Math 254: Introduction to Linear Algebra
Notes #5.3 — Orthogonal Transformations and Orthogonal Matrices

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Outline

SLOs 5.3   Orthogonal Transformations and Orthogonal Matrices

After this lecture you should:

- Know what Orthogonal Transformations are; and their relation to Orthonormal Bases.
- Know the Properties of Orthogonal Matrices.
- Be able to perform an Orthogonal Projection using Orthonormal Basis you have constructed.

Orthogonal Transformations and Orthogonal Matrices
Suggested Problems

Examples, and Fundamental Theorems
Products, Inverses, and Transposes of Orthogonal Matrices
The Matrix of an Ortho. Projection, using an Ortho. Basis

Orthogonal Transformations

For many reasons, we tend to "like" linear transformations that preserve the norm (length) of vectors; and angles between vectors:

Definition (Orthogonal Transformations)

A linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is called orthogonal if it preserves the norm (length) of vectors:

$$\| T(\vec{x}) \| = \| \vec{x} \|, \ \forall \vec{x} \in \mathbb{R}^n.$$

If $T(\vec{x}) = A\vec{x}$ is an orthogonal transformation, we say that A is an orthogonal (or *unitary,* when it has complex entries) matrix.

Related topic: "Isometries" in [MATH 524 (NOTES#7.2)].

Orthogonal Transformations and Orthogonal Matrices
Suggested Problems

Examples, and Fundamental Theorems
Products, Inverses, and Transposes of Orthogonal Matrices
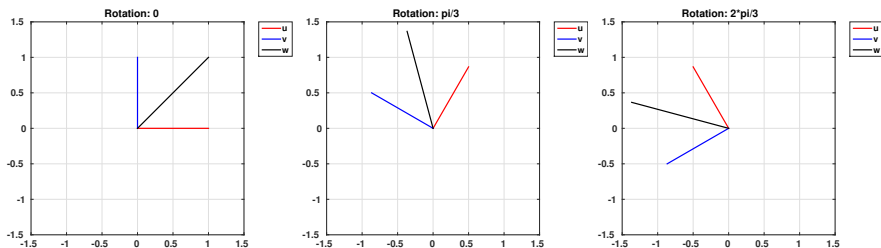The Matrix of an Ortho. Projection, using an Ortho. Basis

Example: Rotations

Example (Rotations)

The rotation

$$T(\vec{x}) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \vec{x}$$

is an orthogonal transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, and $\forall \theta$.
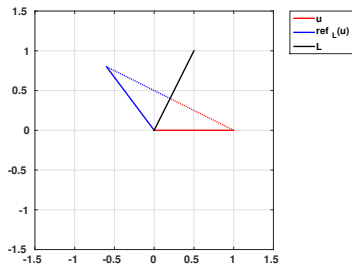
Orthogonal Transformations and Orthogonal Matrices
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Example: Reflections

Example (Reflections)

Consider a subspace V of $\mathbb{R}^n$. For a vector $\vec{x} \in \mathbb{R}^n$, the vector $\mathrm{ref}_V(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp} \equiv 2\mathrm{proj}_V(\vec{x}) - \vec{x}$ is the reflection of $\vec{x}$ in V. We show that reflections are orthogonal transformations:

By the [PYTHAGOREAN THEOREM], we have

$$\|\mathrm{ref}_V(\vec{x})\|^2 = \left\|\vec{x}^{\parallel} - \vec{x}^{\perp}\right\|^2 = \left\|\vec{x}^{\parallel}\right\|^2 + \|-\vec{x}^{\perp}\|^2 = \left\|\vec{x}^{\parallel}\right\|^2 + \|\vec{x}^{\perp}\|^2 = \|\vec{x}\|^2$$

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Preservation of Orthogonality

Theorem (Preservation of Orthogonality)

Consider an orthogonal transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^n$. If the vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are orthogonal, then so are $T(\vec{v})$ and $T(\vec{w})$.

Proof

Proof (Preservation of Orthogonality {*Short: relies on fundamental properties/definitions*})

By the theorem of Pythagoras, we have to show that

$$\| T(\vec{v}) + T(\vec{w}) \|^2 = \| T(\vec{v}) \|^2 + \| T(\vec{w}) \|^2 :$$

$$
\begin{aligned}
\| T(\vec{v}) + T(\vec{w}) \|^2 &= \| T(\vec{v} + \vec{w}) \|^2 && \text{[Linearity of } T\text{]} \\
&= \| \vec{v} + \vec{w} \|^2 && \text{[Orthogonality of } T\text{]} \\
&= \| \vec{v} \|^2 + \| \vec{w} \|^2 && [\vec{v} \perp \vec{w}] \\
&= \| T(\vec{v}) \|^2 + \| T(\vec{w}) \|^2 && \text{[Orthogonality of } T\text{]}
\end{aligned}
$$

Orthogonal Transformations and Orthogonal Matrices
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Orthogonal Transformations and Orthonormal Bases

Theorem (Orthogonal Transformations and Orthonormal Bases)

a. *A linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is orthogonal if and only if the vectors $T(\vec{e}_1), \ldots, T(\vec{e}_n)$ form an orthonormal basis of $\mathbb{R}^n$.*

b. *An $(n \times n)$ matrix A is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^n$.*

Proof

[PROOF IN SUPPLEMENTAL SLIDES]

Orthogonal Transformations and Orthogonal Matrices
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Examples, and Fundamental Theorems
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[Proof] Orthogonal Transformations and Orthonormal Bases [Focus :: Math]

Proof (Part (a))

$\Rightarrow$ If T is orthogonal, then, by definition, the $T(\vec{e}_k)$ are unit vectors, and orthogonal by the previous theorem; hence a basis for $\mathbb{R}^n$.

$\Leftarrow$ Conversely, suppose $T(\vec{e}_1), \ldots, T(\vec{e}_n)$ form an orthonormal basis. Consider a vector $\vec{x} = x_1 \vec{e}_1 + \cdots + x_n \vec{e}_n \in \mathbb{R}^n$. Then

$$
\begin{aligned}
\|T(\vec{x})\|^2 &= \|x_1 T(\vec{e}_1) + \cdots + x_n T(\vec{e}_n)\|^2 && \text{[Linearity]} \\
&= \|x_1 T(\vec{e}_1)\|^2 + \cdots + \|x_n T(\vec{e}_n)\|^2 && \text{[Pythagoras]} \\
&= x_1^2 \|T(\vec{e}_1)\|^2 + \cdots + x_n^2 \|T(\vec{e}_n)\|^2 \\
&= x_1^2 + \cdots + x_n^2 \\
&= \|\vec{x}\|^2 .
\end{aligned}
$$

Orthogonal Transformations and Orthogonal Matrices
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[Proof] Orthogonal Transformations and Orthonormal Bases   [Focus :: Math]

Proof (Part (b))

This follows from the result from [Notes#2.1] restated below...

Theorem (The Columns of the Matrix of a Linear Transformation)

Consider a linear transformation $T : \mathbb{R}^m \mapsto \mathbb{R}^n$. Then, the matrix of T is

$$A = \left[\begin{array}{cccc} | & | & & | \\ T(\vec{e_1}) & T(\vec{e_2}) & \ldots & T(\vec{e_m}) \\ | & | & & | \end{array} \right],$$

where $\vec{e_i} \in \mathbb{R}^m$ is the vector of all zeros, except entry #i which is 1.

Orthogonal Transformations and Orthogonal Matrices
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A Warning

> **WARNING!!!**
>
> A matrix with orthogonal columns need not be an orthogonal matrix, *e.g.*
> $$A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}.$$

Example ($A = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$ has Orthogonal Columns, but is Not Orthogonal)

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \|\vec{x}\| = \sqrt{2}, \quad A\vec{x} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, \quad \|A\vec{x}\| = \sqrt{50}$$

Orthogonal Transformations and Orthogonal Matrices
Suggested Problems

Examples, and Fundamental Theorems
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Products and Inverses of Orthogonal Matrices

Theorem (Products and Inverses of Orthogonal Matrices)

a. The product AB of two orthogonal ($n \times n$) matrices A and B is orthogonal.

b. The inverse A^{-1} of an orthogonal ($n \times n$) matrix A is orthogonal.

Proof ({Short: relies on fundamental properties/definitions})

a. the linear transformation $T(\vec{x}) = AB\vec{x}$ preserves norm (length), since $\|T(\vec{x})\| = \|A(B\vec{x})\| = \|B\vec{x}\| = \|\vec{x}\|$.

b. the linear transformation $T(\vec{x}) = A^{-1}\vec{x}$ preserves norm (length), since $\|A^{-1}\vec{x}\| = \|AA^{-1}\vec{x}\| = \|\vec{x}\|$.

Orthogonal Transformations and Orthogonal Matrices	Examples, and Fundamental Theorems
Suggested Problems	**Products, Inverses, and Transposes of Orthogonal Matrices**
	The Matrix of an Ortho. Projection, using an Ortho. Basis

Example: Properties of the Transpose of an Orthonormal Matrix

Example

Consider the orthogonal matrix A, and the matrix where the ij entry has been shifted to the ji position (B):

$$A = \frac{1}{7} \begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix}, \quad B = \frac{1}{7} \begin{bmatrix} 2 & 3 & 6 \\ 6 & 2 & -3 \\ 3 & -6 & 2 \end{bmatrix}.$$

We compute

$$BA = \frac{1}{49} \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The $k\ell$ entry in BA is the dot product of the kth row of B, and the ℓth column of A; by construction this is the dot product of the kth and ℓ^{th} columns of A; since A is orthogonal this gives 1 when $k = \ell$, and 0 otherwise.

Orthogonal Transformations and Orthogonal Matrices
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Matrix Transpose, Symmetric and Skew-symmetric Matrices

Definition (Matrix Transpose, Symmetric and Skew-symmetric Matrices)

Consider an $(m \times n)$ matrix A.

- The *transpose* A^T of A is the $(n \times m)$ matrix whose ij^{th} entry is the ji^{th} entry of A: The roles of rows and columns are reversed.

- We say that a square matrix A is *symmetric* if $A^T = A$, and

- A is called *skew-symmetric* if $A^T = -A$.

Example (Transpose)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

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[FOCUS :: MATH]   Symmetric (2×2) Matrices   [Linear Spaces]

Example (Symmetric (2×2) Matrices)

The symmetric (2×2) matrices are of the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

They form a **3**-dimensional subspace of $\mathbb{R}^{2 \times 2}$ with basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Note: $\mathbb{R}^{2 \times 2}$ (the collection of all 2-by-2 matrices) is a linear space (formal definition on the next slide)...

Orthogonal Transformations and Orthogonal Matrices
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[FOCUS :: MATH]   Linear Space   Definition   [Linear Spaces]

Definition (Linear Space)

A Linear Space V is a set with a definition (rule) for addition "+", and a definition
(rule) for scalar multiplication; and the following must hold ($\forall u, v, w \in V$, $\forall c, k \in \mathbb{R}$)

   a. $v + w \in V$.

   b. $kv \in V$.

   c. $(u + v) + w = u + (v + w)$.

   d. $u + v = v + u$.

   e. $\exists n \in V$: $u + n = u$, [NEUTRAL ELEMENT, denoted by 0]

   f. $\exists \hat{u}$: $u + \hat{u} = 0$; $\hat{u}$ unique, and denoted by $-u$.

   g. $k(u + v) = ku + kv$.

   h. $(c + k)u = cu + ku$.

   i. $c(ku) = (ck)u$.

   j. $1u = u$.

in $\mathbb{R}^{2\times 2}$, the neutral element is $n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Orthogonal Transformations and Orthogonal Matrices
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[FOCUS :: MATH]   Skew-Symmetric (2×2) Matrices   [Linear Spaces]

Example (Skew-Symmetric (2×2) Matrices)

The symmetric (2×2) matrices are of the form

$$\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

They form a **1**-dimensional subspace of $\mathbb{R}^{2\times 2}$ with basis

$$\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

Note: $\dim(\mathbb{R}^{2\times 2}) = 4$; $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis.

Orthogonal Transformations and Orthogonal Matrices
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Transpose of a Vector

Example (Transpose of a Vector)

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \Rightarrow \quad \vec{v}^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

We use this all the time:

Theorem

If $\vec{v}$ and $\vec{w}$ are two (column) vectors $\in \mathbb{R}^n$, then

$$\vec{v} \cdot \vec{w} \qquad \equiv \qquad \vec{v}^T \vec{w}$$

DOT PRODUCT   "MATRIX" PRODUCT

Orthogonal Transformations and Orthogonal Matrices
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Examples, and Fundamental Theorems
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Orthogonal Matrices: A^T and A^{-1}

Theorem

Consider an $(n \times n)$ matrix A. The matrix A is orthogonal if and only if $A^T A = I_n$ or, equivalently, if $A^{-1} = A^T$.

Proof ({*Short: relies on fundamental properties/definitions*})

Write A in terms of its columns:

$$A = \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_n, \end{bmatrix}$$

then

$$A^T A = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_n^T \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} \vec{v}_1^T \vec{v}_1 & \ldots & \vec{v}_1^T \vec{v}_n \\ \vdots & \ddots & \vdots \\ \vec{v}_n^T \vec{v}_1 & \ldots & \vec{v}_n^T \vec{v}_n \end{bmatrix}$$

this is I_n if and only if A is orthogonal.

Orthogonal Transformations and Orthogonal Matrices
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Orthogonal Matrices: Summary

Summary :: Orthogonal Matrices

Consider an $(n \times n)$ matrix A. The following statements are equivalent:

i. A is an orthogonal matrix.

ii. The transformation $T(\vec{x}) = A\vec{x}$ preserves norm (length), that is, $\|A\vec{x}\| = \|\vec{x}\| \ \forall \vec{x} \in \mathbb{R}^n$.

iii. The columns of A form an orthonormal basis of $\mathbb{R}^n$.

iv. $A^T A = I_n$.

v. $A^{-1} = A^T$.

vi. A preserves the dot product, meaning that $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$ $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$.

Orthogonal Transformations and Orthogonal Matrices
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Properties of the Matrix Transpose

Theorem (Properties of the Transpose)

a. $(A + B)^T = A^T + B^T$ $\quad \forall A, B \in \mathbb{R}^{m \times n}$

b. $(kA)^T = kA^T$ $\quad \forall A \in \mathbb{R}^{m \times n}, \forall k \in \mathbb{R}$

c. $(AB)^T = (B^T A^T)$ $\quad \forall A \in \mathbb{R}^{m \times p}, \forall B \in \mathbb{R}^{p \times n}$

d. $\operatorname{rank}(A) = \operatorname{rank}(A^T)$ $\quad \forall matrices\ A$

e. $(A^T)^{-1} = (A^{-1})^T$ $\quad \forall invertible\ matrices\ A$

Orthogonal Transformations and Orthogonal Matrices
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The Matrix of an Orthogonal Projection

We can use our expanded matrix-notation-language to express orthogonal projections.... First consider

$$\text{proj}_L(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1$$

onto a line L in $\mathbb{R}^n$; where $\vec{u}_1$ is a unit vector in L. Think of this vector as an $(n \times 1)$ matrix, and the scalar $(\vec{u}_1 \cdot \vec{x})$ as an (1×1) matrix; we can rearrange

$$\text{proj}_L(\vec{x}) = \vec{u}_1(\vec{u}_1 \cdot \vec{x}) \overset{①}{=} \vec{u}_1(\vec{u}_1^T \vec{x}) \overset{②}{=} \vec{u}_1 \vec{u}_1^T \vec{x} \overset{③}{=} (\vec{u}_1 \vec{u}_1^T)\vec{x} \overset{④}{=} A\vec{x}$$

where $A = \vec{u}_1 \vec{u}_1^T$.

(!) We derived an expression for A (for action in $\mathbb{R}^2$) back in [NOTES#2.2].

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①Notation; ②Associative property for matrix multiplication; ③Associative property for matrix multiplication; ④"Book-keeping" /interpretation.

Orthogonal Transformations and Orthogonal Matrices
Suggested Problems

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Vector-Vector Products

New: Outer Product $\qquad\qquad \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$

$$\underbrace{A}_{[n \times n]} = \underbrace{\vec{u}\vec{v}^T}_{[n \times 1] \times [1 \times n]} \quad \text{is known as the } \textbf{outer product}.$$

Old: Inner Product / Dot Product $\qquad \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^1$

$$\underbrace{s}_{[1 \times 1]} = \underbrace{\vec{u}^T \vec{v}}_{[1 \times n] \times [n \times 1]}$$

Upcoming: Cross Product $\qquad \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3, \text{ (or } \mathbb{R}^7 \times \mathbb{R}^7 \mapsto \mathbb{R}^7)$

$$\underbrace{\vec{q}}_{[3 \times 1]} = \underbrace{\vec{u}}_{[3 \times 1]} \times \underbrace{\vec{v}}_{[3 \times 1]}, \quad \left(\text{or} \quad \underbrace{\vec{w}}_{[7 \times 1]} = \underbrace{\vec{u}}_{[7 \times 1]} \times \underbrace{\vec{v}}_{[7 \times 1]} \right)$$

Orthogonal Transformations and Orthogonal Matrices
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The Matrix of an Orthogonal Projection

We can apply the same idea to the general projection formula

$$
\begin{aligned}
\mathrm{proj}_V(\vec{x}) &= (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + \cdots + (\vec{u}_n \cdot \vec{x})\vec{u}_n \\
&= \vec{u}_1 \vec{u}_1^T \vec{x} + \cdots + \vec{u}_n \vec{u}_n^T \vec{x} \\
&= \underbrace{(\vec{u}_1 \vec{u}_1^T + \cdots + \vec{u}_n \vec{u}_n^T)}_{A} \vec{x}
\end{aligned}
$$

and we can also write

$$
A = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix} \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_n^T \end{bmatrix}
$$

We summarize on the next slide...

Orthogonal Transformations and Orthogonal Matrices
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Examples, and Fundamental Theorems
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The Matrix of an Orthogonal Projection: Summary

Theorem (The Matrix of an Orthogonal Projection: Summary)

Consider a subspace V of $\mathbb{R}^n$ with orthonormal basis $\vec{q}_1, \ldots, \vec{q}_m$.
The matrix P of the orthogonal projection onto V is

$$P = QQ^T, \quad \text{where } Q = \begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_m \end{bmatrix}.$$

- Note that it is QQ^T **not** $Q^T Q$
- P is symmetric — $P^T = (QQ^T)^T = (Q^T)^T Q^T = QQ^T = P$.

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Example   1 of 2

In (5.2.7) [SEE LEARNING GLASS] we orthogonaliz(ed) the vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix},$$

using the Gram-Schmidt method, and get(got)

$$\vec{q}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \quad \vec{q}_2 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}, \quad \vec{q}_3 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix},$$

Let's define $\{ Q_1 \in \mathbb{R}^{3\times 1}, \ Q_2 \in \mathbb{R}^{3\times 2}, \ Q_3 \in \mathbb{R}^{3\times 3} \}$

$$Q_1 = \begin{bmatrix} \vec{q}_1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{bmatrix}.$$

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Example

Projection Matrices	**Orthonormality Confirmation**
$P_1 = Q_1 Q_1^T = \dfrac{1}{9} \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$	$Q_1^T Q_1 = \begin{bmatrix} 1 \end{bmatrix}$
$P_2 = Q_2 Q_2^T = \dfrac{1}{9} \begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}$	$Q_2^T Q_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$P_3 = Q_3 Q_3^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$Q_3^T Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note, $Q_1 Q_1^T$, $Q_2 Q_2^T$, and $Q_3 Q_3^T$ are the matrices of orthogonal projections onto a line $L = \mathrm{span}(\vec{q}_1)$, a plane $V = \mathrm{span}(\vec{q}_1, \vec{q}_2)$, and $\mathbb{R}^3 = \mathrm{span}(\vec{q}_1, \vec{q}_2, \vec{q}_3)$.

Suggested Problems 5.3

Available on Learning Glass videos:

5.3 — 1, 2, 5, 6, 13, 15, 17, 19, 28, 32, 33, <u>36</u>, 41

Lecture – Book Roadmap

Lecture	Book, [GS5–]
5.1	§4.1, §4.2, **§4.4**
5.2	§4.1, §4.2, **§4.4**
5.3	§4.1, §4.2, **§4.4**

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?

Right After Lecture		

After Thinking / Office Hours / SI-session		

After Reviewing for Quiz/Midterm/Final		

(5.3.1), (5.3.2)

(5.3.1) Is the given matrix Orthogonal?

$$A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$$

(5.3.2) Is the given matrix Orthogonal?

$$A = \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

(5.3.5), (5.3.6)

If the ($n \times n$) matrices A and B are orthogonal, are the following matrices orthogonal as well?

(5.3.5) $C = 3A$

(5.3.6) $D = -B$

(5.3.13), (5.3.15), (5.3.17), (5.3.19)

If the $(n \times n)$ matrices A and B are symmetric, and B is invertible; are the following matrices symmetric as well?

(5.3.13) $C = 3A$

(5.3.15) $D = AB$

(5.3.17) $F = B^{-1}$

(5.3.19) $G = 2I_n + 3A - 4A^2$

(5.3.28)

(5.3.28) Consider an $(n \times n)$ matrix A. Show that A is orthogonal if-and-only-if: A preserves the dot product; *i.e.*

$$(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$.

Hint, show:

1. $A^T A = I_n \Rightarrow (A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$
2. $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y} \Rightarrow L(\vec{x}) = A\vec{x}$ is norm (length)-preserving.

(5.3.32), (5.3.33)

(5.3.32–a) Consider an ($n \times m$) matrix A such that $A^T A = I_m$. Is is necessarily true that $AA^T = I_n$? (Explain!)

(5.3.32–b) Consider an ($n \times n$) matrix A such that $A^T A = I_n$. Is is necessarily true that $AA^T = I_n$? (Explain!)

(5.3.33) Find all orthogonal (2×2) matrices.

(5.3.36)

(5.3.36) Find an orthogonal matrix of the form

$$A = \begin{bmatrix} 2/3 & 1\sqrt{2} & a \\ 2/3 & -1/\sqrt{2} & b \\ 1/3 & 0 & c \end{bmatrix}$$

(5.3.41)

(5.3.41) Find the matrix A of the orthogonal projection onto the line in $\mathbb{R}^n$ spanned by the vector

$$\vec{1}_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$$

Revisiting the "Why?!?"

This section provides one important answer to "why?!" we should care about orthogonality, orthogonal complements, and orthogonal projections.

We will talk about *Least Squares Solutions* to non-consistent linear systems. (From a slightly different point of view than [NOTES#5.2: SUPPLEMENT].)

The least squares formulation is useful for fitting model parameters to data and has applications in a wide range of fields: chemistry, physics, engineering, finance, economics, etc.

It is sometimes (often?) referred to as *"Linear Regression."*

The Orthogonal Complement of the Image

Example (The Orthogonal Complement of $\operatorname{im}(A)$)

Consider a subspace $V = \operatorname{im}(A)$ of $\mathbb{R}^n$, where

$$A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix}.$$

Then the *orthogonal complement* is,

$$
\begin{aligned}
V^{\perp} &= \{\vec{x} \in \mathbb{R}^n : \vec{v} \cdot \vec{x} = 0, \ \forall \vec{v} \in V\} \\
&= \{\vec{x} \in \mathbb{R}^n : \vec{v}_i \cdot \vec{x} = 0, \ i = 1, \ldots, m\} \\
&= \left\{\vec{x} \in \mathbb{R}^n : \vec{v}_i^T \vec{x} = 0, \ i = 1, \ldots, m\right\}.
\end{aligned}
$$

In other words, $V^{\perp}$ is the kernel of the matrix

$$
A^T = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_m^T \end{bmatrix}.
$$

The Orthogonal Complement of the Image

Theorem (The Orthogonal Complement of the Image)

For any matrix A,

$$(\operatorname{im}(A))^{\perp} = \ker\left(A^T\right)$$

A Line in $\mathbb{R}^3$

Example (A Line in $\mathbb{R}^3$)

Consider the line

$$V = \operatorname{im}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$$

Then

$$V^{\perp} = \ker\left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\right)$$

is the plane with equation $x_1 + 2x_2 + 3x_3 = 0$; as usual we can parameterize (to get a basis), and Gram-Schmidt Orthogonalize (to make it orthonormal)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \tilde{s}\frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \tilde{t}\frac{1}{\sqrt{70}} \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}.$$

[FOCUS :: MATH] $\ker(A)$, $\ker(A^T A)$, and Invertibility of $A^T A$

Theorem

a. If A is an $(m \times n)$ matrix, then $\ker(A) = \ker(A^T A)$.

b. If A is an $(m \times n)$ matrix with $\ker(A) = \{\vec{0}\}$, then $A^T A$ is invertible.

Proof (Proof)

a. Clearly, the kernel of A is contained in the kernel of $A^T A$.
Conversely, consider a vector $\vec{x} \in \ker(A^T A)$, so that $A^T A \vec{x} = \vec{0}$.
Then, $A\vec{x}$ is in the image of A and in the kernel of A^T. Since
$\ker(A^T)$ is the orthogonal complement of $\operatorname{im}(A)$ by the previous
theorem, the vector $A\vec{x}$ is $\vec{0}$, [NOTES#5.1], that is, $\vec{x} \in \ker(A)$.

b. Note that $A^T A$ is an $(n \times n)$ matrix. By part (a), $\ker(A^T A) = \{\vec{0}\}$,
and $A^T A$ is therefore invertible. [NOTES#3.3]

Orthogonal Projections

Theorem

*Consider a vector $\vec{x} \in \mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$. Then, the orthogonal projection $\mathrm{proj}_V(\vec{x})$ is the vector in V **closest** to $\vec{x}$, in that*

$$\|\vec{x} - \mathrm{proj}_V(\vec{x})\| < \|\vec{x} - \vec{v}\|, \ \forall \vec{v} \in V \setminus \mathrm{proj}_V(\vec{x}).$$

As usual $\vec{x}^{\|} \equiv \mathrm{proj}_V(\vec{x})$, and $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\|}$ is the orthogonal "left-over" of $\vec{x}$ after the projection. The distance $\|\vec{x}^{\perp}\|$ is the shortest distance from V to $\vec{x}$.

If we move, in V, a distance ϵ away from $\vec{x}^{\|}$, the distance from that point to $\vec{x}$ is $\sqrt{\epsilon^2 + \|\vec{x}^{\perp}\|^2}$. [PYTHAGOREAN THEOREM].

The Error, or Residual

Consider a linear system $A\vec{x} = \vec{b}$, which is inconsistent; meaning that $\vec{b} \notin \text{im}(A)$.

An inconsistent linear system *does not have a solution* (in the traditional sense).

However, we can find the $\vec{x}^*$ which is the best candidate in that it minimizes the distance between $A\vec{x}^*$ and $\vec{b}$ (even though that distance is not zero).

We measure that distance

$$\|A\vec{x} - \vec{b}\| \equiv \|\vec{b} - A\vec{x}\|$$

and call it the *error*, or *residual*.

Least-Squares Solution

Definition (Least-Squares Solution)

Consider a linear system

$$A\vec{x} = \vec{b},$$

where A is an $(m \times n)$ matrix. A vector $\vec{x}^* \in \mathbb{R}^n$ is called a
least-squares solution of this system if

$$\|\vec{b} - A\vec{x}^*\| \leq \|\vec{b} - A\vec{x}\|, \ \forall \vec{x} \in \mathbb{R}^n.$$

The name *least-squares solution* comes from the fact that we a
minimizing the sum-of-squares norm (length) of the error vector
$\vec{e} = \vec{b} - A\vec{x}$.

If/When the system $A\vec{x} = \vec{b}$ is consistent the least-squares solution
is the exact solution, and $\|\vec{b} - A\vec{x}^*\| = 0$.

Finding Least-Squares Solutions

How do we hunt down this wild beast?!

- We want the least-squares solutions $\vec{x}^*$ to $A\vec{x} = \vec{b}$

- By definition we are looking for
  - $\|\vec{b} - A\vec{x}^*\| \leq \|\vec{b} - A\vec{x}\|, \ \forall \vec{x} \in \mathbb{R}^n.$

- Our projection theorem says:
  - $A\vec{x}^* = \text{proj}_V(\vec{b})$, where $V = \text{im}(A)$.

- So, the error is in the orthogonal complement of $\text{im}(A)$:
  - $\vec{b} - A\vec{x}^* \in V^\perp = (\text{im}(A))^\perp = \ker(A^T).$

- Which means:
  - $A^T(\vec{b} - A\vec{x}^*) = 0 \Leftrightarrow A^T A\vec{x} = A^T \vec{b}.$

Finding Least-Squares Solutions   The Normal Equations

Theorem (The Normal Equations)

*The least-squares solutions of the system $A\vec{x} = \vec{b}$, are the exact solutions of the (consistent) system $A^T A \vec{x} = A^T \vec{b}$. The system $A^T A \vec{x} = A^T \vec{b}$ is called the **normal equations** of $A\vec{x} = \vec{b}$.*

The case where $\ker(A) = \{\vec{0}\}$ is of particular importance, since in that case the matrix $A^T A$ is invertible, and we can give a closed form expression for the solution:

Closed Form Least Square Solutions

Theorem (Closed Form Expression for the Least Squares Solution)

If $\ker(A) = \{\vec{0}\}$*, the linear system* $A\vec{x} = \vec{b}$ *has the unique least-squares solution*

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b},$$

and

$$A\vec{x}^* = \text{proj}_{\text{im}(A)}(\vec{b}) = \underbrace{A(A^T A)^{-1} A^T}_{P} \vec{b},$$

where the matrix $P = A(A^T A)^{-1} A^T$ *is the matrix of the orthogonal projection onto* $\text{im}(A)$.

Note: Just because you can write down a mathematical expression, it does
not mean using it for anything practical is a good idea.

A BIG Warning!

WARNING

Whereas the least-squares solution, and orthogonal projection CAN be expressed as

$$(A^T A)^{-1} A^T \vec{b}, \text{ and } A(A^T A)^{-1} A^T \vec{b}, \text{ respectively.}$$

Anyone using these expressions outside of small homework problems are likely to run into **Big Trouble**!!!

We do not have the tools (eigenvalues) to explain why yet, but the warning stands!

So... What Should One Do?

Well, recall the Gram-Schmidt Process, and the QR-factorization...
If we have computed $QR = A$, then the following is true:

THE SOLUTION			
	$A\vec{x}$	$=$	$\vec{b}$
	$QR\vec{x}$	$=$	$\vec{b}$
multiply by Q^T	$Q^T QR\vec{x}$	$=$	$Q^T \vec{b}$
$Q^T Q = I_n$	$R\vec{x}$	$=$	$Q^T \vec{b}$
solve	$\vec{x}^*$	$=$	$\mathbf{R^{-1}Q^T\tilde{b}}$

THE PROJECTION			
	$QR\vec{x}^*$	$=$	$QRR^{-1}Q^T \vec{b}$
	$QR\vec{x}^*$	$=$	$\mathbf{QQ^T\tilde{b}}$

			use	not
$\vec{x}^*$	$=$		$R^{-1}Q^T \vec{b}$	$(A^T A)^{-1} A^T \vec{b}$
$\mathrm{proj}_{\mathrm{im}(A)}(\vec{b})$	$=$		$QQ^T \vec{b}$	$A(A^T A)^{-1} A^T \vec{b}$

More Examples and Discussion???

It makes sense to return to the Least-Squares solutions with more tools (eigenvalues) in hand; but, alas, we will run out of time this semester.

Some additional examples and discussion can be found in [AVAILABLE ONLINE]:

Class	Notes#
Math 541[R.I.P.]	10, 11
Math 524	6
Math 543	8, 14
Math 693a	22, 23, 24

Clearly, there's a lot more to say...