Math 254: Introduction to Linear Algebra

Notes #7.3 — Finding the Eigenvectors of a Matrix

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Outline

1. Student Learning Objectives
   - SLOs: Finding the Eigenvectors of a Matrix
2. Finding the Eigenvectors of a Matrix
   - Eigenvalues $\rightsquigarrow$ Eigenvectors and Eigenvectors
   - Diagonalizing Matrices
   - Complex Eigenvalues / Eigenvectors: Rotations and Scalings
3. Suggested Problems
   - Suggested Problems 7.3 and 7.5
   - Lecture – Book Roadmap
4. Supplemental Material
   - Metacognitive Reflection
   - Problem Statements 7.3 and 7.5
   - Complex Numbers: Quick Review / Crash Course
   - Fundamental Theorem of Algebra

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SLOs 7.3 — Finding the Eigenvectors of a Matrix

After this lecture you should

- Be familiar with Eigenspaces
- Know the definition of, and be able to determine, the Geometric Multiplicity of an Eigenvalue
- Be able to complete the Process:
  1. Identify Eigenvalues — characteristic equation $p_A(\lambda) = 0$.
  2. For each unique Eigenvalue, Identify its Eigenspace — $E(\lambda, A) = \ker(A - \lambda I_n)$.
  3. If an Eigenbasis exists, collect it; then Identify the Diagonalizing Similarity Transform (Matrix S, and Diagonal Matrix B).

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Characterization of Eigenvalues, and Eigenvectors

$\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$

$$\Updownarrow$$

There exists a non-zero vector $\vec{v} \in \mathbb{C}^n$ such that
$A\vec{v} = \lambda\vec{v}$, or $(A - \lambda I_n)\vec{v} = \vec{0}$

$$\Updownarrow$$

$\ker(A - \lambda I_n) \neq \{\vec{0}\}$ *Today $\rightsquigarrow$ Find Eigenvectors.*

$$\Updownarrow$$

The matrix $(A - \lambda I_n)$ is not invertible

$$\Updownarrow$$

$\det(A - \lambda I_n) = 0$ *Last Time $\rightsquigarrow$ Find Eigenvalues.*

Eigenvalues ⇝ Eigenvectors

OK, we have some ideas on how to find eigenvalues (*e.g.* through the roots of the characteristic polynomial); the next step is to identify the associated eigenvectors:

Definition (Eigenspaces, and Eigenvectors)

Consider an eigenvalue λ of an $(n \times n)$ matrix A. Then the kernel of the matrix $(A - \lambda I_n)$ is called the *eigenspace* associated with λ, often denoted $E(\lambda, A)$:

$$E(\lambda, A) = \ker(A - \lambda I_n) = \{\, \vec{v} \in \mathbb{R}^n \;:\; A\vec{v} = \lambda\vec{v} \,\}.$$

All vectors $\vec{w} \in E(\lambda, A)$ are *eigenvectors*.

A (2×2) Example

Example

Find the eigenspaces of the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

Solution: We have already shown that the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -1$. We are looking for

$$E(5, A) = \ker\left(\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\right), \quad E(-1, A) = \ker\left(\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\right)$$

here we can use the famous *method of the eyeball** to see that

$$E(5, A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right), \quad E(-1, A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

* If/when this fails, we get the result by computing $\mathrm{rref}(A - \lambda I_n)$ and finding the basis for the kernel as usual (via parameterization).

A (2 × 2) Example

Example (Checking Our Answer)

The claim is that the eigenvalues and eigenspaces of

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

are

$$\left\{ \lambda_1 = 5,\ E(5,A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right) \right\},\quad \left\{ \lambda_2 = -1,\ E(-1,A) = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \right\},$$

We multiply

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

A (2×2) Example

Example (Diagonalizing A)

If we collect the eigenvectors as columns in S, and the eigenvalues in B:

$E(\lambda_1, A) \leftrightarrow \lambda_1$

$$S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

$E(\lambda_2, A) \leftrightarrow \lambda_2$

then

$$S^{-1}\mathbf{A}S = \mathbf{B}, \quad \mathbf{A}S = S\mathbf{B}:$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 10 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 10 & 1 \end{bmatrix}.$$

$$\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 10 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$$

A (3×3) Example

Example (Identifying The Eigenvalues)

Find the eigenspaces of the matrix A:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{note: } \mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: Since A is upper triangular, we see that the eigenvalues are $\{1_{\mathrm{am}:2}, 0_{\mathrm{am}:1}\}$ — $p_A(\lambda) = (1-\lambda)^2(0-\lambda)$

($1_{\mathrm{am}:2}$ is my home-cooked notation for *"algebraic multiplicity 2."*).

Note: The eigenvalues of a matrix are NOT preserved by row-operations; the matrix we get by subtracting the 2nd from the 1st and 3rd rows has eigenvalues $\{1_{\mathrm{am}:1}, 0_{\mathrm{am}:2}\}$.

A (3×3) Example

Example (Finding the Eigenspaces — $E(0, A)$)

Since 0 is an eigenvalue, and the kernel **is** preserved by row-operations, we have

$$E(0, A) = \ker(A) = \ker(\mathrm{rref}(A)) = \ker\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right),$$

as usual we parameterize the free variable (x_2) and identify

$$E(0, A) = \mathrm{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right)$$

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A (3×3) Example

Example (Finding the Eigenspaces — $E(1, A)$)

Since $1_{\mathrm{am}:2}$ is an eigenvalue, and the kernel **is** preserved by row-operations:

$$E(1,A) = \ker(A - I_3) = \ker\left(\mathrm{rref}\left(\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)\right) = \ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

as usual we parameterize the free variable (x_1) and identify

$$E(1,A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

A (3×3) Example

Example (Discussion)

We notice that both $E(0, A)$ and $E(1, A)$ are 1-dimensional subspaces of $\mathbb{R}^3$; for $\lambda = 0_{\mathrm{am:1}}$, this is not a big surprise. However, for $\lambda = 1_{\mathrm{am:2}}$ it is a bit disturbing; it feels like something is missing?

Theorem (Geometric Multiplicity)

Consider an eigenvalue of an $(n \times n)$ matrix A. The dimension of the eigenspace $E(\lambda, A) = \ker(A - \lambda I_n)$ is called the **geometric multiplicity** *of eigenvalue λ; we have*

$$\mathrm{Geometric\_Multiplicity}(\lambda) = \mathrm{nullity}^*(A - \lambda I_n) = n - \mathrm{rank}(A - \lambda I_n).$$

* $\mathrm{nullity}(A - \lambda I_n) \equiv \dim\left(\ker\left(A - \lambda I_n\right)\right) \equiv \dim\left(E(\lambda, A)\right).$

Geometric vs. Algebraic Multiplicity

Theorem (Geometric vs. Algebraic Multiplicity)

$$\textit{Geometric\_Multiplicity}(\lambda) \leq \textit{Algebraic\_Multiplicity}(\lambda)$$

Theorem (Eigenbases and Geometric Multiplicities)

a. *Consider and $(n \times n)$ matrix A. If we find a basis for each eigenspace of A and concatenate all these bases, then the resulting eigenvectors $\vec{v}_1, \ldots, \vec{v}_s$ will be linearly independent.*

Note: *s is the sum of the geometric multiplicities of the eigenvalues of A.*

(!) *This means that $s \leq n$.*

b. *Matrix A is diagonalizable if and only if the geometric multiplicities of the eigenvalues add up to n (i.e. $s = n$ in part* **a.***)*

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An $(n \times n)$ Matrix with n Distinct Eigenvalues

Theorem (An $(n \times n)$ Matrix with n Distinct Eigenvalues)

If an $(n \times n)$ matrix has n distinct eigenvalues, then A is diagonalizable. We can construct the eigenbasis by finding an eigenvector for each eigenvalue.

Note: "All the Eigenvalues are Distinct"
$\Leftrightarrow$ "All Eigenvalues have algebraic multiplicity 1"
$\Rightarrow$ "All Eigenvalues have geometric multiplicity 1"
$\Leftrightarrow$ Each Eigenspace has a single [eigen]vector.

Note: When λ is an eigenvalue, there is *at least* one eigenvector, therefore $\mathbf{1} \leq \mathrm{gm}(\lambda) \leq \mathrm{am}(\lambda)$.

The Eigenvalues of Similar Matrices — IMPORTANT!!!

Theorem (The Eigenvalues of Similar Matrices)

Suppose matrix A is similar to matrix B. Then

a. *A and B has the same characteristic polynomial,* $p_A(\lambda) = p_B(\lambda)$.

b. $\text{rank}(A) = \text{rank}(B)$, $\text{nullity}(A) = \text{nullity}(B)$.

c. *A and B have the same eigenvalues, with the same algebraic and geometric multiplicities. However, the eigenvectors need not be the same.*

d. *A and B have the same determinant, and trace:* $\det(A) = \det(B)$, $\text{trace}(A) = \text{trace}(B)$.

Similar Matrices?

Example (Similar Matrices?)

Is $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ similar to $B = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$?

Solution: We have an easy way to show that the answer is "no!"

- $\mathrm{trace}(A) = 9$, but $\mathrm{trace}(B) = 8$.

Note that is it possible to have two matrices for which $\det(A) = \det(B)$, and $\mathrm{trace}(A) = \mathrm{trace}(B)$ that are NOT similar, *e.g.*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \frac{5+\sqrt{10}}{2} & 0 \\ 0 & 0 & \frac{5-\sqrt{10}}{2} \end{bmatrix}$$

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Strategy for Diagonalization

Theorem (Strategy for Diagonalization)

Given an $(n \times n)$ matrix A: in order to determine whether it is diagonalizable, we seek S and B (diagonal) such that $S^{-1}AS = B$:

a. *Find the eigenvalues of A by solving the characteristic equation $p_A(\lambda) = \det(A - \lambda I_n) = 0$.*

b. *For each eigenvalue, find a basis for the eigenspace $E(\lambda, A) = \ker(A - \lambda I_n)$.*

c. *The matrix is diagonalizable if and only if the dimensions of the eigenspaces add up to n; in which case we collect the eigenspaces as columns in the matrix S, and place the corresponding eigenvalues on the diagonal of B:*

$$S = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}, \quad S^{-1}AS = B = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

(Modified) A (3 × 3) Example

Example (Identifying The Eigenvalues)

Find the eigenspaces of the matrix A:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{note: } \mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Solution: Since A is upper triangular, we see that the eigenvalues are $\{1_{\text{am:2}}, 0_{\text{am:1}}\}$.

($1_{\text{am:2}}$ is my home-cooked notation for *"algebraic multiplicity 2."*).

Note: The eigenvalues of a matrix are NOT preserved by row-operations; the matrix we get by swapping the 2nd and the 3rd row has eigenvalues $\{1_{\text{am:1}}, 0_{\text{am:2}}\}$.

(Modified) A (3×3) Example

Example (Finding the Eigenspaces — $E(0, A)$)

Since 0 is an eigenvalue, and the kernel **is** preserved by row-operations, we have

$$E(0, A) = \ker(A) = \ker(\text{rref}(A)) = \ker\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right),$$

as usual we parameterize the free variable (x_2) and identify

$$E(0, A) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right), \rightsquigarrow \lambda_1 = 0 \quad \text{has am:1, and gm:1.}$$

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(Modified) A (3×3) Example

Example (Finding the Eigenspaces — $E(1, A)$)

Since 1 is an eigenvalue, and the kernel **is** preserved by row-operations, therefore

$$E(1,A) = \ker(A - I_3) = \ker\left(\mathrm{rref}\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\right) = \ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

as usual we parameterize the free variables (x_1, x_3) and identify

$$E(1,A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \rightsquigarrow \lambda_2 = 1 \quad \text{has am:2, and gm:2.}$$

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(Modified) A (3×3) Example

Example

Now, since we have matching algebraic and geometric multiplicities for ALL eigenvalues, the matrix **is** diagonalizable.

$E(0, A) \leftrightarrow \lambda_1 = 0$

$$S = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$E(1, A) \leftrightarrow \lambda_2 = 1$

Note that the ordering of eigenspaces and eigenvalues must match.

Revisiting Rotations and Scalings with Complex Eigenvalue / Eigenvectors

Example (Rotations and Scalings — Complex Eigenvalues)

The matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad a, b \in \mathbb{R}$$

represents a combined rotation/scaling. We now diagonalize this matrix, allowing for complex eigenvalues...

Solution: We get the eigenvalues from the characteristic polynomial

$$p_A(\lambda) = \det\left(\begin{bmatrix} a-\lambda & -b \\ b & a-\lambda \end{bmatrix}\right) = (a-\lambda)^2 + b^2 = 0$$

$$(a-\lambda)^2 = -b^2 \quad \Leftrightarrow \quad a-\lambda = \pm ib \quad \Leftrightarrow \quad \lambda = \mathbf{a \pm ib}$$

Revisiting Rotations and Scalings with Complex Eigenvalue / Eigenvectors

Example (Rotations and Scalings — Complex Diagonalization)

Next, we find the eigenspaces

$$E(a+ib,A) = \ker\left(\begin{bmatrix} -ib & -b \\ b & -ib \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$$

$$E(a-ib,A) = \ker\left(\begin{bmatrix} ib & -b \\ b & ib \end{bmatrix}\right) = \operatorname{span}\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$$

If we let

$$R = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad R^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix}$$

then

$$\mathbf{R}^{-1}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\mathbf{R} = \begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$$

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Revisiting Rotations and Scalings with Complex Eigenvalue / Eigenvectors

Example (Rotations and Scalings — Alternative Book-keeping)

Let us ponder the $R \in \mathbb{C}^{2\times 2}$ which defined the diagonalizing similarity transform — we split it into its real and imaginary parts:

$$R = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + i \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

now, let

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \left(\text{clearly} \begin{cases} \mathrm{span}(\vec{v}) & = & \mathrm{im}(\mathrm{real}(R)) \\ \mathrm{span}(\vec{w}) & = & \mathrm{im}(\mathrm{imag}(R)) \end{cases}\right)$$

which means

$$R = \underbrace{\begin{bmatrix} \vec{v} + i\vec{w} & \vec{v} - i\vec{w} \end{bmatrix}}_{\text{Call this form } P}$$

Revisiting Rotations and Scalings with Complex Eigenvalue / Eigenvectors

Example (Rotations and Scalings — Alternative Book-keeping)

We now have two equivalent expressions for the diagonalization:

$$R^{-1}AR = P^{-1}AP \qquad \left(P \text{ is just another way of building } R\ldots\right)$$

Pre-multiply by R and post-multiply by R^{-1}, then

$$A = RR^{-1}ARR^{-1} = (RP^{-1})A(PR^{-1})$$

Let $S = PR^{-1}$; $S^{-1} = RP^{-1}$, then

$$S = \begin{bmatrix} \vec{v} + i\vec{w} & \vec{v} - i\vec{w} \end{bmatrix} \frac{1}{2} \begin{bmatrix} -i & 1 \\ i & 1 \end{bmatrix} = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

Formalizing...

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Revisiting Rotations and Scalings with Complex Eigenvalue / Eigenvectors

Theorem (Complex Eigenvalues and Rotation-Scaling Matrices)

If $A \in \mathbb{R}^{2\times 2}$ with eigenvalues $a \pm ib$ (where $b \neq 0$), and if $\vec{v} + i\vec{w}$ is an eigenvector of A with eigenvalue $a + ib$, then

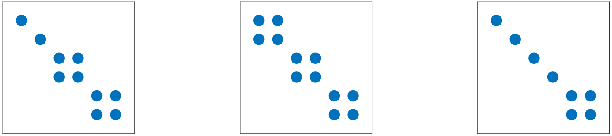
$$S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \textit{where } S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$$

Note that $A, S \in \mathbb{R}^{2\times 2}$, and $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in \mathbb{R}^{2\times 2}$.

The matrix A is similar to a rotation-scaling matrix.

Revisiting Rotations and Scalings with Complex Eigenvalue / Eigenvectors

Real (2×2)-Block Diagonalization vs. Complex Diagonalization



For a complex pair of eigenvalues $\lambda = a \pm ib$ —

- if we keep the similarity-transform-matrix $S = \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}$ *real* we can get similarity to a rotation-scaling matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$; and
- if we allow $S = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$ to be *complex* we can get similarity to a diagonal matrix (with complex entries) $\begin{bmatrix} a+ib & 0 \\ 0 & a-ib \end{bmatrix}$

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Complex Diagonalization vs. Real Block-Diagonalization

This holds for any size matrices:

- if a real matrix $A_{\mathbb{R}} \in \mathbb{R}^{n \times n}$ is complex-diagonalizable $A_{\mathbb{R}} \sim S_{\mathbb{C}} D_{\mathbb{C}} S_{\mathbb{C}}^{-1}$, then
- it can alternatively be similarity-transformed into a real block-diagonal matrix $A_{\mathbb{R}} \sim S_{\mathbb{R}} B_{\mathbb{R}} S_{\mathbb{R}}^{-1}$; where each diagonal complex-pair-block (in $D_{\mathbb{C}}$) $\begin{bmatrix} a_k + ib_k & 0 \\ 0 & a_k - ib_k \end{bmatrix}$ is replaced by a (2×2)-block $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (in $B_{\mathbb{R}}$); $-b$ is in the first super-diagonal, and b in the first sub-diagonal.

See illustration on next slide...

Complex Diagonalization vs. Real Block-Diagonalization

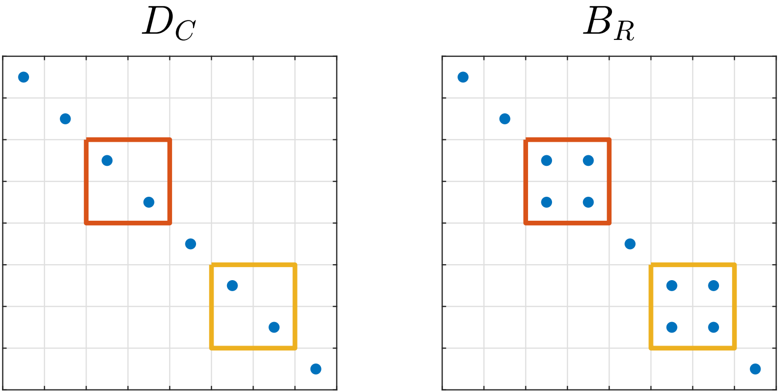

Figure: The (2×2) blocks in $D_{\mathbb{C}} \in \mathbb{C}^{n \times n}$ contain complex pairs of eigenvalues; and the corresponding blocks in $B_{\mathbb{R}} \in \mathbb{R}^{n \times n}$ contain "rotation blocks."

Suggested Problems 7.3

Available on Learning Glass videos:

7.3 — 1, 3, 5, 9, 13, 17, 23, 27, 31, 35

7.5 — 13, 15, 17, 21, 23

Lecture – Book Roadmap

Lecture	Book, [GS5–]
7.1	§6.1
7.2	§6.1, §6.2
7.3	§6.1, §6.2

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned	Almost there	Huh?!?
Right After Lecture		
After Thinking / Office Hours / SI-session		
After Reviewing for Quiz/Midterm/Final		

(7.3.1), (7.3.3), (7.3.5)

(7.3.1) Find all (real) eigenvalues; then find a basis of each eigenspace, and diagonalize A, if you can.

$$A = \begin{bmatrix} 7 & 8 \\ 0 & 9 \end{bmatrix}$$

(7.3.3) Find all (real) eigenvalues; then find a basis of each eigenspace, and diagonalize A, if you can.

$$A = \begin{bmatrix} 6 & 3 \\ 2 & 7 \end{bmatrix}$$

(7.3.5) Find all (real) eigenvalues; then find a basis of each eigenspace, and diagonalize A, if you can.

$$A = \begin{bmatrix} 4 & 5 \\ -2 & -2 \end{bmatrix}$$

(7.3.9), (7.3.13)

(7.3.9) Find all (real) eigenvalues; then find a basis of each eigenspace, and diagonalize A, if you can.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(7.3.13) Find all (real) eigenvalues; then find a basis of each eigenspace, and diagonalize A, if you can.

$$A = \begin{bmatrix} 3 & 0 & -2 \\ -7 & 0 & 4 \\ 4 & 0 & -3 \end{bmatrix}$$

(7.3.17), (7.3.23), (7.3.27)

(7.3.17) Find all (real) eigenvalues; then find a basis of each eigenspace, and diagonalize A, if you can.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(7.3.23) Find all eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Is there an eigenbasis? Interpret your result geometrically.

(7.3.27) Consider a (2×2) matrix A. Suppose that $\mathrm{trace}(A) = 5$ and $\det(A) = 6$. Find the eigenvalues of A.

(7.3.31), (7.3.35)

(7.3.31) Suppose there is an eigenbasis for a matrix A. What is the relationship between the algebraic and geometric multiplicities of its eigenvalues?

(7.3.35) Is the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ similar to $\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$.

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(7.5.13, 15, 17, 21, 23)

For each of the the given matrices, find an invertible matrix S such that $S^{-1}AS = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

(7.5.13)

$$A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$$

(7.5.15)

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}$$

(7.5.17)

$$A = \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}$$

For each of the the given matrices, find all (real and complex) eigenvalues

(7.5.21)

$$A = \begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}$$

(7.5.23)

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Definition, Complex Addition

Definition (Complex Numbers)

With $a, b \in \mathbb{R}$, we define the complex value $z \in \mathbb{C}$:

$$z = a + ib$$

where i is the imaginary unit $+\sqrt{-1}$. a is the *Real Part* ($a = \mathrm{Re}\, z$), and b the *Imaginary Part* ($b = \mathrm{Im}\, z$) of z.

Definition (Complex Addition)

Let $z_1, z_2 \in \mathbb{C}$, then

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

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Complex Multiplication

Definition (Complex Multiplication)

Let $z_1, z_2 \in \mathbb{C}$, then

$$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

this follows from the fact that $i^2 = -1$.

Note: $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$

Let $T : \mathbb{R}^2 \to \mathbb{C}$ be the linear transformation:

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a + ib, \quad T^{-1}(a + ib) = \begin{bmatrix} a \\ b \end{bmatrix},$$

that is we can interpret vectors in $\mathbb{R}^2$ as complex numbers (and the other way around).

Multiplication by $i \rightsquigarrow$ Rotation

Example (Multiplication by i)

Consider $z = a + ib$, and let $a, b > 0$ so that the corresponding vector lives in the first quadrant.

z		$a + ib$
iz	$i(a+ib) = ia + i^2 b$	$-b + ia$
$i^2 z$	$i(-b+ia) = -ib + i^2 a$	$-a - ib$
$i^3 z$	$i(-a-ib) = -ia + i^2 b$	$b - ia$
$i^4 z$	$i(b-ia) = ib - i^2 a$	$a + ib$

We see that $z = -i^2 z = i^4 z$, and since

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} -b \\ a \end{bmatrix} = a(-b) + ba = 0, \quad \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} b \\ -a \end{bmatrix} = ab + b(-a) = 0$$

we can interpret multiplication by i as a ccw-rotation by $\pi/2$ ($90°$).

Complex numbers solve our issue of "no real eigenvalues" for rotations!

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Complex Conjugate

Definition (Complex Conjugate)

Given $z = (a + ib) \in \mathbb{C}$, the complex conjugate is defined by

$$\overline{z} = (a - ib), \quad \text{sometimes } z^* = (a - ib)$$

(reversing the sign on the imaginary part). Note that this is a reflection across the real axis in the complex plane.

Hey! It's a reflection across the real axis!

z and z^* form a *conjugate pair* of complex numbers, and $zz^* = (a + ib)(a - ib) = a^2 + b^2$.

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Polar Coordinate Representation

Polar Coordinate Representation (Modulus and Argument)

We can represent $z = a + ib$ in terms of its length r (*modulus*) and angle θ (*argument*); where

$$r = \mathrm{mod}(z) = |z| = \sqrt{a^2 + b^2}, \quad \theta = \arg(z) \in [0, 2\pi)$$

where

$$\theta = \arg(z) = \begin{cases} \arctan(\frac{b}{a}) & \text{if } a > 0 \\ \arctan(\frac{b}{a}) + \pi & \text{if } a < 0 \text{ and } b \geq 0 \\ \arctan(\frac{b}{a}) - \pi & \text{if } a < 0 \text{ and } b < 0 \\ \frac{\pi}{2} & \text{if } a = 0 \text{ and } b > 0 \\ -\frac{\pi}{2} & \text{if } a = 0 \text{ and } b < 0 \\ \text{indeterminate} & \text{if } a = 0 \text{ and } b = 0. \end{cases}$$

Polar Coordinate Representation

Polar form of z

Given r and θ we let

$$z = r(\cos\theta + i\sin\theta) \equiv re^{i\theta},$$

where the identity

$$e^{i\theta} = (\cos\theta + i\sin\theta)$$

is known as *Euler's Formula*.

Once we restrict the range of θ to an interval of length 2π, the representation is unique. Common choices are $\theta \in [0, 2\pi)$ [we will use this here], or $\theta \in [-\pi, \pi)$; but $\theta \in [\xi, \xi + 2\pi)$ for any $\xi \in \mathbb{R}$ works (but why make life harder than necessary?!)

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Multiplying in Polar Form

Example

Given $z_1, z_2 \in \mathbb{C}$, then

$$z_1 z_2 = \begin{cases} (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \\ r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)} \\ r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) = \\ (r_1 r_2)((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)) \end{cases}$$

these three expressions are equivalent.

Since Euler's formula says $e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$, we can restate some old painful memories:

$$\begin{aligned} \cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1+\theta_2) &= \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 \end{aligned}$$

Bottom line, for $z = z_1 z_2$, we have

$$|z| = |z_1|\,|z_2|, \quad \arg(z) = \arg(z_1) + \arg(z_2) \pmod{2\pi}.$$

From Euler to De Moivre

From Euler's Identity $e^{i\theta} = (\cos\theta + i\sin\theta)$ we see that

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta),$$

which is known as *De Moivre's Formula*.

OK, we have enough fragments of Complex Analysis to state the key result we need prior to revisiting our Eigenvalue/Eigenvector problem space.

Fundamental Theorem of Algebra

Theorem (Fundamental Theorem of Algebra)

Any nth degree polynomial $p_n(\lambda)$ with complex coefficients can be written as a product of linear factors*

$$p_n(\lambda) = k(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

for some complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ and k. (The λ_k's need not be distinct).

Therefore a polynomial $p_n(\lambda)$ of degree n has precisely n complex roots if they are counted with their multiplicity.

* Note that real coefficients are complex coefficients with zero imaginary part.