Math 254: Introduction to Linear Algebra

Notes #6.1 — Determinants :: Executive Summary

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Outline

SLOs 6.1 — Determinants

After this lecture you should:

- Know that a Square Matrix has a Non-Zero Determinant if and only if it is Invertible
- Be familiar with the Connection between the determinant of a (3×3) matrix and the Cross Product (especially for Engineering / Physics students)
- Be able to compute the determinant using
  - Laplace (co-factor) Expansion Method [“TRADITIONAL” WAY].
  - Row Reductions,
- Know the Impact of Row Divisions/Swaps/Additions on the value of the Determinant
- Be familiar with computation of the Determinant of Products, Powers, Transposes, and Inverses of matrices: $\det(AB)$, $\det(A^k)$, $\det(A^T)$, and $\det(A^{-1})$
- Forget about Cramer’s Rule: Don’t Use It! (Only for use in Joe Mahaffy’s Math 237/337 class)

Determinants — Modern Goals

"Almost all linear algebra books use determinants to prove that every linear operator on a finite-dimensional complex vector space has an eigenvalue. **Determinants are difficult, non-intuitive, and often defined without motivation.** *To prove the theorem about existence of eigenvalues on complex vector spaces, most books must define determinants, prove that a linear map is not invertible if and only if its determinant equals 0, and then define the characteristic polynomial. This tortuous (torturous?) path gives students little feeling for why eigenvalues exist."*

— Sheldon Axler, "Linear Algebra Done Right."

"Further, the computation of determinants (from the given definitions) for non-tiny matrices are frequently computationally intractable, even on modern computers. Therefore, with diminishing theoretical and practical need, we seek to de-emphasize matrix determinants in this class."

— Peter Blomgren, "Notes 6.1"

Determinants — Modern Goals

We look at the minimum "essentials" / "executive summary" of determinants; mostly because the "traditional" view is prevalent, and you are likely to encounter small-matrix determinants in other classes.

Comment

For small matrices we will [NOTES#7.2] use the determinant to find the *characteristic polynomial*, which will give us the *eigenvalues*. For non-small matrices, the *minimal polynomial* (which also reveals the eigenvalues) can be identified without the use of determinants.

Much of the material has been "banished" to the supplements, which make for an interesting read on a dark-and-stormy night.

Determinants: Introduction

(Time Travel to [Notes#2.4])

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if [Notes#2.4]

$$\det(A) = ad - bc \neq 0.$$

The quantity $(ad - bc)$ is called the *determinant* of the matrix A.

? It is natural to ask: *Can we assign a number $\det(A)$ to any square matrix A, such that A is invertible if-and-only-if $\det(A) \neq 0$?*

To our euphoric joy, the answer is "yes!"

The Determinant of a (3×3) Matrix

First, we "upsize" to the (3×3) case; let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$$

The matrix is not invertible if the three column vectors are contained in a same plane $\Leftrightarrow$ they are linearly dependent.

In the (3×3) case it is popular to express the determinant in terms of the *cross product*...

The cross product can also be defined for (7×7) matrices [SEE SUPPLEMENTS]. For other n we can define something "cross-product like" (*e.g.* the exterior, or "wedge", product).

Slight Detour: The Cross Product in $\mathbb{R}^3$ — Definition

Definition (Cross Product in $\mathbb{R}^3$)

The cross product $(\vec{a} \times \vec{b})$ for $\vec{a}, \vec{b} \in \mathbb{R}^3$ is the vector in $\mathbb{R}^3$ with the following properties:

- $(\vec{a} \times \vec{b})$ is orthogonal to both $\vec{a}$ and $\vec{b}$.
- $\|\vec{a} \times \vec{b}\| = \sin(\theta)\,\|\vec{a}\|\,\|\vec{b}\|$; θ is the angle between $\vec{a}$ and $\vec{b}$; $\theta \in [0, \pi]$.*
- The direction of $(\vec{a} \times \vec{b})$ follows the *right-hand-rule*

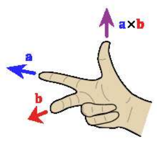

Figure: The right-hand-rule.

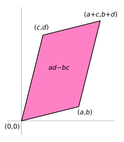

Figure: *This means that $\|\vec{a} \times \vec{b}\|$ is the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.

Slight Detour: The Cross Product in $\mathbb{R}^3$ — Properties

Theorem (Properties of the Cross Product)

The following equations hold $\forall \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, *and* $\forall k \in \mathbb{R}$:

- **a.** $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$. *(anti-commutative)*
- **b.** $(k\vec{v}) \times \vec{w} = k(\vec{v} \times \vec{w}) = \vec{v} \times (k\vec{w})$.
- **c.** $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$
- **d.** $\vec{v} \times \vec{w} = \vec{0}$ *if and only if* $\vec{v} \parallel \vec{w}$.
- **e.** $\vec{v} \times \vec{v} = \vec{0}$.
- **f.** $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$, $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$, $\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$. *(right-hand-rule)*
- **g.** *the Jacobi Identity:* *(this is the property that "makes" a cross product)*
  $\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = 0$

Note: Associativity is "missing."

Slight Detour: The Cross Product in $\mathbb{R}^3$ — Dot, Cross, and Outer Product

- The Dot ("inner") Product, $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$
  - $\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$ is a scalar
  - $\vec{a} \cdot \vec{b} = \cos\theta \, \|\vec{a}\| \, \|\vec{b}\|$
- The Cross Product, $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$
  - $\vec{a} \times \vec{b}$ is a vector $\perp \text{span}\left(\vec{a}, \vec{b}\right)$
  - $\|\vec{a} \times \vec{b}\| = \sin(\theta) \, \|\vec{a}\| \, \|\vec{b}\|$
- The Outer Product, $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$
  - $\vec{a}\vec{b}^T = M$ is a matrix, $M \in \mathbb{R}^{n \times n}$
  - $m_{ij} = a_i b_j$

θ is the angle between $\vec{a}$ and $\vec{b}$; $\theta \in [0, \pi]$.

$\exists$ Movie: `Cross_Dot.mpeg`

Slight Detour: The Cross Product in $\mathbb{R}^3$ — Computation

Theorem (The Cross Product in Components)

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

OK, back to determinants...

The (3×3) Determinant via the Cross Product

In the context of (3×3) matrices, we can compute

$$\det\left(\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}\right) = \vec{u} \cdot (\vec{v} \times \vec{w})$$

When $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then they “live” in the same plane; and $(\vec{v} \times \vec{w})$ is orthogonal to that plane, so the determinant is zero (as desired).

When $\vec{u}, \vec{v}, \vec{w}$ are linearly independent, then $(\vec{v} \times \vec{w})$ is NOT orthogonal to $\vec{u}$, and the determinant is non-zero.

Formula Overload!

So ponder:

$$
\begin{aligned}
\det(A) &= \vec{u} \cdot (\vec{v} \times \vec{w}) \\
&= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \cdot \left(\begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \times \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \right) \\
&= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \cdot \begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} \\ a_{32}a_{13} - a_{12}a_{33} \\ a_{12}a_{23} - a_{22}a_{13} \end{bmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{21}(a_{32}a_{13} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \\
&= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
&\quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\end{aligned}
$$

Committing the above to memory is not a good use of brain-power.

The Road Forward How to Compute $\det(A)$ $\mathbb{R}^{1\times1}$ and $\mathbb{R}^{2\times2}$ Cases

$\mathbb{R}^{1\times1}$: The matrix (and its associated linear transformation $T : \mathbb{R}^1 \mapsto \mathbb{R}^1$)

$$A = \begin{bmatrix} a_{11} \end{bmatrix}$$

is invertible if and only if

$$\det(A) = a_{11} \neq 0.$$

$\mathbb{R}^{2\times2}$: The matrix (and $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$)

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is invertible if and only if [Notes#2.4]

$$\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

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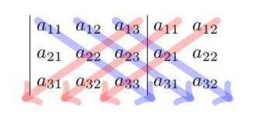
Sarrus' Rule — How to Compute $\det(A)$ — Only for $\mathbb{R}^{3\times 3}$

Theorem (Sarrus' Rule, for $\mathbb{R}^{3\times 3}$ only)

To find the determinant of a (3×3) matrix A, write the first 2 columns of A to the right of A, then multiply entries along the six diagonals shown:



Add the (blue / right-down) and subtract the (red / left-down) products, to get

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Unfortunately, Sarrus' Rule **does not generalize*** to $(n \times n)$ matrices $(n \geq 4)$.

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* This tends to lead to lost points on midterms and finals...

The Road Forward — How to Compute $\det(A)$ — $\mathbb{R}^{n\times n}$, $n \geq 4$

Revisiting the (3×3) case, we recall the result of Sarrus' formula

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

3 (of the 12) multiplications can be "saved" by writing

$$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{21}(a_{13}a_{32} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

We can recognize this as

$$\det(A) = a_{11}\det\left(\begin{bmatrix} * & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix}\right) - a_{21}\det\left(\begin{bmatrix} & a_{12} & a_{13} \\ * & & \\ & a_{32} & a_{33} \end{bmatrix}\right) + a_{31}\det\left(\begin{bmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ * & & \end{bmatrix}\right)$$

$$= a_{11}\det\left(\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}\right) - a_{21}\det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}\right) + a_{31}\det\left(\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}\right)$$

We introduce a bit of notation and language...

Matrix Minors — How to Compute $\det(A)$ — $\mathbb{R}^{n\times n}$, $n \geq 4$

Definition (Minors)

For an $(n \times n)$ matrix A, let A_{ij} be the matrix obtained by omitting the i^{th}row, and j^{th}column of A. The determinant of this $((n-1) \times (n-1))$ matrix A_{ij} is called a *minor* of A.

With this language, the determinant of the (3×3) matrix A:

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21}) + a_{31}\det(A_{31}).$$

This is known as the *Laplace expansion*, or *co-factor expansion* of $\det(A)$ down the first column.

We generalize this common strategy...

Laplace (co-factor) Expansion — $\mathcal{O}(n!)$ Work

Theorem (Laplace (co-factor) Expansion)

We can compute the determinant of an $(n \times n)$ matrix A by Laplace expansion down **any column**, *or along any row:*

- *Expansion down the j^{th} column:*

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j}\, a_{ij} \det(A_{ij}).$$

- *Expansion along the i^{th} row:*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij} \det(A_{ij}).$$

Laplace (co-factor) Expansion — How to Compute $\det(A)$ — $\mathbb{R}^{n\times n}$, $n \geq 4$

Best Practice: *Select the row/column with the MOST zeros:*

$$\det\left(\begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 0 & 3 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 0 & 1 & 3 \end{bmatrix}\right) = (-1)\det\left(\begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 4 \\ 3 & 1 & 3 \end{bmatrix}\right) + (-2)\det\left(\begin{bmatrix} 1 & 2 & 5 \\ 1 & 3 & 4 \\ 3 & 1 & 3 \end{bmatrix}\right)$$

$(-1)^{(1+2)} \cdot 1$

$(-1)^{(3+2)} \cdot 2$

Call Uncle Sarrus!

$$(-1)\Big(+ \text{mult}(1,2,3) + \text{mult}(3,4,3) + \text{mult}(4,2,1) - \text{mult}(1,4,1) - \text{mult}(3,2,2) - \text{mult}(4,2,3)\Big)$$

$$(-2)\Big(+ \text{mult}(1,3,3) + \text{mult}(2,4,3) + \text{mult}(5,1,1) - \text{mult}(1,4,1) - \text{mult}(2,1,3) - \text{mult}(5,3,3)\Big)$$

$$= (-1)(6+36+8-4-18-24) + (-2)(9+24+5-4-6-45) = (-1)(4) + (-2)(-17) = 30$$

In general: An $(n \times n)$ determinant is computed using n $((n-1) \times (n-1))$ determinants... The means the work growns as n-factorial.

$\mathbb{R}^{5\times 5}$ Number Crunching Bonanza! The Madness Begins... [SUPPLEMENT]

For no particular reaso, we do the Laplace co-factor expansion down the 3rd column...

$$\det\left(\begin{bmatrix}1 & 6 & 11 & 16 & 21\\ 2 & 7 & 12 & 17 & 22\\ 3 & 8 & 13 & 18 & 23\\ 4 & 9 & 14 & 19 & 24\\ 5 & 10 & 15 & 20 & 25\end{bmatrix}\right) = (+11)\det\left(\begin{bmatrix}2 & 7 & 17 & 22\\ 3 & 8 & 18 & 23\\ 4 & 9 & 19 & 24\\ 5 & 10 & 20 & 25\end{bmatrix}\right) + (-12)\det\left(\begin{bmatrix}1 & 6 & 16 & 21\\ 3 & 8 & 18 & 23\\ 4 & 9 & 19 & 24\\ 5 & 10 & 20 & 25\end{bmatrix}\right)$$

$$+(+13)\det\left(\begin{bmatrix}1 & 6 & 16 & 21\\ 2 & 7 & 17 & 22\\ 4 & 9 & 19 & 24\\ 5 & 10 & 20 & 25\end{bmatrix}\right) + (-14)\det\left(\begin{bmatrix}1 & 6 & 16 & 21\\ 2 & 7 & 17 & 22\\ 3 & 8 & 18 & 23\\ 5 & 10 & 20 & 25\end{bmatrix}\right) + (+15)\det\left(\begin{bmatrix}1 & 6 & 16 & 21\\ 2 & 7 & 17 & 22\\ 3 & 8 & 18 & 23\\ 4 & 9 & 19 & 24\end{bmatrix}\right)$$

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The 1st $\mathbb{R}^{4\times4}$ Determinant — "2nd Generation Problem" [SUPPLEMENT]

Let's just go down the 1st column...

$$\det\left(\begin{bmatrix} 2 & 7 & 17 & 22 \\ 3 & 8 & 18 & 23 \\ 4 & 9 & 19 & 24 \\ 5 & 10 & 20 & 25 \end{bmatrix}\right) = (+2)\det\left(\begin{bmatrix} 8 & 18 & 23 \\ 9 & 19 & 24 \\ 10 & 20 & 25 \end{bmatrix}\right) + (-3)\det\left(\begin{bmatrix} 7 & 17 & 22 \\ 9 & 19 & 24 \\ 10 & 20 & 25 \end{bmatrix}\right)$$

$$+(+4)\det\left(\begin{bmatrix} 7 & 17 & 22 \\ 8 & 18 & 23 \\ 10 & 20 & 25 \end{bmatrix}\right) + (-5)\det\left(\begin{bmatrix} 7 & 17 & 22 \\ 8 & 18 & 23 \\ 9 & 19 & 24 \end{bmatrix}\right)$$

$$= (+2)\big((8\cdot19\cdot25) + (18\cdot24\cdot10) + (23\cdot9\cdot20) - (8\cdot24\cdot20) - (18\cdot9\cdot25) - (23\cdot19\cdot10)\big)$$

$$+(-3)\big((7\cdot19\cdot25) + (17\cdot24\cdot10) + (22\cdot9\cdot20) - (7\cdot24\cdot20) - (17\cdot9\cdot25) - (22\cdot19\cdot10)\big)$$

$$+(+4)\big((7\cdot18\cdot25) + (17\cdot23\cdot10) + (22\cdot8\cdot20) - (7\cdot23\cdot20) - (17\cdot8\cdot25) - (22\cdot18\cdot10)\big)$$

$$+(-5)\big((7\cdot18\cdot24) + (17\cdot23\cdot9) + (22\cdot8\cdot19) - (7\cdot23\cdot19) - (17\cdot8\cdot24) - (22\cdot18\cdot9)\big)$$

$$= (+2)\cdot0 + (-3)\cdot0 + (+4)\cdot0 + (-5)\cdot0 = \mathbf{0}$$

The 2nd $\mathbb{R}^{4\times 4}$ Determinant "2nd Generation Problem" [SUPPLEMENT]

If we grind through the 4 remaining $\mathbb{R}^{4\times 4}$ determinants, we get 4 more zeros! That is massive amount of integer algebra to get 0 zeros...

$$\det\left(\begin{bmatrix}1 & 6 & 16 & 21\\ 3 & 8 & 18 & 23\\ 4 & 9 & 19 & 24\\ 5 & 10 & 20 & 25\end{bmatrix}\right) = 0, \quad \det\left(\begin{bmatrix}1 & 6 & 16 & 21\\ 2 & 7 & 17 & 22\\ 4 & 9 & 19 & 24\\ 5 & 10 & 20 & 25\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}1 & 6 & 16 & 21\\ 2 & 7 & 17 & 22\\ 3 & 8 & 18 & 23\\ 5 & 10 & 20 & 25\end{bmatrix}\right) = 0, \quad \det\left(\begin{bmatrix}1 & 6 & 16 & 21\\ 2 & 7 & 17 & 22\\ 3 & 8 & 18 & 23\\ 4 & 9 & 19 & 24\end{bmatrix}\right) = 0.$$

Therefore,

$$\det\left(\begin{bmatrix}1 & 6 & 11 & 16 & 21\\ 2 & 7 & 12 & 17 & 22\\ 3 & 8 & 13 & 18 & 23\\ 4 & 9 & 14 & 19 & 24\\ 5 & 10 & 15 & 20 & 25\end{bmatrix}\right) = 0$$

Triangular Matrices

[Simplifying Cases]

Example (Upper/Lower Triangular Matrix)

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

Then $\det(A) = a_{11}a_{22}a_{33}$, and $\det(B) = b_{11}b_{22}b_{33}$

Theorem (Determinant of a Triangular Matrix)

The determinant of a triangular matrix is the product of the diagonal entries of the matrix.

In particular, the determinant of a diagonal matrix is the product of its diagonal entries.

Row-Reductions and Determinants

Our three fundamental row-operations are:

1. *Row division:* Dividing a row by a non-zero scalar k.
2. *Row swap:* swapping two rows.
3. *Row addition:* adding (subtracting) a multiple of one row to another.

It is natural to ask how these operations change the value of the determinant.

Row-Reductions and Determinants: The (2×2) Case

First, consider the (2×2) case $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $\det(A) = ad - bc$:

1. *Row division:* If $B = \begin{bmatrix} a/k & b/k \\ c & d \end{bmatrix}$, then
$\det(B) = ad/k - bc/k = \det(A)/k$.
⇝ *Scaling of the Determinant*

Row-Reductions and Determinants: The (2×2) Case

First, consider the (2×2) case $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $\det(A) = ad - bc$:

1. *Row division:* If $B = \begin{bmatrix} a/k & b/k \\ c & d \end{bmatrix}$, then $\det(B) = ad/k - bc/k = \det(A)/k$.
   ⇝ *Scaling of the Determinant*
2. *Row swap:* If $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$, then $\det(B) = cb - da = -\det(A)$.
   ⇝ *Sign Change of the Determinant*

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Row-Reductions and Determinants: The (2×2) Case

First, consider the (2×2) case $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $\det(A) = ad - bc$:

1. *Row division:* If $B = \begin{bmatrix} a/k & b/k \\ c & d \end{bmatrix}$, then $\det(B) = ad/k - bc/k = \det(A)/k$.
   ⇝ *Scaling of the Determinant*
2. *Row swap:* If $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$, then $\det(B) = cb - da = -\det(A)$.
   ⇝ *Sign Change of the Determinant*
3. *Row addition:* If $B = \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$, then $\det(B) = (a+kc)d - (b+kd)c = (ad-bc) + k(cd-dc) = \det(A)$.
   ⇝ *No Change to the Determinant*

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Row-Reductions and Determinants — Properties for all $(n \times n)$-matrices

Theorem (Elementary Row Operations and Determinants)

a. *If B is obtained from A by dividing a row of A by a scalar k, then*

$$\det(B) = \frac{1}{k}\det(A)$$

b. *If B is obtained from A by a row swap, then*

$$\det(B) = -\det(A)$$

c. *If B is obtained from A by adding a multiple of a row of A to another row, then*

$$\det(B) = \det(A)$$

Analogous results hold for elementary column operations.

Row-Reductions and Determinants

Relating $\det(\mathrm{rref}(A))$ and $\det(A)$

Now, if we *in the process* of computing the reduced-row-echelon-form of a matrix A

- count the number of row-swaps: s, and
- keep track of scalar divisions $k_1, \ldots, k_r$.

then:

$$\det(\mathrm{rref}(A)) = (-1)^s \frac{1}{k_1 k_2 \ldots k_r} \det(A),$$

or

$$\det(A) = (-1)^s k_1 k_2 \ldots k_r \det(\mathrm{rref}(A))$$

This can save *a lot* of work for large matrics; Computing $\mathrm{rref}(A)$ requires $\sim \mathcal{O}\left(n^3\right)$ work, which is a grows a lot slower than $\mathcal{O}\left(n!\right)$.

n^3 Growth vs. $n!$ Growth

Computing $\det(A)$ for $A \in \mathbb{R}^{n\times n}$

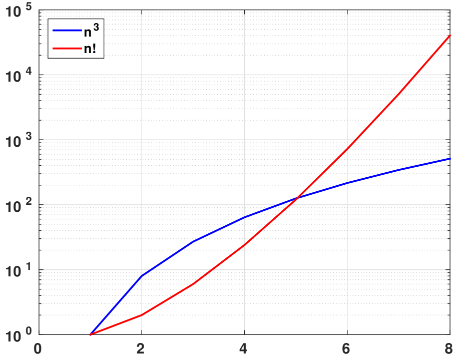

Figure: For small problems ($n \leq 4$), it is faster to use the Laplace co-factor expansion, but for large ($n \geq 6$) problems using row-reductions is faster. This creates a very real and unfortunate disconnect between typical homework/test problems, and real-world size problems!

Gauss-Jordan Elimination and the Determinant

If instead of computing $\det(\text{rref}(A))$, we perform elementary row operations on A to transform it into some matrix B, where $\det(B)$ is easy to compute; the same rules apply; if we performed s row swaps, and scaled rows by the factors $k_1, \dots, k_r$, then

$$\det(A) = (-1)^s k_1 k_2 \dots k_r \det(B)$$

Transforming A into upper triangular form U is a popular choice, since

$$\det(U) = \prod_{k=1}^{n} u_{kk}.$$

This approach saves about half the work vs. computing $\det(\text{rref}(A))$. **This is the clever thing to do!**

$\mathbb{R}^{5\times5}$ Number Crunching Bonanza — Redux! Row-reductions... [SUPPLEMENT]

Once again, we compute:

$$\det\left(\begin{bmatrix} 1 & 6 & 11 & 16 & 21 \\ 2 & 7 & 12 & 17 & 22 \\ 3 & 8 & 13 & 18 & 23 \\ 4 & 9 & 14 & 19 & 24 \\ 5 & 10 & 15 & 20 & 25 \end{bmatrix}\right)$$

First, we eliminate the first column:

$$\begin{bmatrix} 1 & 6 & 11 & 16 & 21 \\ 0 & -5 & -10 & -15 & -20 \\ 0 & -10 & -20 & -30 & -40 \\ 0 & -15 & -30 & -45 & -60 \\ 0 & -20 & -40 & -60 & -80 \end{bmatrix}$$

We divide the rows by $(-5, -10, -15, -20)$; and eliminate down the 2nd column...

$$\begin{bmatrix} 1 & 6 & 11 & 16 & 21 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 6 & 11 & 16 & 21 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbb{R}^{5\times5}$ Number Crunching Bonanza — Redux! Row-reductions... [SUPPLEMENT]

Ok, we have used row-reductions to transform A into upper triangular form U:

$$U = \begin{bmatrix} 1 & 6 & 11 & 16 & 21 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\det(A) = (-1)^{s=0}(-5)(-10)(-15)(-20)\det(U),$$

$(s = 0)$ since we did not swap any rows; the row-division factors are back as multiplications; and $\det(U)$ is just the product of the diagonal entries

$$\det(U) = 1 \cdot 1 \cdot 0 \cdot 0 \cdot 0 = 0...$$

and again it follows that $\det(A) = 0$.

Determinant of the Transpose

Theorem (Determinant of the Transpose)

If A is a square matrix, then

$$\det\left(A^T\right) = \det(A).$$

- This means that any property expressed in terms of columns/rows is also true for rows/columns;

Linearity of the Determinant in the Columns (Rows) [Focus :: Math]

Theorem (Linearity of the Determinant in the Columns)

Consider fixed column vectors

$$\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n \in \mathbb{R}^n.$$

Then the function $T : \mathbb{R}^n \mapsto \mathbb{R}$ *defined by*

$$T(\vec{x}) = \det\left(\begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_{i-1} & \vec{x} & \vec{v}_{i+1} & \ldots & \vec{v}_n \end{bmatrix}\right)$$

is a linear transformation.

Proof

We can convince ourselves that the theorem is indeed true...

Linearity of the Determinant in the Columns [Focus :: Math]

"Proof"

- Compute the determinant using the Laplace co-factor expansion down the "function" column; we get a n $((n-1)\times(n-1))$ determinants, each multiplied by a scalar factor $(x_k + y_k)$; just split into one x-expression, and one y-expression.

We express $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$, and $T(k\vec{x}) = kT(\vec{x})$:

$$\det\left(\begin{bmatrix}\vec{v}_1 & \dots & \vec{x}+\vec{y} & \dots & \vec{v}_n\end{bmatrix}\right) = \det\left(\begin{bmatrix}\vec{v}_1 & \dots & \vec{x} & \dots & \vec{v}_n\end{bmatrix}\right) + \det\left(\begin{bmatrix}\vec{v}_1 & \dots & \vec{y} & \dots & \vec{v}_n\end{bmatrix}\right)$$

$$\det\left(\begin{bmatrix}\vec{v}_1 & \dots & k\vec{x} & \dots & \vec{v}_n\end{bmatrix}\right) = k\det\left(\begin{bmatrix}\vec{v}_1 & \dots & \vec{x} & \dots & \vec{v}_n\end{bmatrix}\right)$$

The formal proof that scaled column/row addition does not change the deterimant follows from this property...

Invertibility and Determinant

If A is invertible, then $\mathrm{rref}(A) = I_n$, so that $\det(\mathrm{rref}(A)) = \det(I_n) = 1$, and

$$\det(A) = (-1)^s k_1 k_2 \dots k_r \neq 0.$$

If A is non-invertible, then the last row of $\mathrm{rref}(A)$ is all zeros, and by linearity $\det(\mathrm{rref}(A)) = 0$; so that $\det(A) = 0$.

Theorem (Invertibility and Determinant)

A square matrix A is invertible if and only if $\det(A) \neq 0$.

! **Yes!** *we can assign a number $\det(A)$ to any square matrix A, such that A is invertible if and only if $\det(A) \neq 0$!*

Determinants of Products and Powers

Theorem (Determinants of Products and Powers)

If A and B are $(n \times n)$ matrices, and $m \in \mathbb{Z}^+$ is a positive integer, then

a. $\det(AB) = (\det(A))\,(\det(B))$, *and*

b. $\det(A^m) = (\det(A))^m$.

Proof

[PROOF IN THE SUPPLEMENTS]

Determinants of Similar Matrices

Example (Similar Matrices)

Consider two similar matrices A, B; where S is an invertible matrix so that

$$AS = SB.$$

The previous theorems then implies that

$$\det(A)\,\det(S) = \det(S)\,\det(B),$$

so that $\det(A) = \det(B)$.

Determinants of Similar Matrices

Theorem (Determinants of Similar Matrices)

If a matrix A is similar to B, then $\det(A) = \det(B)$.

Theorem (Determinant of an Inverse)

If a matrix A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof: *{Short: relies on fundamental properties/definitions}.*

Since $I_n = A^{-1}A$, $1 = \det(I_n) = \det(A^{-1})\det(A)$. □

Cramer's Rule — The Second Worst Idea in Linear Algebra

Theorem (Cramer's Rule)

Consider the linear system

$$A\vec{x} = \vec{b}$$

where A is an invertible $(n \times n)$ system. The components x_i of the solution vector $\vec{x}$ are

$$x_i = \frac{\det(A_{\vec{b},i})}{\det(A)},$$

where $A_{\vec{b},i}$ is the matrix obtained by replacing the i^{th} column of A by $\vec{b}$.

Peter's Postulate

Solving linear systems using Cramer's Rule is a BAD IDEA. — We need to compute $(n+1)$ determinants of size $(n \times n)$.

The Worst Idea in Linear Algebra...

Rewind (Definition: Minors)

For an $(n \times n)$ matrix A, let A_{ij} be the matrix obtained by omitting the i^{th}row, and j^{th}column of A. The determinant of the $((n-1) \times (n-1))$ matrix A_{ij} is called a *minor* of A.

Now:

Definition (The Classical Adjoint)

The classical adjoint $M = \text{adj}(A)$ of an invertible $(n \times n)$ matrix, is the matrix whose ij^{th} entry $m_{ij} = (-1)^{i+j} \det(A_{ji})$. Yes, we have to compute n^2 $((n-1) \times (n-1))$ determinants to build the adjoint! With "only" one more $(n \times n)$ determinant, we can express the inverse:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$

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Suggested Problems 6.1, 6.2, 6.3

Available on Learning Glass videos:
6.1 — 1, 5, 11, 13, 23, 25, 27, 31, 43, 45
6.2 — 1, 5, 7, 9, 11, 12, 13, 15
6.3 — 1, 3, 5, 7, 9, 11, 13, 19 - 20 - 21

The structure of the lecture has changed greatly (Fall 2018–Spring 2019), but most of the problems are still doable $\sim$ just substitute either the Laplace co-factor expansion, or row-reductions to upper triangular form where necessary.

For the area/volume problems in 6.3, is is helpful to review the 5 supplemental slides on *"Geometric Interpretation: n-Dimensional Expansion Factor;"* however, it is sufficient to interpret the determinant of a (3×3) matrix as the volume of a parallelepiped defined by the 3 columns of the matrix. The supplemental slides also define the 2-volume in $\mathbb{R}^4$ (6.3.13).

Examples...

1. Compute det(A), where $A \in \mathbb{R}^{n \times n}$; $n = 1, 2, 3$; or $\{\, n > 3 :$ with "structure" $\}$ (lots of zeros, or some other greatly simplifying characteristic)
   - using row-reductions
   - using Laplace co-factor expansion
   - using appropriate "short-cut rule(s)"
2. Given matrices A, B compute det(A), det(B), det(AB), det(A^{4006}), det(B^{-1}), det($A^T B^{-1} A B^T$); etc...
3. Given the value of det(A) is the matrix invertible?
4. Given a matrix A, with det(A); how does det(A) change under row/column operations

Lecture – Book Roadmap

Lecture	Book, [GS5–]
6.1*	§5.1, §5.2, §5.3
6.2	§5.1, §5.2, §5.3
6.3	§5.1, §5.2, §5.3

* Strang does not talk about the combinatorial (pattern) definition of the determinant.

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

Outline

(6.1.1), (6.1.5)

(6.1.1) Is the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

invertible?

(6.1.5) Is the matrix

$$A = \begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

invertible?

(6.1.11), (6.1.13)

(6.1.11) For which values of k is the matrix

$$A = \begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix}$$

invertible?

(6.1.13) For which values of k is the matrix

$$A = \begin{bmatrix} k & 3 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

invertible?

(6.1.23), (6.1.25)

(6.1.23) Use the determinant to find out for which values of λ the matrix $(A - \lambda I_n)$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$$

is invertible?

(6.1.25) Use the determinant to find out for which values of λ the matrix $(A - \lambda I_n)$

$$A = \begin{bmatrix} 4 & 2 \\ 4 & 6 \end{bmatrix}$$

is invertible?

(6.1.27), (6.1.31)

(6.1.27) Use the determinant to find out for which values of λ the matrix $(A - \lambda I_n)$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 7 & 6 & 4 \end{bmatrix}$$

is invertible?

(6.1.31) Find the determinant of

$$A = \begin{bmatrix} 1 & 9 & 8 & 7 \\ 0 & 2 & 9 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(6.1.43), (6.1.45)

(6.1.43) If A is an $(n \times n)$ matrix, what is the relationship between

$$\det(A) \quad \text{and} \quad \det(-A)?$$

(6.1.45) If A is an (2×2) matrix, what is the relationship between

$$\det(A) \quad \text{and} \quad \det(A^T)?$$

(6.2.1), (6.2.5)

(6.2.1) Use Gaussian Elimination (Row Reductions) to find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{bmatrix}$$

(6.2.5) Use Gaussian Elimination (Row Reductions) to find the determinant of the matrix

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

(6.2.7), (6.2.9)

Use Gaussian Elimination (Row Reductions) to find the determinants of the matrices

$$\textbf{(6.2.7)}\ A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}, \quad \textbf{(6.2.9)}\ B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}.$$

(6.2.11), (6.2.12), (6.2.13), (6.2.15)

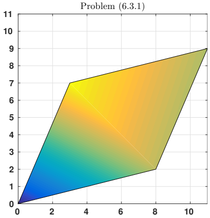
Consider a (4×4) matrix A with rows $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$. If $\det(A) = 8$, what is:

(6.2.11) $\det\left(\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ -9\,\vec{v}_3 \\ \vec{v}_4 \end{bmatrix}\right)$, **(6.2.12)** $\det\left(\begin{bmatrix} \vec{v}_4 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \end{bmatrix}\right)$,

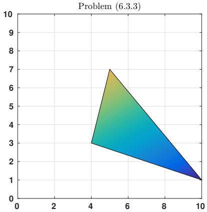
(6.2.13) $\det\left(\begin{bmatrix} \vec{v}_2 \\ \vec{v}_3 \\ \vec{v}_1 \\ \vec{v}_4 \end{bmatrix}\right)$, and **(6.2.15)** $\det\left(\begin{bmatrix} \vec{v}_1 \\ \vec{v}_1 + \vec{v}_2 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 \\ \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4 \end{bmatrix}\right)$

(6.3.1), (6.3.3)

(6.3.1) Find the area of the parallelogram defined by $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$.



(6.3.3) Find the area of the triangle with corners in $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 10 \\ 1 \end{bmatrix}$.



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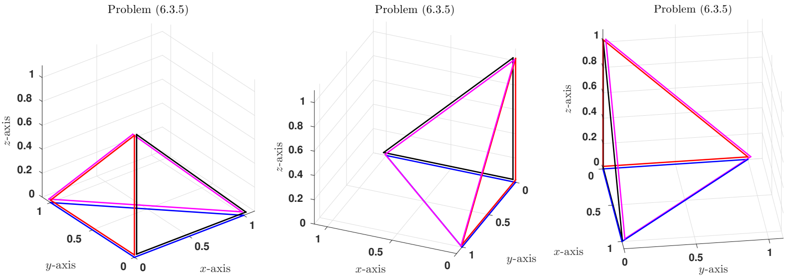
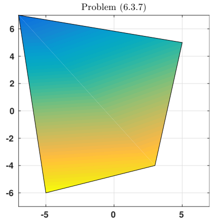
(6.3.5)



(6.3.5) The tetrahedron defined by three vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3 \in \mathbb{R}^3$ is the set of all vectors of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, where $c_i \geq 0$, and $c_1 + c_2 + c_3 \leq 1$. Explain why the volume is one sixth of the volume of the parallelepiped defined by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

(6.3.7), (6.3.9), (6.3.11)

(6.3.7) Find the area of the region with corners in $\begin{bmatrix}-7\\7\end{bmatrix}$, $\begin{bmatrix}-5\\-6\end{bmatrix}$, $\begin{bmatrix}3\\-4\end{bmatrix}$, and $\begin{bmatrix}5\\5\end{bmatrix}$.



(6.3.9) If $\vec{v}_1$ and $\vec{v}_2$ are linearly independent vectors in $\mathbb{R}^2$, what is the relationship between $\det([\vec{v}_1 \;\; \vec{v}_2])$ and $\det([\vec{v}_1 \;\; \vec{v}_2^{\perp}])$, where $\vec{v}_2^{\perp}$ is the component of $\vec{v}_2$ orthogonal to $\vec{v}_1$.

(6.3.11) Consider a linear transformation $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Suppose for two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^2$ we have $T(\vec{v}_1) = 3\vec{v}_1$ and $T(\vec{v}_2) = 4\vec{v}_2$. What can you say about $\det(A)$? Explain in detail.

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(6.3.13), (6.3.19), (6.3.20)

(6.3.13) Find the 2-volume (aka "area") of the 2-parallelepiped (parallelogram) defined by the two vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

(6.3.19) A basis $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ of $\mathbb{R}^3$ is called *positively oriented* if $\vec{v}_1$ encloses an acute angle with $(\vec{v}_2 \times \vec{v}_3)$. Illustrate with a sketch. Show that the basis is positively oriented if-and-only-if $\det([\vec{v}_1\ \vec{v}_2\ \vec{v}_3])$ is positive.

(6.3.20) We say that a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ *preserves orientation* if it transforms any positively oriented basis into another positively oriented basis. Explain why a linear transformation $T(\vec{x}) = A\vec{x}$ preserves orientation if-and-only-if $\det(A) > 0$.

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(6.3.21)

(6.3.21) Arguing geometrically, determine whether the following orthogonal transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ preserve, or reverse orientation:

- **a.** Reflection about a plane.
- **b.** Reflection about a line.
- **c.** Reflection about the origin.

The 7-Dimensional Cross Product — Crazy-Math-Interlude — [Focus :: Math]

This is not meant to be useful... it's just for "fun!"

The 7D cross product is a *bilinear* operation on vectors in seven-dimensional Euclidean space. For any $\vec{a}, \vec{b} \in \mathbb{R}^7$ it assigns a vector $\vec{v} = (\vec{a} \times \vec{b})$, $\vec{v} \in \mathbb{R}^7$.

Like the 3D cross product, the 7D cross product is *anticommutative* and $(\vec{a} \times \vec{b}) \perp \mathrm{span}(\vec{a}, \vec{b})$.

Unlike in three dimensions, it does not satisfy the *Jacobi identity,* and while the 3D cross product is unique up to a sign, there are many seven-dimensional cross products.

The 7D cross product has the same relationship to the *octonions* as the three-dimensional product does to the *quaternions* (a number system that extends to complex numbers).

This is not meant to be useful... it's just for "fun!"

The 7D cross product is one way of generalizing the cross product to other than 3D, and it is the only other non-trivial bilinear product of two vectors that is
(i) vector-valued,
(ii) anticommutative, and
(iii) orthogonal.

In other dimensions there are vector-valued products of three or more vectors that satisfy these conditions, and binary products with bivector results.

This is not meant to be useful... but it may be

Binary Operation: *(MATHEMATICS) a function of two variables. Examples* $+$, $-$, $*$, $/$...

Bilinear Operation: *a function which combines two arguments, and is linear in each of its arguments (when the other argument is kept fixed). Old examples: Matrix multiplication, Dot product.*

Anticommutative Operation: *a function of two arguments; which changes sign of the order of the arguments is reversed.* $f(x, y) = -f(y, x)$. *Old example: subtraction* $(a - b) = -(b - a)$.

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This is not meant to be useful... but it may be

Jacobi Identity: *a property of a binary operation which describes how the order of evaluation (the placement of parentheses in a multiple product) affects the result of the operation:*

A binary operation $\circ$ on a set S possessing a binary operation $+$ with an additive identity denoted by 0 satisfies the Jacobi identity if:

$$a \circ (b \circ c) \; + \; b \circ (c \circ a) \; + \; c \circ (a \circ b) \; = \; 0 \quad \forall \; a, b, c \in S.$$

By contrast, for operations with the associative property, any order of evaluation gives the same result (parentheses in a multiple product are not needed).

Named after the German mathematician Carl Gustav Jakob Jacobi.

This is not meant to be useful... but it may be

Real Numbers: *basis* — $\{1\}$, *with multiplication table:*

$\times$	1
1	1

Complex Numbers: *basis* — $\{1, i\}$, *with multiplication table:*

$\times$	1	i
1	1	i
i	i	-1

Quarternions: *basis* — $\{1, i, j, k\}$, *with multiplication table:*

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

This is not meant to be useful... it's just for "fun!"

Octonions: *basis* — $\{1, i, j, k, \ell, m, n, o\}$, *with multiplication table:*

$\times$	1	i	j	k	ℓ	m	n	o
1	1	i	j	k	ℓ	m	n	o
i	i	-1	k	$-j$	m	$-\ell$	$-o$	n
j	j	$-k$	-1	i	n	o	$-\ell$	$-m$
k	k	j	$-i$	-1	o	$-n$	m	$-\ell$
ℓ	ℓ	$-m$	$-n$	$-o$	-1	i	j	k
m	m	ℓ	$-o$	n	$-i$	-1	$-k$	j
n	n	o	ℓ	$-m$	$-j$	k	-1	$-i$
o	o	$-n$	m	ℓ	$-k$	$-j$	i	-1

This is “our” 3D Cross product, expressed in the “basis” notation...

3D Cross Product: *basis* — $\{i, j, k\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, *with multiplication table:*

$\times$	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$
$\vec{e}_1$	0	$\vec{e}_3$	$-\vec{e}_2$
$\vec{e}_2$	$-\vec{e}_3$	0	$\vec{e}_1$
$\vec{e}_3$	$\vec{e}_2$	$-\vec{e}_1$	0

This is not meant to be useful... it's just for "fun!"

7D Cross Product: *basis —*
$\{i, j, k, \ell, m, n, o\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5, \vec{e}_6, \vec{e}_7\}$, *with multiplication table:*

$\times$	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$	$\vec{e}_4$	$\vec{e}_5$	$\vec{e}_6$	$\vec{e}_7$
$\vec{e}_1$	0	$\vec{e}_3$	$-\vec{e}_2$	$\vec{e}_5$	$-\vec{e}_4$	$-\vec{e}_7$	$\vec{e}_6$
$\vec{e}_2$	$-\vec{e}_3$	0	$\vec{e}_1$	$\vec{e}_6$	$\vec{e}_7$	$-\vec{e}_4$	$-\vec{e}_5$
$\vec{e}_3$	$\vec{e}_2$	$-\vec{e}_1$	0	$\vec{e}_7$	$-\vec{e}_6$	$\vec{e}_5$	$-\vec{e}_4$
$\vec{e}_4$	$-\vec{e}_5$	$-\vec{e}_6$	$-\vec{e}_7$	0	$\vec{e}_1$	$\vec{e}_2$	$\vec{e}_3$
$\vec{e}_5$	$\vec{e}_4$	$-\vec{e}_7$	$\vec{e}_6$	$-\vec{e}_1$	0	$-\vec{e}_3$	$\vec{e}_2$
$\vec{e}_6$	$\vec{e}_7$	$\vec{e}_4$	$-\vec{e}_5$	$-\vec{e}_2$	$\vec{e}_3$	0	$-\vec{e}_1$
$\vec{e}_7$	$-\vec{e}_6$	$\vec{e}_5$	$\vec{e}_4$	$-\vec{e}_3$	$-\vec{e}_2$	$\vec{e}_1$	0

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Determinants of Products and Powers

[Focus :: Math]

Theorem (Determinants of Products and Powers)

If A and B are $(n \times n)$ matrices, and $m \in \mathbb{Z}^+$ is a positive integer, then

a. $\det(AB) = (\det(A))\,(\det(B))$, *and*

b. $\det(A^m) = (\det(A))^m$.

Proof

Proof in the supplements...

[Proof] Determinants of Matrix Products, and Powers

(a.)

(i) First we assume A is invertible: The row-operations required to transform A to I_n applied to the augmented system $\left[\ A \mid A\,B\ \right]$ gives:

$$\mathrm{rref}\left(\left[\ A \mid A\,B\ \right]\right) = \left[\ I_n \mid I_n\,B\ \right] = \left[\ I_n \mid B\ \right]$$

i.e. they are equivalent to multiplying both sides of the augmentation by A^{-1}.

Keeping track of the s row-swaps, and row divisions (scalings) $k_1, \ldots, k_r$ required to transform A into its RREF-form, we get
$\det(A) = (-1)^s k_1 k_2 \ldots k_r$, and

$$\det(AB) = (-1)^s k_1 k_2 \ldots k_r \, \det(B) = (\det(A))\,(\det(B)).$$

□

[Proof] Determinants of Matrix Products, and Powers

(a.)

(ii — When A is not invertible): If A is not invertible, then neither is AB (remember $(AB)^{-1} = B^{-1}A^{-1}$ which makes sense if and only if both A and B are invertible) so

$$(\det(A))\,(\det(B)) = 0 \cdot \det(B) = \det(AB).$$

□

(b.)

Apply part (a.) $(m-1)$ times to get:

$$\det(A^m) = \det(AA^{m-1}) = \det(A)\det(A^{m-1}) = \cdots = (\det(A))^m$$

□

Technique of Variation of Parameters

Consider the nonhomogeneous problem

$$y'' + p(t)y' + q(t)y = g(t),$$

where $p(t)$, $q(t)$, and $g(t)$ are given continuous functions.

Assume we know the **homogeneous solution** (for the case $g(t) = 0$): $y_h(t) = c_1 y_1(t) + c_2 y_2(t)$

We try a **general solution** of the form $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, where the functions $u_1(t)$ and $u_2(t)$ are to be determined

Differentiating gives:

$$y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t) + u_1'(t)y_1(t) + u_2'(t)y_2(t)$$

ODEs: Variation of Parameters [Focus :: Math]

We relate $u_1(t)$ and $u_2(t)$ (one degree of freedom), be setting

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0,$$

this simplifies the derivative of the general solution.

We use the fact that $y_1(t)$ and $y_2(t)$ satisfy the homogeneous equation... "(t)"):

$$y'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

$$y'' + py' + qy = \left[u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'\right] + p\left[u_1 y_1' + u_2 y_2'\right] + q\left[u_1 y_1 + u_2 y_2\right] = g$$

$$y'' + py' + qy = \left[u_1' y_1' + u_2' y_2'\right] + u_1 \underbrace{\left[y_1'' + py_1' + qy_1\right]}_{0} + u_2 \underbrace{\left[y_2'' + py_2' + qy_2\right]}_{0} = g$$

... and we are left with

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

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ODEs: Variation of Parameters Linear Algebra Connection! [Focus :: Math]

This gives two **linear algebraic equations** in u_1' and u_2'

$$\begin{aligned} u_1'(t)y_1(t) + u_2'(t)y_2(t) &= 0 \\ u_1'(t)y_1'(t) + u_2'(t)y_2'(t) &= g(t), \end{aligned}$$

or in matrix form

$$\underbrace{\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}}_{A(t)} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

As long as the columns of $A(t)$ are linearly independent, we get solutions [Notes#2.4]:

$$\begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \frac{1}{\det(A(t))} \begin{bmatrix} y_2'(t) & -y_2(t) \\ -y_1'(t) & y_1(t) \end{bmatrix} \begin{bmatrix} 0 \\ g(t) \end{bmatrix} = \frac{1}{\det(A(t))} \begin{bmatrix} -y_2(t)\,g(t) \\ y_1(t)\,g(t) \end{bmatrix}$$

ODEs: Variation of Parameters Determinant / "Wronskian" [Focus :: Math]

In the "ODE Universe," $\det(A(t)) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is usually referred to as the **"Wronskian"**, and sometimes denoted $W[y_1, y_2](t)$.

We can now integrate

$$u_1(t) = -\int^t \frac{y_2(\tau)g(\tau)}{\det(A(\tau))}\, d\tau + \mathcal{C}_1, \quad u_2(t) = \int^t \frac{y_1(\tau)g(\tau)}{\det(A(\tau))}\, d\tau + \mathcal{C}_2,$$

and the general solution is given by

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t);$$

where $u_1(t)$ and $u_2(t)$ be be given explicitly, or in integral form.

ODEs: Variation of Parameters — Theorem — [Focus :: Math]

Theorem (Variation of Parameters)

Consider the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t),$$

If the functions $p(t)$, $q(t)$, and $g(t)$ are continuous on an open interval $\mathcal{I} \subset \mathbb{R}^n$, and if y_1 and y_2 form a fundamental set of solutions of the homogeneous equation. Then a particular solution of the nonhomogeneous problem is

$$y_p(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\, ds,$$

where $t_0 \in I$. The **general solution** *is*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t).$$

ODEs: Variation of Parameters — Example — [Focus :: Math]

We consider the nonhomogeneous problem

$$y'' + 4y = 3\sin(t)$$

The homogeneous problem

$$y'' + 4y = 0$$

has two **linearly independent solutions**

$$y_1(t) = \cos(2t), \quad \text{and} \quad y_2(t) = \sin(2t).$$

Variation of Parameters suggests

$$y(t) = u_1(t)\cos(2t) + u_2(t)\sin(2t),$$

where the functions $u_1(t)$ and $u_2(t)$ are to be determined.

ODEs: Variation of Parameters — Example — [Focus :: Math]

Using the "recipe" we get

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}}_{A(t)} \begin{bmatrix} u_1'(t) \\ u_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 3\sin(t) \end{bmatrix}$$

The **Wronskian** satisfies:

$$W[\cos(2t), \sin(2t)](t) = \det(A(t)) = 2\cos^2(2t) + 2\sin^2(2t) \equiv 2$$

So that

$$u_1'(t) = \frac{-3\sin(t)\sin(2t)}{2}, \quad u_2'(t) = \frac{3\sin(t)\cos(2t)}{2}$$

The rest is "just" trig-identities and integration...

ODEs: Variation of Parameters — Example — [Focus :: Math]

Things we have forgotten:

$$\sin(2t) = 2\cos(t)\sin(t), \quad \cos(2t) = 2\cos^2(t) - 1$$

$$u_1'(t) = \frac{-3\sin(t)\sin(2t)}{2} = -3\cos(t)\sin^2(t)$$

$$u_2'(t) = \frac{3\sin(t)\cos(2t)}{2} = 3\cos^2(t)\sin(t) - \frac{3}{2}\sin(t)$$

Which gives us

$$u_1(t) = -\sin^3(t) + \mathcal{C}_1, \quad u_2(t) = \frac{3}{2}\cos(t) - \cos^3(t) + \mathcal{C}_2$$

ODEs: Variation of Parameters — Example — [Focus :: Math]

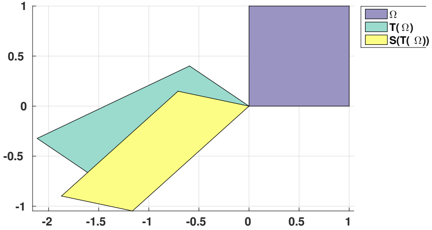
The General Solution is

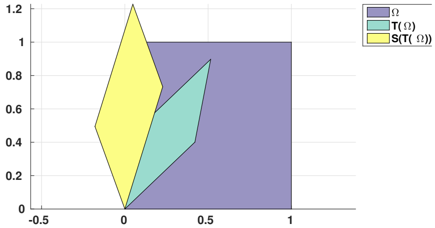
$$\begin{aligned} y(t) &= u_1(t)\cos(2t) + u_2(t)\sin(2t) \\ &= (-\sin^3(t) + \mathcal{C}_1)\cos(2t) + (\tfrac{3}{2}\cos(t) - \cos^3(t) + \mathcal{C}_2)\sin(2t) \\ &= -\sin^3(t)\cos(2t) + \tfrac{3}{2}\cos(t)\sin(2t) - \cos^3(t)\sin(2t) \\ &\quad + \mathcal{C}_1\cos(2t) + \mathcal{C}_2\sin(2t) \\ &= \sin(t) + \mathcal{C}_1\cos(2t) + \mathcal{C}_2\sin(2t) \end{aligned}$$

Somebody tried to convince me that Cramer's rule was a nice way to solve the linear (2×2) linear system... but, no, I still prefer the route taken in these notes.

Disclaimer: I'm sure I "lost" a minus-sign somewhere...

The Determinant as Expansion Factor





Consider a linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$. We have discussed how such a transform impacts lengths, and angles. For a transform $\mathbb{R}^2 \mapsto \mathbb{R}^2$ it also makes sense to think about the 2-volume (aka "The Area"); and for $\mathbb{R}^n \mapsto \mathbb{R}^n$ we can discuss the m-Volume(s).

The Determinant as Expansion Factor

We start in the $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ case, and let the "input area" (Ω) be the unit square (with area 1), described by the two vectors $\vec{e}_1$, and $\vec{e}_2$.

The "output area" ($T(\Omega)$), is then described by $A\vec{e}_1 = \vec{v}_1$, and $A\vec{e}_2 = \vec{v}_2$, *i.e.* the parallelepiped spanned by the columns of A; here the area is $|\det(A)|$.

The expansion factor is

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)|}{1} = |\det(A)|.$$

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The Determinant as Expansion Factor

If the input parallelepiped is described by two vectors $\vec{w}_1$, and $\vec{w}_2$, then the original area is $|\det(B)|$, where $B = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 \end{bmatrix}$.

The "output area" $(T(\Omega))$, is then described by $A\vec{w}_1 = \vec{v}_1$, and $A\vec{w}_2 = \vec{v}_2$; so the area of $T(\Omega)$ is given by

$$|\det(\begin{bmatrix} A\vec{w}_1 & A\vec{w}_2 \end{bmatrix})| = |\det(AB)| = |\det(A)|\,|\det(B)|.$$

The expansion factor is

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)|\,|\det(B)|}{|\det(B)|} = |\det(A)|.$$

The Determinant as Expansion Factor

Theorem (Expansion Factor)

Consider a linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ *defined by* $T(\vec{x}) = A\vec{x}$*. Then* $|\det(A)|$ *is the expansion factor*

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega}$$

of T *on parallelograms* Ω*.*

Likewise, for linear transformations $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ *defined by* $T(\vec{x}) = A\vec{x}$*,* $|\det(A)|$ *is the expansion factor of* T *on* n*-parallelepipeds:*

$$V(A\vec{v}_1, \ldots, A\vec{v}_n) = |\det(A)|\, V(\vec{v}_1, \ldots, \vec{v}_n),$$

for all vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$*.*

The Determinant as Expansion Factor

Stating the "Obvious"

Since $\det(AB) = \det(A)\det(B)$, which of course holds for $B = A^{-1}$, the expansion factors satisfy:

$$|\det(AB)| = |\det(A)|\,|\det(B)|,$$

and

$$|\det(A^{-1})| = \frac{1}{|\det(A)|}.$$

Stating the "Less Obvious"

It is possible to show that the expansion factor associated with a linear transformation $T(\vec{x}) = A\vec{x}$ (as we have defined it) holds for *any* region Ω (not just parallelograms or n-parallelepipeds).

A "Fun" Definition of Determinants

The following discussion introduces the

- "Pattern," or
- "Combinatorial" ← USEFUL FOR INTERNET SEARCHING...

definition of matrix determinants.

The Determinant of an $(n \times n)$ Matrix — "Patterns"

For an $(n \times n)$ matrix A:

- Let a **pattern**, P, of A be a subset (vector) containing n entries a_{ij} selected from the matrix A so that we have exactly one entry from each row and column.
  (There are $n(n-1)(n-2)\cdots 1 = n!$ "n-factorial" such patterns.)
- Let $\text{prod}(P)$ be the product of all entries in a pattern P.
- An *inversion* in a pattern occurs when an entry is to the right, and above another; *e.g.* if the pattern contains a_{12}, and a_{31} we have an inversion.
- Let $\text{sgn}(P)$ be the signature of the pattern, defined as $(-1)^{\#\text{inversions of } P}$.

What the @@@#%&***???!!! Let's go back and apply this to the (3×3) case.

Note: Fun with Factorials — They Grow **FAST!**

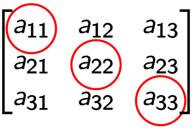
n	n!
1	1
2	2
3	6
4	24
5	120
6	720
7	5,040
8	40,320
9	362,880
10	3,628,800

Bottom line: there are *lots* of patterns for (not very) large matrices.

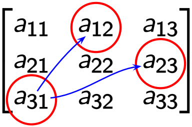
Patterns, Inversions, Products, and Signatures in the (3×3) case...

We have established that

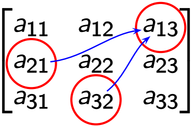
$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$



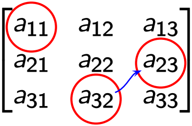
No inversions.



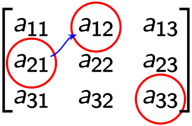
2 Inversions:
$a_{31} : a_{12}$, $a_{31} : a_{23}$



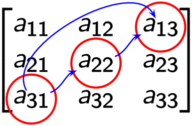
2 Inversions:
$a_{21} : a_{13}$, $a_{32} : a_{13}$



1 Inversion:
$a_{32} : a_{23}$



1 Inversion:
$a_{21} : a_{12}$



3 Inversions:
$a_{31} : a_{22}$, $a_{31} : a_{13}$, $a_{22} : a_{31}$

The Determinant of an $(n \times n)$ Matrix

We can now define the determinant of an $(n \times n)$ matrix using its associated patterns (and the products, inversions and signatures of those patterns):

Definition (The Determinant of an $(n \times n)$ Matrix)

$$\det(A) = \sum_{\text{All } n! \text{ patterns of } A} \operatorname{sgn}(P)\,\operatorname{prod}(P)$$

Is This Computationally Useful? .... Collecting Examples

All of what we have stated is true, but a big chunk of it is not really computationally practical for general matrices.

However, for matrices with lots of zeros *("Sparse Matrices")* this approach can be time-saving, since any pattern containing a zero will not contribute to the determinant.

Further, most properties (which we will discuss) of determinants can be traced back to the combinatorial definition; and some computational strategies leverage those properties.

Determinant of a Sparse (4×4) Matrix

The combinatorial "pattern" approach can be computationally efficient when there are lots of zeros in the matrix; patterns with a 0 will not contribute to the value of the determinant, so we can skip exploring those patterns...

We consider computing the determinant of

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

Patterns including $a_{31} = 1$**:**

$$\begin{bmatrix} x & 0 & 1 & 2 \\ x & 1 & 1 & 3 \\ \textcircled{1} & x & x & x \\ x & 1 & 0 & 0 \end{bmatrix}$$

Determinant of a Sparse (4×4) Matrix

[continued] Patterns including $a_{31} = 1$:

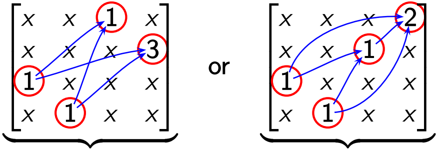
$$\underbrace{\begin{bmatrix} x & x & 1 & 2 \\ x & \textcircled{1} & x & x \\ \textcircled{1} & x & x & x \\ x & x & 0 & 0 \end{bmatrix}}_{P=\{a_{31}, a_{22}, \dots\}}, \quad \text{or} \quad \underbrace{\begin{bmatrix} x & x & 1 & 2 \\ x & x & 1 & 3 \\ \textcircled{1} & x & x & x \\ x & \textcircled{1} & x & x \end{bmatrix}}_{P=\{a_{31}, a_{42}, \dots\}}$$

We notice that for the left pattern, the next non-zero choice is $a_{13} = 1$, but that leaves $a_{44} = 0$ for the final selection; so this is a zero-end pattern. The right pattern can be competed in two ways:

$$\underbrace{\begin{bmatrix} x & x & \textcircled{1} & x \\ x & x & x & \textcircled{3} \\ \textcircled{1} & x & x & x \\ x & \textcircled{1} & x & x \end{bmatrix}}_{P=\{a_{31}, a_{42}, a_{13}, a_{24}\}} \quad \text{or} \quad \underbrace{\begin{bmatrix} x & x & x & \textcircled{2} \\ x & x & \textcircled{1} & x \\ \textcircled{1} & x & x & x \\ x & \textcircled{1} & x & x \end{bmatrix}}_{P=\{a_{31}, a_{42}, a_{23}, a_{14}\}}$$

Determinant of a Sparse (4×4) Matrix

[continued] Patterns including $a_{31} = 1$:



$P=\{a_{31},a_{42},a_{13},a_{24}\}$ $P=\{a_{31},a_{42},a_{23},a_{14}\}$

The pattern $P = \{a_{31}, a_{42}, a_{13}, a_{24}\}$ has 4 inversions, so $\mathrm{sgn}(P) = 1$, and $\mathrm{prod}(P) = 1 \cdot 1 \cdot 1 \cdot 3 = 3$. Contribution to $\det(A)$ :: $(+3)$.

The pattern $P = \{a_{31}, a_{42}, a_{23}, a_{14}\}$ has 5 inversions, so $\mathrm{sgn}(P) = -1$, and $\mathrm{prod}(P) = 1 \cdot 1 \cdot 1 \cdot 2 = 2$. Contribution to $\det(A)$ :: (-2).

Determinant of a Sparse (4×4) Matrix

Next, we have to consider **Patterns including $a_{41} = 2$:**

$$\begin{bmatrix} x & 0 & 1 & 2 \\ x & 1 & 1 & 3 \\ x & 0 & 0 & 0 \\ \textcircled{2} & x & x & x \end{bmatrix}$$

There's only 1 non-zero choice in the 2nd column; and then in third: which forces us to pick a zero in the 4th column...

$$\begin{bmatrix} x & x & 1 & 2 \\ x & \textcircled{1} & x & x \\ x & x & 0 & 0 \\ \textcircled{2} & x & x & x \end{bmatrix}, \quad \begin{bmatrix} x & x & \textcircled{1} & x \\ x & \textcircled{1} & x & x \\ x & x & x & 0 \\ \textcircled{2} & x & x & x \end{bmatrix}, \quad \begin{bmatrix} x & x & \textcircled{1} & x \\ x & \textcircled{1} & x & x \\ x & x & x & \textcircled{0} \\ \textcircled{2} & x & x & x \end{bmatrix}$$

Since the pattern $P = \{a_{41}, a_{22}, a_{13}, a_{34}\} = \{2, 1, 1, 0\}$ contains a zero, we have $\text{prod}(P) = 2 \cdot 1 \cdot 1 \cdot 0 = 0$, it does not contribute to $\det(A)$.

Determinant of a Sparse (4×4) Matrix

In summary, the determinant of

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}$$

is given by

$$\det(A) = \operatorname{sgn}(P_1)\operatorname{prod}(P_1) + \operatorname{sgn}(P_2)\operatorname{prod}(P_2) = (+1)(+3) + (-1)(+2) = 1$$

where

$$P_1 = \{a_{31}, a_{42}, a_{13}, a_{24}\} = \{1, 1, 1, 3\},$$
$$P_2 = \{a_{31}, a_{42}, a_{23}, a_{14}\} = \{1, 1, 1, 2\}$$

are the only 2 (out of 24) patterns with non-zero contributions.

Block Matrices

Example (The Determinant of a Block Matrix)

Find

$$\det(M) = \det\left(\begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & c_{21} & c_{22} \end{bmatrix}\right) = \det\left(\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\right)$$

There are only 4 patterns with non-zero products:

$$\begin{bmatrix} \circled{a_{11}} & a_{12} & b_{11} & b_{12} \\ a_{21} & \circled{a_{22}} & b_{21} & b_{22} \\ 0 & 0 & \circled{c_{11}} & c_{12} \\ 0 & 0 & c_{21} & \circled{c_{22}} \end{bmatrix}, \quad \begin{bmatrix} \circled{a_{11}} & a_{12} & b_{11} & b_{12} \\ a_{21} & \circled{a_{22}} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & \circled{c_{12}} \\ 0 & 0 & \circled{c_{21}} & c_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & \circled{a_{12}} & b_{11} & b_{12} \\ \circled{a_{21}} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & \circled{c_{11}} & c_{12} \\ 0 & 0 & c_{21} & \circled{c_{22}} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & \circled{a_{12}} & b_{11} & b_{12} \\ \circled{a_{21}} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & \circled{c_{12}} \\ 0 & 0 & \circled{c_{21}} & c_{22} \end{bmatrix}$$

Block Matrices

Now,

$$\begin{aligned}
\det(M) &= a_{11}a_{22}c_{11}c_{22} - a_{11}a_{22}c_{12}c_{21} - a_{12}a_{21}c_{11}c_{22} + a_{12}a_{21}c_{12}c_{21} \\
&= (a_{11}a_{22} - a_{12}a_{21})(c_{11}c_{22} - c_{12}c_{21}) \\
&= \det(A)\det(C).
\end{aligned}$$

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Block Matrices

Theorem (Determinant of a Block Matrix)

If A and C are square matrices (not necessarily of the same size), and B and 0 are matrices of appropriate size, then

$$\det\left(\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\right) = \det(A)\det(C), \quad \det\left(\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}\right) = \det(A)\det(C).$$

WARNING WARNING WARNING WARNING WARNING WARNING WARNING

The formula

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det(A)\det(D) - \det(B)\det(C)$$

does NOT always hold.

See https://en.wikipedia.org/wiki/Determinant#Block_matrices

Computational Feasibility of the Determinant

Say you are faced with computing the determinant of a (32×32) matrix (not very large by modern standards).

Using the "pattern" (or Laplace co-factor) method, such a computation would require $31 \cdot 32! \approx 8.16 \cdot 10^{36}$ multiplications.

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Now say you have access to an "exaflop computer" — which can perform 10^{18} operations / second; then your computation would only take about 10^{18} seconds... how long is that?

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Now say you have access to an "exaflop computer" — which can perform 10^{18} operations / second; then your computation would only take about 10^{18} seconds... how long is that?

- 100 seconds/minute $\rightsquigarrow 10^{16}$ minutes
- 100 minutes/hour $\rightsquigarrow 10^{14}$ hours
- 100 hours/day $\rightsquigarrow 10^{12}$ days
- 1000 days/year $\rightsquigarrow 10^{9}$ years.
- so... only about 7% of the age of the universe.

Perfect midterm question, yeah?!?

How Fast is 10^{18} operations/s?

As of November 2019, according to https://www.top500.org/, the fastest supercomputer in the world:

Site	Oak Ridge National Laboratory
System	Summit - IBM Power System AC922, IBM POWER9 22C 3.07GHz, NVIDIA Volta GV100, Dual-rail Mellanox EDR Infiniband
Cores	2,414,592
Rmax	148,600.0 TFlop/s
Rpeak	200,794.9 TFlop/s (0.201×10^{18} Flop/s)
Power	10,096 kW

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