Math 254: Introduction to Linear Algebra

Notes #7.1 — Eigenvalues and Eigenvectors: Diagonalization

Peter Blomgren
⟨blomgren@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

Spring 2022
(Revised: January 18, 2022)

Outline

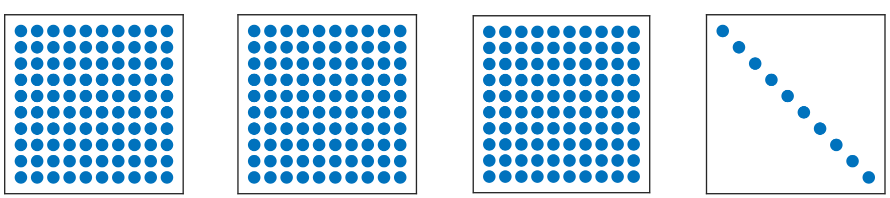
SLOs 7.1 Eigen-values and vectors: Diagonalization

After this lecture you should know how

- Matrix Diagonalization
- Similarity Transformation
- Eigenvalues, Eigenvectors, and Eigenbases

are inter-related.

S^{-1} $\quad A \quad$ S $\quad D$



Introduction "Intellectual Set-up of Eigenvalues and Eigenvectors Computations"

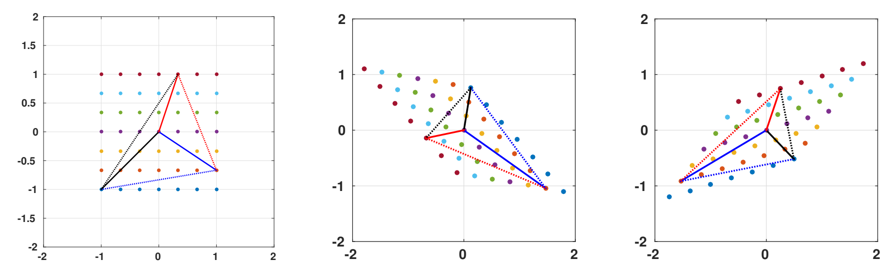


Figure: Consider linear transformations $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$. What do we know? — They rotate, reflect, and stretch our input space (LEFT PANEL) in various ways; two examples shown in the CENTER and RIGHT PANELS.

An **eigenvector**, $\vec{v}$, of a linear transformation is a vector whose orientation is preserved by the transformation, *i.e.* $\vec{v} \parallel A\vec{v}$, *i.e.*

$$A\vec{v} = \lambda\vec{v},$$

and the scalar λ is the **eigenvalue** associated with $\vec{v}$.

Why Should We Care About Eigenvalues? Applications

From a "pure" linear algebra perspective, operations on eigenvectors are easy, since they are just (multiplicative) scalings.

In applications the eigenvector-eigenvalue pair describe some fundamental property of a "system" (something we are using a mathematical model to describe):

- **Vibrations**, either in strings (guitars, pianos, etc) or other structures (bridges, tall buildings): the eigenvalue is the frequency, and the eigenvector is the deformation. [Tacoma Bridge]
  ((Link sponsored by "C+ Engineering LLC."))
- In statistics, **Principal Component Analysis**, is an eigenvector-eigenvalue analysis of the correlation matrix, and is used to study large data sets, such as those encountered in data mining, chemical research, psychology, and in marketing.
  Buzzwords: "Data Science," "Big Data," and "Page Rank."

Why Should We Care About Eigenvalues? Cooking

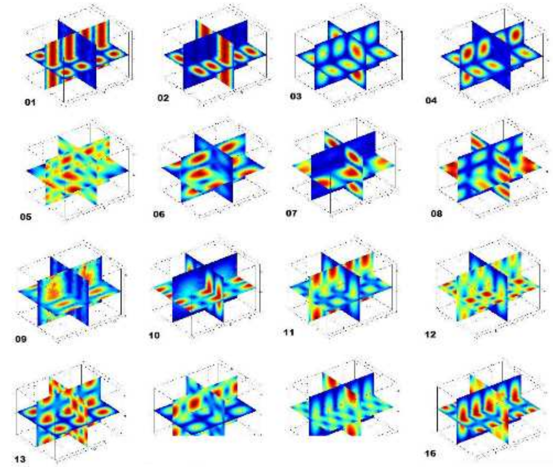


Figure: "Eigenvalue Analysis of Microwave Oven." *Haider, Siddique, Abbas, and Ahmed* — International Journal of Scientific & Engineering Research, Volume 4, Issue 9, September-2013, pp. 2473.

Why Should We Care About Eigenvalues? Combustion

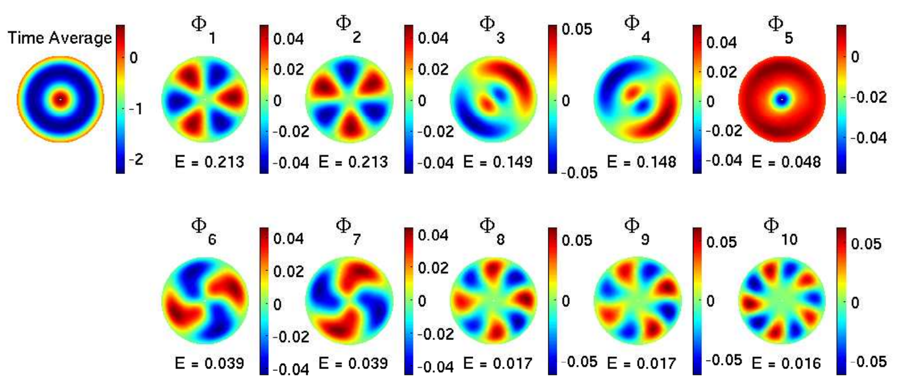
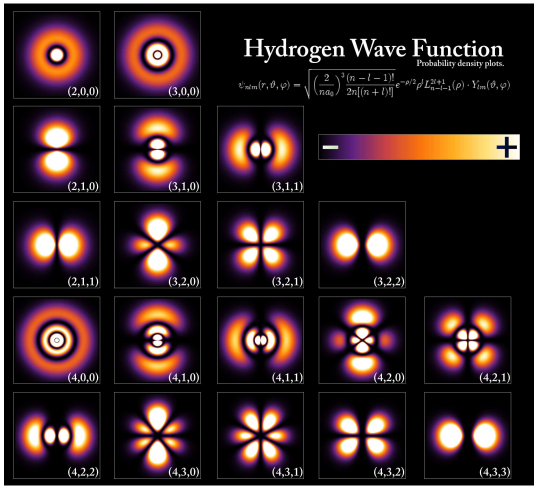


Figure: Space and time evolution of a three-cells hopping state found in simulations of the Kuramoto-Sivashinsky equation. The cells move non-uniformly and their shapes change periodically. Parameter values are: $\varepsilon = 0.32$, $\eta_1 = 1.0$, $\eta_2 = 0.013$, and $R = 7.7475$. — *Blomgren, Gasner, and Palacios*, "Hopping Behavior in the Kuramoto-Sivashinsky Equation." Chaos: An Interdisciplinary Journal of Nonlinear Science, volume 15; March 28, 2005.

Example: Energy Modes in the Hydrogen Wave Function



OK, we may not be quite ready to identify the energy states in the hydrogen atom... yet. Let us, momentarily, retreat to our safe Linear Algebra "universe."

Motivating Example

Given the matrices

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix}$$

ponder the "fun" of

1. computing:
   - A^5, $\text{rank}(A)$, $\det(A)$, the basis of $\ker(A)$, and the basis of $\text{im}(A)$
2. computing:
   - B^5, $\text{rank}(B)$, $\det(B)$, the basis of $\ker(B)$, and the basis of $\text{im}(B)$

Motivating Example (A)

For Matrix A we can write down the answers quickly:

$$\text{rank}(A) = 3, \quad \det(A) = 0$$

$$A^5 = \begin{bmatrix} (-1)^5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1^5 & 0 \\ 0 & 0 & 0 & 2^5 \end{bmatrix} = \begin{bmatrix} (-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 32 \end{bmatrix}$$

$$\ker(A) \in \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \text{im}(A) \in \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Motivating Example (B)

Now, for matrix B it's a "bit" more work... and maybe not immediately obvious that:

$$\text{rank}(B) = 2, \quad \det(B) = 0$$

$$B^5 = \begin{bmatrix} 412,928 & 413,184 & 413,440 & 413,696 \\ 1,052,928 & 1,053,184 & 1,053,440 & 1,053,696 \\ 1,187,072 & 1,186,816 & 1,186,560 & 1,186,304 \\ 547,072 & 546,816 & 546,560 & 546,304 \end{bmatrix}$$

$$\ker(B) \in \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad \text{im}(B) \in \text{span}\left\{ \begin{bmatrix} 1 \\ 5 \\ 9 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 8 \\ 4 \end{bmatrix} \right\}$$

We come to the realization that diagonal matrices are our friends!

Diagonalizable Matrices

Remember "Coordinates"?

Definition (Diagonalizable Matrices)

Consider a linear transformation $T(\vec{x}) = A\vec{x}$; $(T : \mathbb{R}^n \mapsto \mathbb{R}^n)$. Then A (and/or T) is said to be *diagonalizable* if the matrix B of T with respect to some basis, $\mathfrak{B}(\mathbb{R}^n)$ is diagonal. [MATH 524 NOTATION]: $B = \mathcal{M}(T, \mathfrak{B}(\mathbb{R}^n))$

By previous discussion [NOTES#3.4], the matrix A is diagonalizable if and only if it is similar to some diagonal matrix B; meaning that there exists some invertible matrix S, so that

$$S^{-1}AS = B \text{ is a diagonal matrix.}$$

Definition (Diagonalization of a Matrix)

To *diagonalize* a square matrix A means to find an invertible matrix S and a diagonal matrix B such that $S^{-1}AS = B$.

Rewind — [Notes#3.4] Standard Matrix vs. $\mathfrak{B}$-matrix ((Change of Basis))

Visualizing the Theorem:

$$\begin{tikzcd}
\text{standard coordinates} & \vec{x} \arrow[r, "A\vec{x}"] \arrow[d, shift left, "{[\vec{x}]_{\mathfrak{B}} = S^{-1}\vec{x}}"] & T(\vec{x}) \arrow[d, shift left, "{[T(\vec{x})]_{\mathfrak{B}} = S^{-1}T(\vec{x})}"] \\
\mathfrak{B}\text{-coordinates} & {[\vec{x}]_{\mathfrak{B}}} \arrow[r, "{B[\vec{x}]_{\mathfrak{B}}}"] \arrow[u, shift left, "{\vec{x} = S[\vec{x}]_{\mathfrak{B}}}"] & {[T(\vec{x})]_{\mathfrak{B}}} \arrow[u, shift left, "{T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}}}"]
\end{tikzcd}$$

$$S = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}, \quad \mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_n)$$

$$\vec{x} = S[\vec{x}]_{\mathfrak{B}}, \; S^{-1}\vec{x} = [\vec{x}]_{\mathfrak{B}}; \; T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}}, \; S^{-1}T(\vec{x}) = [T(\vec{x})]_{\mathfrak{B}}$$

Therefore

$$A\vec{x} = T(\vec{x}) = S[T(\vec{x})]_{\mathfrak{B}} = SB[\vec{x}]_{\mathfrak{B}} = SBS^{-1}\vec{x}$$

Eigenvectors, Eigenvalues, and Eigenbases

Definition (Eigenvectors, Eigenvalues, and Eigenbases)

Consider a linear transformation $T(\vec{x}) = A\vec{x}$; $(T : \mathbb{C}^n \mapsto \mathbb{C}^n)$. A **non-zero** vector $\vec{v} \in \mathbb{C}^n$ is called an **eigenvector** of A (and/or T) if

$$A\vec{v} = \lambda\vec{v},$$

for some $\lambda \in \mathbb{C}$. This λ is called the **eigenvalue** associated with the eigenvector $\vec{v}$.

A basis $\vec{v}_1, \dots, \vec{v}_n$ of $\mathbb{C}^n$ is called an **eigenbasis** for A (and/or T) if the vectors $\vec{v}_1, \dots, \vec{v}_n$ are eigenvectors of A, *i.e.*

$$A\vec{v}_k = \lambda_k\vec{v}_k, \quad k = 1, \dots, n$$

for some scalars $\lambda_1, \dots, \lambda_n$.

Repeated Multiplication by A

Example (Repeated Multiplication by A)

If $\vec{v}$ is an eigenvector of A, then $A\vec{v} = \lambda\vec{v}$; and

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda^2\vec{v}.$$

It does not take a lot of imagination to realize

$$A^k\vec{v} = \lambda^k\vec{v}$$

Example $A^k\vec{v}$ in $\mathbb{R}^2$

Example (Life in $\mathbb{R}^2$)

Assume we have an eigenbasis $\{\vec{v}_1, \vec{v}_2\}$, then any $\vec{w} \in \mathbb{R}^2$ can be written as $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2$ for unique scalars a_1 and a_2; now

$$A\vec{w} = A(a_1\vec{v}_1 + a_2\vec{v}_2) = a_1A\vec{v}_1 + a_2A\vec{v}_2 = a_1\lambda_1\vec{v}_1 + a_2\lambda_2\vec{v}_2$$

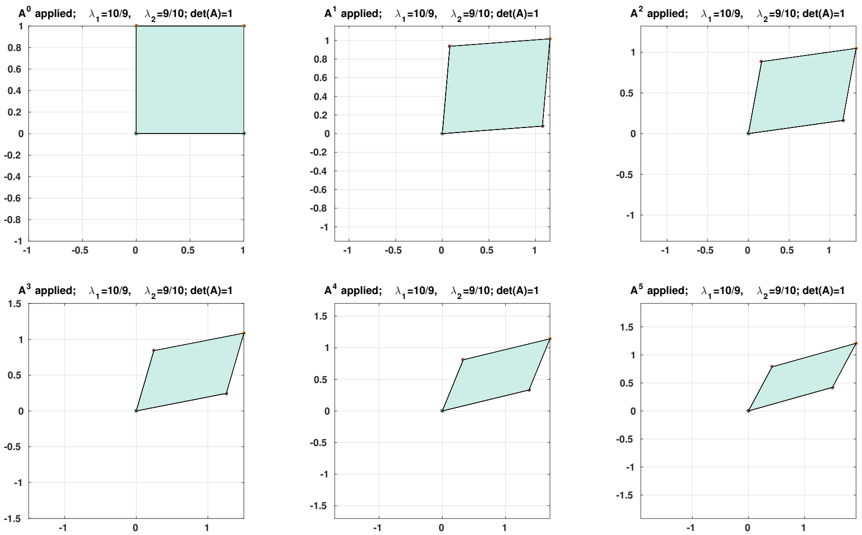
and

$$A^k\vec{w} = A^{k-1}(A(a_1\vec{v}_1 + a_2\vec{v}_2)) = A^{k-1}(a_1\lambda_1\vec{v}_1 + a_2\lambda_2\vec{v}_2) = a_1\lambda_1^k\vec{v}_1 + a_2\lambda_2^k\vec{v}_2$$

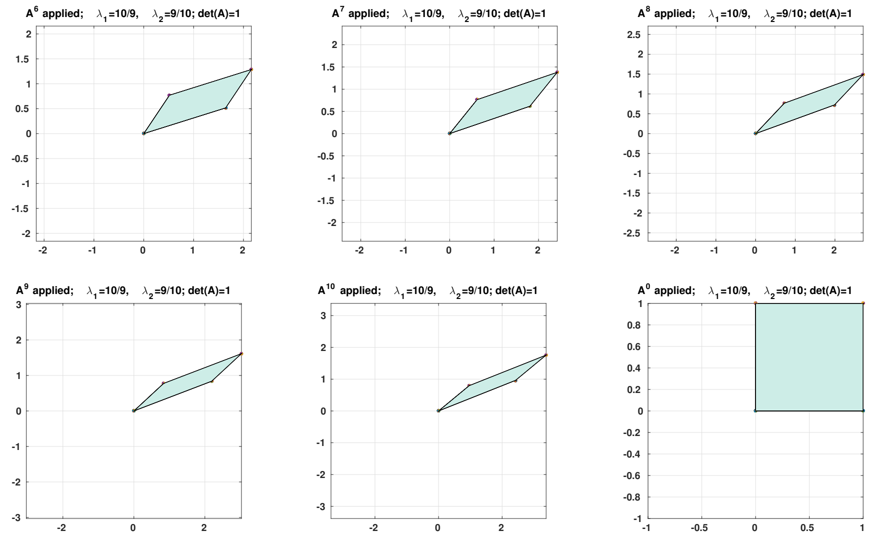
"Future-Proofing:"

$\mathbb{R}^n$: if $\vec{v}_1, \vec{v}_2$ are eigenvectors of A, and $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2$ we have $A^k\vec{w} = a_1\lambda_1^k\vec{v}_1 + a_2\lambda_2^k\vec{v}_2$. That is the "action" is the "same" as above when restricted to $V = \mathrm{span}(\vec{v}_1, \vec{v}_2)$.

Example $A^k\vec{v}$ in $\mathbb{R}^2$



Example $A^k\vec{v}$ in $\mathbb{R}^2$



Eigenbases and Diagonalization

Theorem (Eigenbases and Diagonalization)

The matrix A is diagonalizable if and only if there exists an eigenbasis for A. If $\vec{v}_1, \ldots, \vec{v}_n$ is an eigenbasis for A, with $A\vec{v}_1 = \lambda_1\vec{v}_1, \ldots, A\vec{v}_n = \lambda_n\vec{v}_n$, then the matrices

$$S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

will diagonalize A, meaning $S^{-1}AS = B$.

Conversely, if the matrices S and B diagonalize A, then the columns of S will form an eigenbasis for A, and the diagonal entries of B will be the associated eigenvalues.

Eigenvalues of the Identity Transformation

Example ($T(\vec{x}) = I_n\vec{x} = \vec{x}$)

Find all the eigenvalues and eigenvectors of the identity matrix I_n.

Solution: Since

$$I_n\vec{x} = 1\vec{x} \quad \forall \vec{x} \in \mathbb{R}^n,$$

it follows that *all* $\vec{x} \in \mathbb{R}^n$ are eigenvectors of I_n with associated eigenvalues 1.

Therefore, all bases of $\mathbb{R}^n$ are eigenbases for I_n, and clearly the already diagonal matrix I_n is diagonalizable. Any invertible S will do the trick: $S^{-1}I_nS = S^{-1}S = I_n$.

Eigenvalues of a Projection

Example (Projection: Setup)

Let $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, and consider the projection onto the line $L = \mathrm{span}\,\{\vec{w}\}$:

$$T(\vec{x}) = \mathrm{proj}_L(\vec{x}) = \left(\frac{\vec{x} \circ \vec{w}}{\|\vec{w}\|^2} \right) \vec{w} = P\vec{x}$$

where [NOTES#2.2] the projection matrix P is

$$P = \frac{1}{25} \begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}$$

Eigenvalues of a Projection

Example (Projection: Getting Started...)

RESTATING OLD RESULTS IN THIS NEW CONTEXT —

Any vector $\parallel$ to $L = \{k\vec{w},\ k \in \mathbb{R}\}$ will be projected onto itself (hence it is an eigenvector with eigenvalue 1); and any vector $\perp$ to $L = \{k\vec{w},\ k \in \mathbb{R}\}$ will be projected onto $\vec{0}$, so it is an eigenvector with eigenvalue 0;

$$P\vec{x}^{\parallel} = 1\vec{x}^{\parallel}, \quad P\vec{x}^{\perp} = 0\vec{x}^{\perp}$$

one eigenbasis is

$$\mathfrak{B} = (\vec{x}^{\parallel}, \vec{x}^{\perp}), \quad \text{where} \quad \vec{x}^{\parallel} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \vec{x}^{\perp} = \begin{bmatrix} -3 \\ 4 \end{bmatrix},$$

Eigenvalues of a Projection

Example (Projection: ...Moving Along)

The B-matrix (which expresses T in the $\mathfrak{B}$-basis) is

$$B = \mathcal{M}(T; \mathfrak{B}(\mathbb{R}^2)) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

Recall that the first column is the coefficients, $(1, 0)$ of $T(\vec{x}^{\parallel})$ in the basis $\mathfrak{B}(\mathbb{R}^2) = (\vec{x}^{\parallel}, \vec{x}^{\perp})$; and the second column is the coefficients $(0, 0)$ of $T(\vec{x}^{\perp})$.

The matrices B and

$$S = \left[\vec{x}^{\parallel} \quad \vec{x}^{\perp}\right] = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$$

diagonalize P.

Notation: $\mathcal{M}(T; \mathfrak{B}(\mathbb{R}^2))$ — "The matrix of T with respect to the basis $\mathfrak{B}$ of $\mathbb{R}^2$."
(Future-proofed for [MATH 524])

What About Rotations?

Example (Rotation by $\pi/2$ (90°))

Let $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $T(\vec{x}) = R\vec{x}$ be the rotation transformation.

Now, given any $\vec{x} \in \mathbb{R}^2$, $\vec{x}$ is not parallel to $R\vec{x}$.

As long as we insist on REAL eigenvectors and REAL eigenvalues, we find none...

Matrices with real entries may have *Complex eigenvalues*.

$\mathbb{C}$omplex: The (complex) eigenvalues of the matrix above are $0+1i$, and $0-1i$; where $i = \sqrt{-1}$.

Eigenvalues of Orthogonal Matrices

Example (Orthogonal $A \in \mathbb{R}^{n\times n}$)

Let A be an orthogonal matrix; then $T(\vec{x}) = A\vec{x}$ preserves length, so if/when $\vec{v}$ is an eigenvector

$$\|\vec{v}\| = \|A\vec{v}\| = \|\lambda\vec{v}\| = |\lambda|\,\|\vec{v}\|$$

therefore, we must have $|\lambda| = 1$.

Theorem

The only possible real eigenvalues of an orthogonal matrix are 1 and -1.

$\mathbb{C}$omplex: When a matrix as above has $|\lambda| = 1$, and λ is allowed to be complex; there are infinitely many possibilities $\lambda = e^{i\theta} = \cos\theta + i\sin\theta$. A length-preserving matrix with complex eigenvalues is usually called a *Unitary Matrix*; the *Orthogonal Matrices* are special cases of Unitary Matrices (with real eigenvalues).

Matrices with Eigenvalue 0

Core Property — Zero Eigenvalue

By definition 0 is an eigenvalue if and only if we can find a non-zero $\vec{x} \in \mathbb{R}^n$ so that $A\vec{x} = 0\vec{x} = \vec{0}$.

That means 0 is an eigenvalue of A if and only if $\ker(A) \neq \{\vec{0}\}$, *i.e.* A is non-invertible.

We add this to our list from [Notes#2.4], [Notes#3.1], and [Notes#3.3].

Characteristics of Invertible Matrices — Important!

Equivalent Statements: Invertible Matrices

For an $(n \times n)$ matrix A, the following statements are equivalent:

- **i.** A is invertible
- **ii.** The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, $\forall \vec{b} \in \mathbb{R}^n$
- **iii.** $\mathrm{rref}(A) = I_n$
- **iv.** $\mathrm{rank}(A) = n$
- **v.** $\mathrm{im}(A) = \mathbb{R}^n$
- **vi.** $\ker(A) = \{\vec{0}\}$
- **vii.** The column vectors of A form a basis of $\mathbb{R}^n$
- **viii.** The column vectors of A span $\mathbb{R}^n$
- **ix.** The column vectors of A are linearly independent
- **x.** $\det(A) \neq 0$.
- **xi.** 0 is not an eigenvalue of A.

Suggested Problems 7.1

Available on Learning Glass videos:

7.1 — 1, 3, 5, 7, 15, 17, 21

Lecture – Book Roadmap

Lecture	Book, [GS5–]
7.1	§6.1
7.2	§6.1, §6.2
7.3	§6.1, §6.2
7.5	§6.1, §6.2

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(7.1.1), (7.1.3), (7.1.5), (7.1.7)

(7.1.1) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix, and $\vec{v}$ an eigenvector of A, with associated eigenvalue λ. Is $\vec{v}$ an eigenvector of A^3? If so, what is the eigenvalue?

(7.1.3) Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix, and $\vec{v}$ an eigenvector of A, with associated eigenvalue λ. Is $\vec{v}$ an eigenvector of $A + 2I_n$? If so, what is the eigenvalue?

(7.1.5) If a vector $\vec{v}$ is an eigenvector of both $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$, is $\vec{v}$ necessarily an eigenvector of $A + B$?

(7.1.7) If $\vec{v}$ is an eigenvector of $A \in \mathbb{R}^{n\times n}$, with eigenvalue λ, what can you say about $\ker(A - \lambda I_N)$?

(7.1.15), (7.1.17), (7.1.21)

(7.1.15) Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformation — *Reflection about a line L in* $\mathbb{R}^2$ — find the eigenbasis (if possible) and determine whether the transformation is diagonalizable?

(7.1.17) Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformation — *Counterclockwise rotation through an angle of* 45° $(\pi/4)$ *followed by a scaling by 2 in* $\mathbb{R}^2$ — find the eigenbasis (if possible) and determine whether the transformation is diagonalizable?

(7.1.21) Arguing geometrically, find all eigenvectors and eigenvalues of the linear transformation — *Scaling by 5 in* $\mathbb{R}^3$ — find the eigenbasis (if possible) and determine whether the transformation is diagonalizable?

Definition, Complex Addition

Definition (Complex Numbers)

With $a, b \in \mathbb{R}$, we define the complex value $z \in \mathbb{C}$:

$$z = a + ib$$

where i is the imaginary unit $\sqrt{-1}$. a is the *Real Part* ($a = \text{Re}(z)$), and b the *Imaginary Part* ($b = \text{Im}(z)$) of z.

Definition (Complex Addition)

Let $z_1, z_2 \in \mathbb{C}$, then

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

Complex Multiplication

Definition (Complex Multiplication)

Let $z_1, z_2 \in \mathbb{C}$, then

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

this follows from the fact that $i^2 = -1$.

Note: $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$

Let $T : \mathbb{R}^2 \mapsto \mathbb{C}$ be the linear transformation:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + ib, \quad T^{-1}(a + ib) = \begin{bmatrix} a \\ b \end{bmatrix},$$

that is we can interpret vectors in $\mathbb{R}^2$ as complex numbers (and the other way around).

Multiplication by $i \rightsquigarrow$ Rotation

Example (Multiplication by i)

Consider $z = a + ib$, and let $a, b > 0$ so that the corresponding vector lives in the first quadrant.

z		$a + ib$
iz	$i(a + ib) = ia + i^2 b$	$-b + ia$
$i^2 z$	$i(-b + ia) = -ib + i^2 a$	$-a - ib$
$i^3 z$	$i(-a - ib) = -ia + i^2 b$	$b - ia$
$i^4 z$	$i(b - ia) = ib - i^2 a$	$a + ib$

We see that $z = -i^2 z = i^4 z$, and since

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} -b \\ a \end{bmatrix} = a(-b) + ba = 0, \quad \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} b \\ -a \end{bmatrix} = ab + b(-a) = 0$$

we can interpret multiplication by i as a ccw-rotation by $\pi/2$ (90°).

Complex numbers can solve our issue of "no real eigenvalues" for rotations!

Complex Conjugate

Definition (Complex Conjugate)

Given $z = (a + ib) \in \mathbb{C}$, the complex conjugate is defined by

$$\overline{z} = (a - ib), \quad \text{sometimes } z^* = (a - ib)$$

(reversing the sign on the imaginary part). Note that this is a reflection across the real axis in the complex plane.

Hey! It's a reflection across the real axis!

z and z^* form a *conjugate pair* of complex numbers, and

$$z\, z^* = (a + ib)(a - ib) = a^2 + b^2.$$

Polar Coordinate Representation

Polar Coordinate Representation (Modulus and Argument)

We can represent $z = a + ib$ in terms of its length r (*modulus*) and angle θ (*argument*); where

$$r = \mathrm{mod}(z) = |z| = \sqrt{a^2 + b^2}, \quad \theta = \mathrm{arg}(z) \in [0, 2\pi)$$

where

$$\theta = \arg(z) = \begin{cases} \arctan(\frac{b}{a}) & \text{if } a > 0 \\ \arctan(\frac{b}{a}) + \pi & \text{if } a < 0 \text{ and } b \geq 0 \\ \arctan(\frac{b}{a}) - \pi & \text{if } a < 0 \text{ and } b < 0 \\ \frac{\pi}{2} & \text{if } a = 0 \text{ and } b > 0 \\ -\frac{\pi}{2} & \text{if } a = 0 \text{ and } b < 0 \\ \text{indeterminate} & \text{if } a = 0 \text{ and } b = 0. \end{cases}$$

Polar Coordinate Representation

Polar form of z

Given r and θ we let

$$z = r(\cos\theta + i\sin\theta) \equiv re^{i\theta},$$

where the identity

$$e^{i\theta} = (\cos\theta + i\sin\theta)$$

is known as *Euler's Formula*.

Once we restrict the range of θ to an interval of length 2π, the representation is unique. Common choices are $\theta \in [0, 2\pi)$ [we will use this here], or $\theta \in [-\pi, \pi)$; but $\theta \in [\xi, \xi + 2\pi)$ for any $\xi \in \mathbb{R}$ works (but why make life harder than necessary?!)

Multiplying in Polar Form

Example

Given $z_1, z_2 \in \mathbb{C}$, then

$$z_1 z_2 = \begin{cases} (a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) \\ r_1e^{i\theta_1} r_2 e^{i\theta_2} = (r_1r_2)e^{i(\theta_1+\theta_2)} \\ r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2) = \\ (r_1r_2)((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \end{cases}$$

these three expressions are equivalent.

Since Euler's formula says $e^{i(\theta_1+\theta_2)} = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$, we can restate some old painful memories:

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1 + \theta_2) &= \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \end{aligned}$$

Bottom line, for $z = z_1z_2$, we have

$$|z| = |z_1|\,|z_2|, \quad \arg(z) = \arg(z_1) + \arg(z_2) \pmod{2\pi}.$$

From Euler to De Moivre

From Euler's Identity $e^{i\theta} = (\cos\theta + i\sin\theta)$ we see that

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta),$$

which is known as *De Moivre's Formula*.

OK, we have enough fragments of Complex Analysis to state the key result we need prior to revisiting our Eigenvalue/Eigenvector problem space.

Fundamental Theorem of Algebra

Theorem (Fundamental Theorem of Algebra)

Any nth degree polynomial $p_n(\lambda)$ with complex coefficients can be written as a product of linear factors*

$$p_n(\lambda) = k(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

for some complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and k. (The λ_k's need not be distinct).

Therefore a polynomial $p_n(\lambda)$ of degree n has precisely n complex roots if they are counted with their multiplicity.

* Note that real coefficients are complex coefficients with zero imaginary part.