & \times \\ 1/\sqrt{2} & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

- We move on to $\vec{v}_2$...

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$

- $\vec{q}_1 \cdot \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \left((-1)^2 + 1 \times 0 + 0 \times 1 + 0 \times 0 \right) = \frac{1}{\sqrt{2}}$
- $\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{q}_1 \cdot \vec{v}_2)\vec{q}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right) = \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$
- $\|\vec{v}_2^{\perp}\| = \frac{1}{2}\sqrt{1+1+4+0} = \frac{\sqrt{6}}{2}$
- $\vec{q}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|} \vec{v}_2^{\perp} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$
- $$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & \times \\ 1/\sqrt{2} & -1/\sqrt{6} & \times \\ 0 & 2/\sqrt{6} & \times \\ 0 & 0 & \times \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \times \\ 0 & \sqrt{6}/2 & \times \\ 0 & 0 & \times \end{bmatrix}$$

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$

- $$\vec{q}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left((-1)^2 + 1 \times 0 + 0 \times 1 + 0 \times 0 \right) = \frac{1}{\sqrt{2}}$$

- $$\vec{q}_2 \cdot \vec{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \left((-1)^2 + (-1) \times 0 + 0 \times 1 + 0 \times 1 \right) = \frac{1}{\sqrt{6}}$$

- $$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & \times \\ 1/\sqrt{2} & -1/\sqrt{6} & \times \\ 0 & 2/\sqrt{6} & \times \\ 0 & 0 & \times \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 & 1/\sqrt{6} \\ 0 & 0 & \times \end{bmatrix}$$

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$

- $\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{q}_1 \cdot \vec{v}_3)\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_3)\vec{q}_2$:

$$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) - \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$$

- $\|\vec{v}_3^{\perp}\| = \frac{1}{3}\sqrt{1+1+1+9} = \frac{\sqrt{12}}{3}$
- $\vec{q}_3 = \frac{1}{\|\vec{v}_3^{\perp}\|}\vec{v}_3^{\perp} = \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$
- $$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} \\ 0 & 2/\sqrt{6} & -1/\sqrt{12} \\ 0 & 0 & 3/\sqrt{12} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{6}/2 & 1/\sqrt{6} \\ 0 & 0 & \sqrt{12}/3 \end{bmatrix}$$

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$ Projections!

- $\vec{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
- $\vec{q}_1 \cdot \vec{y}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (-1 + 1 + 0 + 0) = 0$
- $\vec{q}_2 \cdot \vec{y}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} (-1 - 1 + 2 + 0) = 0$
- $\vec{q}_3 \cdot \vec{y}_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{12}} (-1 - 1 - 1 + 3) = 0$
- $\mathrm{proj}_V(\vec{y}_1) = \vec{0}$
- Of course! We constructed $B_V = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ by finding all vectors orthogonal to $\vec{y}_1$ ((Solving $[1\ 1\ 1\ 1]\vec{x} = \vec{0}$))

$V = \{ x_1 + x_2 + x_3 + x_4 = 0 \} \subset \mathbb{R}^4$ Projections! 8 of 8

- $\vec{y}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$
- $\vec{q}_1 \cdot \vec{y}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}} (-1 + 2 + 0 + 0) = \frac{1}{\sqrt{2}}$
- $\vec{q}_2 \cdot \vec{y}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{6}} (-1 - 2 + 6 + 0) = \frac{3}{\sqrt{6}}$
- $\vec{q}_3 \cdot \vec{y}_2 = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{12}} (-1 - 2 - 3 + 12) = \frac{6}{\sqrt{12}}$
- $\mathrm{proj}_V(\vec{y}_2) = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{3}{6} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} + \frac{6}{12} \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

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5.2.35 and Beyond

What is your problem?!?

Given A, find an orthonormal basis for $\mathrm{im}(A)$, and the QR-factorization $QR = A$:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} . & . & . \\ . & . & . \\ . & . & . \end{bmatrix}, \quad R = \begin{bmatrix} . & . & . \\ 0 & . & . \\ 0 & 0 & . \end{bmatrix}$$

$$\vec{v}_1 :: \quad \|\vec{v}_1\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3; \quad \vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/3 & . & . \\ 2/3 & . & . \\ 2/3 & . & . \end{bmatrix}, \quad R = \begin{bmatrix} 3 & . & . \\ 0 & . & . \\ 0 & 0 & . \end{bmatrix}$$

5.2.35 and Beyond

$$\vec{v}_2 :: \quad \vec{v}_2^{\perp} = \vec{v}_2 - (\vec{q}_1 \cdot \vec{v}_2)\,\vec{q}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} - \underbrace{\left(\frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right)}_{0} \left(\frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

$$\|\vec{v}_2^{\perp}\| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3, \quad \vec{q}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1/3 & 2/3 & . \\ 2/3 & 1/3 & . \\ 2/3 & -2/3 & . \end{bmatrix}, \quad R = \begin{bmatrix} 3 & 0 & . \\ 0 & 3 & . \\ 0 & 0 & . \end{bmatrix}$$

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5.2.35 and Beyond

$$
\begin{aligned}
\vec{v}_3^{\perp} &= \vec{v}_3 - (\vec{q}_1 \cdot \vec{v}_3)\,\vec{q}_1 - (\vec{q}_2 \cdot \vec{v}_3)\,\vec{q}_2 \\
&= \begin{bmatrix}1\\1\\0\end{bmatrix} - \underbrace{\left(\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}\right)}_{1} \left(\frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix}\right) - \underbrace{\left(\frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix} \cdot \begin{bmatrix}1\\1\\0\end{bmatrix}\right)}_{1} \left(\frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix}\right) \\
&= \frac{1}{3}\begin{bmatrix}3\\3\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\2\\2\end{bmatrix} - \frac{1}{3}\begin{bmatrix}2\\1\\-2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}
\end{aligned}
$$

$$
\|\vec{v}_3^{\perp}\| = 0, \quad \vec{q}_q = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \begin{bmatrix}?\\?\\?\end{bmatrix}
$$

$$
Q = \begin{bmatrix}1/3 & 2/3 & ?\\ 2/3 & 1/3 & ?\\ 2/3 & -2/3 & ?\end{bmatrix}, \qquad R = \begin{bmatrix}3 & 0 & 1\\ 0 & 3 & 1\\ 0 & 0 & 0\end{bmatrix}
$$

5.2.35 and Beyond

- $\vec{v}_3^\perp = 0$ means that $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.
- Therefore $\mathrm{im}(A) = \mathrm{span}\,(\vec{v}_1, \vec{v}_2) = \mathrm{span}\,(\vec{q}_1, \vec{q}_2)$
- We have 2 options for the QR-factorization:

$$A = \underbrace{\begin{bmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}}_{\text{"Economy Size" } QR\text{-factorization}}, \text{ or } \underbrace{\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{"Full" } QR\text{-factorization}}.$$

Note that in the second version, we have added a third orthonormal vector to the Q-matrix, and a row of zeros to the R-matrix.