Math 254: Introduction to Linear Algebra

Notes #7.2 — Finding the Eigenvalues of a Matrix

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Outline

1. Student Learning Objectives
   - SLOs: Eigen-values and vectors: Diagonalization
2. Finding the Eigenvalue of a Matrix
   - Determinants $\rightsquigarrow$ Characteristic Equation/Polynomial
3. Suggested Problems
   - Suggested Problems 7.2
   - Lecture – Book Roadmap
4. Supplemental Material
   - Metacognitive Reflection
   - Problem Statements 7.2
   - Complex Eigenvalues

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SLOs 7.2 Finding the Eigenvalues of a Matrix

After this lecture you should

- Know how to find the Eigenvalues of a Triangular Matrix
- Be able to derive the Characteristic Polynomial (of a Matrix), and understand its relation to the Eigenvalues
    - Be able to use the Characteristic Polynomial to find the Eigenvalues of a matrix.
- Be familiar with the Algebraic Multiplicity of an Eigenvalue
- Be able to express the Determinant, and Trace of a matrix in terms of the eigenvalues

Characterization of Eigenvalues :: Equivalences

$\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{R}^{n\times n}$

$\Updownarrow$

There exists a non-zero vector $\vec{v} \in \mathbb{C}^n$ such that
$$A\vec{v} = \lambda\vec{v} \quad \Leftrightarrow \quad (A - \lambda I_n)\vec{v} = \vec{0}$$

$\Updownarrow$

$$\ker(A - \lambda I_n) \neq \{\vec{0}\}$$

$\Updownarrow$

The matrix $(A - \lambda I_n)$ is not invertible

$\Updownarrow$ $\Updownarrow$

$$\det(A - \lambda I_n) = 0$$

$\Leftrightarrow$

Columns of $(A - \lambda I_n)$ are linearly dependent

The Characteristic Equation

Theorem (Eigenvalues and Determinants: The Characteristic Equation)

Consider an $(n \times n)$ matrix A and a scalar λ; then λ is an eigenvalue of A if and only if

$$\det(A - \lambda I_n) = 0.$$

This is called the **Characteristic Equation** *of the matrix A.*

Note that the Characteristic Equation is a *polynomial* in λ... the **Characteristic Polynomial** — $p_A(\lambda)$. We are looking for roots (zeros) of this polynomial.

BOTTOM LINE: The Eigenvalue problem can be solved as *("is equivalent to")* a polynomial root-finding problem.

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Example: Eigenvalues of a (2×2)-matrix

Example

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution: We solve the characteristic equation $\det(A - \lambda I_2) = 0$:

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)(3-\lambda) - 2 \cdot 4 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0.$$

We get two solutions: $\lambda_1 = 5$, and $\lambda_2 = -1$.

Example : Alternative (Determinant Free) Approach

[FOCUS :: MATH]

Example (Revisited: "Minimal Polynomial Approach" [MATH 524 (NOTES#8)])

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Solution: We select a vector $\vec{v} \in \mathbb{R}^2$; here $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; we form the set $\mathbb{K} = \{\vec{v}, A\vec{v}, A^2\vec{v}\}$. The vectors in $\mathbb{K}$ must be linearly dependent; we look for the first non-leading column in the matrix $M = \begin{bmatrix} \vec{v} & A\vec{v} & A^2\vec{v} \end{bmatrix}$:

$$\mathbb{K} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ 16 \end{bmatrix} \right\}, \; M = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 4 & 16 \end{bmatrix}, \; \mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \end{bmatrix}$$

This means $5I_2\vec{v} + 4A\vec{v} - A^2\vec{v} = \vec{0}$. We rearrange: $(5I_2 + 4A - A^2)\vec{v} = \vec{0}$. With the convention $A^0 = I_n$, we can let $p^{(m)}(\lambda) = 5 + 4\lambda - \lambda^2$; this is the *minimal polynomial* (which in this case is also the characteristic polynomial). We have $p^{(m)}(A)\vec{v} = \vec{0}$, and the roots of $p^{(m)}(\lambda)$ are the eigenvalues $\{-1, 5\}$.

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Example: Eigenvalues of a (3×3) Triangular Matrix

Example

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution: We solve the characteristic equation $\det(A - \lambda I_3) = 0$:

$$\det(A - \lambda I_3) = \det \left(\begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 3-\lambda & 4 \\ 0 & 0 & 4-\lambda \end{bmatrix} \right)$$

$$= (2-\lambda)(3-\lambda)(4-\lambda).$$

We get three solutions: $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = 4$.

Example : Alternative (Determinant Free) Approach

[Focus :: Math]

Example (Revisited: "Minimal Polynomial Approach" [Math 524 (Notes#8)])

Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution: We select a vector $\vec{v} \in \mathbb{R}^3$; here $\vec{v} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$; we form the set $\mathbb{K} = \{\vec{v}, A\vec{v}, A^2\vec{v}, A^3\vec{v}\}$. The vectors in $\mathbb{K}$ must be linearly dependent; we look for the first non-leading column in the matrix $M = \begin{bmatrix} \vec{v} & A\vec{v} & A^2\vec{v} & A^3\vec{v} \end{bmatrix}$:

$$\mathbb{K} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 36 \\ 28 \\ 16 \end{bmatrix}, \begin{bmatrix} 220 \\ 148 \\ 64 \end{bmatrix} \right\}, \quad \mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & 0 & 24 \\ 0 & 1 & 0 & -26 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$

This means $24I_2\vec{v} - 26A\vec{v} + 9A^2\vec{v} - A^3 = \vec{0}$. We identify the *minimal polynomial* $p^{(m)}(\lambda) = 24 - 26\lambda + 9\lambda^2 - \lambda^3$. We have $p^{(m)}(A)\vec{v} = \vec{0}$, and the roots of $p^{(m)}(\lambda)$ are the eigenvalues $\{2, 3, 4\}$.

A Note on the Minimal Polynomial Approach

#ThingsThatCanGoWrong

The selection of $\vec{v}$ must be such that (at a minimum) the set $\mathbb{K}$ has non-zero entries in all rows (where A has non-zeros).

Ponder the previous problem with $\vec{v} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$:

$$\mathbb{K} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix} \right\}, \ \mathrm{rref}(M) = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The second column is linearly dependent on the first, so $2\vec{v} - A\vec{v} = \vec{0} \rightsquigarrow p(\lambda) = 2 - \lambda$, which only reveals one eigenvalue (2); we see that $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ is an *eigenvector*.

We try again...

A Note on the Minimal Polynomial Approach

#ThingsThatCanGoWrong

Yet again, ponder the previous problem with $\vec{v} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$:

$$\mathbb{K} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 15 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 57 \\ 27 \\ 0 \end{bmatrix} \right\}, \ \mathrm{rref}(M) = \begin{bmatrix} 1 & 0 & -6 & -30 \\ 0 & 1 & 5 & 19 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The third column is linearly dependent on the first two, so $-6\vec{v} + 5A\vec{v} - A^2\vec{v} = \vec{0} \rightsquigarrow p(\lambda) = -6 + 5\lambda = \lambda^2$, which reveals the eigenvalues $\{2, 3\}$.

"Unlucky" choices of $\vec{v}$ may not capture all the eigenvalues.

Obvious Question: How do we know that we have all of them???
#WeHaveWorkToDo #Math524

Key Observation: Eigenvalues of Triangular Matrices

Theorem (Eigenvalues of a Triangular Matrix)

The eigenvalues of a triangular matrix are its diagonal elements.

Again, this "special" structure of the matrix makes eigenvalue computation easy [FOR THIS "TYPE" OF MATRICES].

WARNING WARNING!!! WARNING

Don't get any ideas... We CANNOT use row-reductions to transform a general matrix to upper triangular form, and then extract the eigenvalues from the diagonal. Bummer.

Example: Eigenvalues of a General (2×2)-matrix

Example

Find the characteristic equation of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Solution: We solve the characteristic equation $\det(A - \lambda I_2) = 0$:

$$\det(A - \lambda I_2) = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\right)$$

$$= (a-\lambda)(d-\lambda) - bc = \lambda^2 - (\mathbf{a}+\mathbf{d})\lambda + \underbrace{(ad-bc)}_{\det(A)}$$

Note that $(\mathbf{a}+\mathbf{d})$ is the sum of the diagonal elements of A; this quantity shows up frequently in linear algebra... and it has its own name ...

The Trace of a Matrix

Definition (Trace)

The sum of the diagonal entries of a square matrix A is called the **trace** of A, denoted by $\mathrm{trace}(A)$. For $A \in \mathbb{R}^{n\times n}$

$$\mathrm{trace}(A) = \sum_{i=1}^{n} a_{ii}$$

From the previous example we have:

Theorem (Characteristic Equation of a (2×2) matrix A)

$$\det(A - \lambda I_2) = \lambda^2 - \mathrm{trace}(A)\,\lambda + \det(A) = 0.$$

The Characteristic Polynomial

Theorem (Characteristic Polynomial)

For an $(n \times n)$ *matrix* A*,* $\det(A - \lambda I_n) = p_A(\lambda)$ *is a polynomial of degree* n*, of the form*

$$p_A(\lambda) = (-\lambda)^n + \textbf{trace}(A)\,(-\lambda)^{n-1} + \cdots + \textbf{det}(A).$$

Note: $\mathrm{trace}(A)$ is always the coefficient for the $(-\lambda)^{n-1}$ term, and $\det(A)$ is always the constant term.

It is possible (but not necessarily useful), and "somewhat" tedious to develop expressions for the remaining coefficients... we'll leave that as an "Exercise for the motivated student."

Finding Eigenvalues $\Leftrightarrow$ Solving $p_A(\lambda) = 0$

What do we know about polynomials?

Well, a polynomial of degree n has at most n real roots/zeros. Therefore an $(n \times n)$ matrix has at most n real eigenvalues.

Heads-Up: Allowing for complex roots: every nth degree polynomial has exactly n roots/zeros (counting repeats, a.k.a "multiplicity"); therefore an $(n \times n)$ matrix has exactly n eigenvalues.

Special case: When n is **odd**

[Recall: $p_A(\lambda) = (-\lambda)^n + \mathrm{trace}(A)\,(-\lambda)^{n-1} + \cdots + \det(A)$]

$$\lim_{\lambda \to \infty} p_A(\lambda) = -\infty, \quad \lim_{\lambda \to -\infty} p_A(\lambda) = \infty,$$

by the *intermediate value theorem*, there's at least one $\lambda_\# \in \mathbb{R}$ so that $p_A(\lambda_\#) = 0$.

Algebraic Multiplicity of an Eigenvalue

Example (Algebraic Multiplicity)

Find all eigenvalues of

$$A = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 0 & 4 & 3 & 2 & 1 \\ 0 & 0 & 5 & 2 & 1 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Solution: The characteristic polynomial is given by $\det(A - \lambda I_5)$ $\rightsquigarrow$ $p_A(\lambda) = (5-\lambda)^3(4-\lambda)^2$, so the eigenvalues are 5 and 4. $\lambda = 5$ is a root of multiplicity 3, and $\lambda = 4$ is a root of multiplicity 2; we say that the eigenvalue $\lambda_1 = 5$ has *algebraic multiplicity 3*, and $\lambda_2 = 4$ has *algebraic multiplicity 2*.

Algebraic Multiplicity of an Eigenvalue

Theorem (Algebraic Multiplicity of an Eigenvalue)

We say than an eigenvalue λ_ℓ of a square matrix A has **algebraic multiplicity** *k if λ_ℓ is a root of multiplicity k of the characteristic polynomial $p_A(\lambda)$, meaning that we can write*

$$p_A(\lambda) = (\lambda_\ell - \lambda)^k g(\lambda),$$

where $g(\lambda)$ is some polynomial (of order $(n-k)$) such that $g(\lambda_\ell) \neq 0$.

Flash-Forward: In [NOTES#7.3] we will discuss the *Geometric Multiplicity* of eigenvalues. (The Geometric Multiplicity is the count of the number of linearly indenpendent eigenvectors associated with the eigenvalue)

Number of Eigenvalues

Theorem (Number of Eigenvalues)

- *An $(n \times n)$ matrix A has at most n real eigenvalues, counted with algebraic multiplicities.*
- *An $(n \times n)$ matrix A has exactly n (possibly complex) eigenvalues, counted with algebraic multiplicities.*
- *If n is odd, then an $(n \times n)$ matrix has at least one real eigenvalue.*

Example (No Real Eigenvalues / Purely Imaginary Eigenvalues)

Consider the rotation matrix

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \text{with } p_A(\lambda) = \lambda^2 + 1.$$

The roots are $\pm\sqrt{-1} = \pm i$, which are not real...
Here, we have two complex eigenvalues $\lambda_1 = i$, and $\lambda_2 = -i$.

The 4 Cases for (3×3) Matrices

Example (4 Cases for Eigenvalues for (3×3) Matrices)

For (3×3) matrices with real entries, the characteristic polynomial $p_A(\lambda)$ is 3rd order, and takes either the form

$$p_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda),$$

or

$$p_A(\lambda) = (\lambda_1 - \lambda) g_2(\lambda), \text{ where } g_2(\lambda) \neq 0 \ \forall \lambda \in \mathbb{R}.$$

We end up with 4 possibilities: ...

The 4 Cases for (3×3) Matrices

Example (4 Cases for (3×3) Matrices with Real Entries)

#1 λ_1, λ_2, and λ_3 are distinct real eigenvalues, each with algebraic multiplicity 1.

#2 $\lambda_1 = \lambda_2$, and $\lambda_1 \neq \lambda_3$; *i.e.* one eigenvalue with algebraic multiplicity 2, and one *e.v.* with algebraic multiplicity 1.

#3 $\lambda_1 = \lambda_2 = \lambda_3$; *i.e.* one eigenvalue with algebraic multiplicity 3.

#4 $\lambda_1 \in \mathbb{R}$, $\lambda_2 = \lambda_3^* \in \mathbb{C}$; each with algebraic multiplicity 1.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{p_A(\lambda)=(1-\lambda)(2-\lambda)(3-\lambda)} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{p_A(\lambda)=(1-\lambda)^2(3-\lambda)} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{p_A(\lambda)=(1-\lambda)^3} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{p_A(\lambda)=(1-\lambda)(\lambda^2+1)}$$

What's the Trace Got to Do With It?

Example

Let A be a (2×2) matrix with eigenvalues λ_1 and λ_2 (allowing for algebraic multiplicity 2: $\lambda_1 = \lambda_2$). We have two expressions for the characteristic polynomial:

$$p_A(\lambda) = \lambda^2 - \text{trace}(A)\lambda + \det(A),$$

and

$$p_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2,$$

which means

$$\det(A) = \lambda_1\lambda_2, \quad \text{trace}(A) = (\lambda_1 + \lambda_2).$$

It turns out this generalizes to $(n \times n)$ matrices...

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Eigenvalues, Determinant, and Trace

Theorem (Eigenvalues, Determinant, and Trace)

If an $(n \times n)$ matrix A has the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, listed with their algebraic multiplicities, then

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n, \quad \textit{the product of the eigenvalues},$$

and

$$\mathrm{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \quad \textit{the sum of the eigenvalues},$$

Implication: We can know the product and the sum of the eigenvalues, without computing them. In particular, the trace is quick-and-easy to compute. (The determinant requires more work)

Is Finding Eigenvalues Easy?

We know that identifying the eigenvalues of an $(n \times n)$ matrix A reduces to finding the roots of the characteristic polynomial

$$p_A(\lambda) = \det(A - \lambda I_n).$$

How hard is this?

When $n = 2$, we have the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

giving the roots to $p_A(\lambda) = a\lambda^2 + b\lambda + c$.

Is Finding Eigenvalues Easy?

Similar expressions exist for $n = 3$ (*e.g.* `Cardano's Formula`) and $n = 4$ [NEXT TWO SLIDES: $n = 3$, $n = 4$]), but are not very useful in general.

However, it possible to manufacture (2×2), (3×3), and (4×4) examples, where identifying the eigenvalues "by hand" is achievable with reasonable effort. [$\Rightarrow \exists$ test questions!]

For $n \geq 5$, the **Abel-Ruffini theorem*** says there are no general algebraic solutions (expressed and nth roots).

Often it is impossible to find the exact eigenvalues of a (large) matrix. There are numerical methods which can be used to identify good approximations of the eigenvalues (see *e.g.* [MATH 543]).

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* The theorem is named after Paolo Ruffini, who provided an incomplete proof in 1799, and Niels Henrik Abel, who provided a proof in 1824. (Galois later proved more general statements, and provided a construction of a polynomial of degree 5 whose roots cannot be expressed in radicals from its coefficients.)

One Root for $p(x) = ax^3 + bx^2 + cx + d = 0$ (one case)

Let

$$u = \frac{9abc - 2b^3 - 27a^2 d}{54a^3},$$

$$v = u^2 + \left[\frac{3ac - b^2}{9a^2}\right]^3$$

If $v \geq 0$, then

$$x_1 = -\frac{b}{3a} + \sqrt[3]{u + \sqrt{v}} + \sqrt[3]{u - \sqrt{v}}$$

is one of the roots of $p(x)$; and a "simple" factoring gives a quadratic polynomial, to which we can apply the quadratic formula...

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Roots for $p(x) = ax^4 + bx^3 + cx^2 + dx + e = 0$ — Note: Does NOT Cover All Cases

$$x_{1,2} = -\frac{b}{4a} - S \pm \frac{1}{2}\sqrt{-4S^2 - 2p + \frac{q}{S}}, \quad x_{3,4} = -\frac{b}{4a} + S \pm \frac{1}{2}\sqrt{-4S^2 - 2p - \frac{q}{S}}$$

where

$$p = \frac{8ac - 3b^2}{8a^2}, \quad q = \frac{b^3 - 4abc + 8a^2 d}{8a^3}$$

where

$$S = \frac{1}{2}\sqrt{-\frac{2}{3}\,p + \frac{1}{3a}\left(Q + \frac{\Delta_0}{Q}\right)}, \quad Q = \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}$$

with

$$\Delta_0 = c^2 - 3bd + 12ae$$
$$\Delta_1 = 2c^3 - 9bcd + 27b^2 e + 27ad^2 - 72ace$$

Suggested Problems 7.2

Available on Learning Glass videos:
7.2 — 1, 3, 5, 11, 15, 17, 19, 23

Lecture – Book Roadmap

Lecture	Book, [GS5–]
7.1	§6.1
7.2	§6.1, §6.2
7.3	§6.1, §6.2
7.5	§6.1, §6.2

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(7.2.1), (7.2.3), (7.2.5)

(7.2.1) Use the characteristic polynomial $f_A(\lambda) = \det(A - \lambda I_n)$ to find all the *real* eigenvalues of the matrix, with their algebraic multiplicities; where

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

(7.2.3) Use the characteristic polynomial $f_A(\lambda) = \det(A - \lambda I_n)$ to find all the *real* eigenvalues of the matrix, with their algebraic multiplicities; where

$$A = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix}$$

(7.2.5) Use the characteristic polynomial $f_A(\lambda) = \det(A - \lambda I_n)$ to find all the *real* eigenvalues of the matrix, with their algebraic multiplicities; where

$$A = \begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$$

(7.2.11), (7.2.15), (7.2.17)

(7.2.11) Use the characteristic polynomial $f_A(\lambda) = \det(A - \lambda I_n)$ to find all the *real* eigenvalues of the matrix, with their algebraic multiplicities; where

$$A = \begin{bmatrix} 5 & 1 & -5 \\ 2 & 1 & 0 \\ 8 & 2 & -7 \end{bmatrix}$$

(7.2.15) Consider the matrix $A = \begin{bmatrix} 1 & k \\ 1 & 1 \end{bmatrix}$, where k is an arbitrary (real) constant. For which values of k does A have two distinct real eigenvalues? Where is there no real eigenvalue?

(7.2.17) Consider the matrix $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where a and b are arbitrary (real) constants. Find all eigenvalues of A. Explain in terms of the geometric interpretation of the linear tranformation $T(\vec{x}) = A\vec{x}$.

(7.2.19), (7.2.23)

(7.2.19) *True of False?* If the determinant of a matrix $A \in \mathbb{R}^{2\times 2}$ is negative, then A has two distinct real eigenvalues.

(7.2.23) Suppose a matrix A is similar to a matrix B ($A \sim B$). What is the relationship between the characteristic polynomials of A and B? What does that imply for the eigenvalues of A and B?

$\mathbb{C}$: Revisiting Rotations and Scalings

Example (Rotations and Scalings)

The matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad a, b \in \mathbb{R}$$

represents a combined rotation/scaling. What are the eigenvalues?

Solution: We get the eigenvalues from the characteristic polynomial

$$p_A(\lambda) = \det\left(\begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix}\right) = (a-\lambda)^2 + b^2 = 0$$

$$(a-\lambda)^2 = -b^2 \quad \Leftrightarrow \quad a - \lambda = \pm ib \quad \Leftrightarrow \quad \lambda = a \pm ib$$

$\mathbb{C}$: Two Theorems

Theorem

A complex $(n \times n)$ matrix has n complex eigenvalues if they are counted with their algebraic multiplicities.

Theorem (Trace, Determinant, and Eigenvalues)

Consider an $(n \times n)$ matrix A with complex eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, listed with their algebraic multiplicities. Then

$$\mathrm{trace}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

and

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$