

Math 254: Introduction to Linear Algebra

Notes #7.E — Examples

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Complex Diagonalization vs. Real (2 × 2) Block Diagonalization

(7.5.13)

$$A_{(7.5.13)} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}, \det(A) = 4, \text{trace}(A) = 0$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det \left(\begin{bmatrix} -\lambda & -4 \\ 1 & -\lambda \end{bmatrix} \right) = \lambda^2 + 4$$

Eigenvalues = Roots of $p_A(\lambda) = \pm 2i$.

Eigenspaces:

$$E_{2i} = \ker(A - 2iI) = \ker \left(\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \right) = \ker \left(\begin{bmatrix} -2i & -4 \\ 2i & 4 \end{bmatrix} \right) = \ker \left(\begin{bmatrix} -2i & -4 \\ 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 2i \\ 1 \end{bmatrix} \right)$$

$$E_{-2i} = \ker(A + 2iI) = \ker \left(\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix} \right) = \ker \left(\begin{bmatrix} 2i & -4 \\ 2i & -4 \end{bmatrix} \right) = \ker \left(\begin{bmatrix} 2i & -4 \\ 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} -2i \\ 1 \end{bmatrix} \right)$$



Outline

1 Examples 7.5

- (2 × 2)-Examples
- (3 × 3)-Examples



Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization

(7.5.13)

Complex Diagonalization: use the vectors from E_{2i} and E_{-2i} as columns in S :

$$S = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix}, S^{-1} = \frac{1}{4} \begin{bmatrix} -i & 2 \\ i & 2 \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 2i & \\ & -2i \end{bmatrix}$$

Real (2 × 2)-Block Diagonalization: Use $\text{Imag}(E_{2i})$ and $\text{Real}(E_{2i})$ *in that order* as columns in S :

$$S = \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}, S^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ & 1 \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

Note: By convention we always use vectors from the eigenspace with **POSITIVE** imaginary part.

If we use $\text{Imag}(E_{-2i})$ and $\text{Real}(E_{-2i})$ as columns in S :

$$S = \begin{bmatrix} 2 & \\ & -1 \end{bmatrix}, S^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ & -1 \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

the off-diagonal minus sign ends up in the wrong (“non-convention”) location.



Complex Diagonalization vs. Real 2 × 2 Block Diagonalization

(7.5.15)

$$A_{(7.5.15)} = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}, \det(A) = 5, \text{trace}(A) = 4$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -5 & 4 - \lambda \end{pmatrix} = \lambda(\lambda - 4) + 5 = \lambda^2 - 4\lambda + 5$$

Eigenvalues = Roots of $p_A(\lambda)$:

$$\lambda^2 - 4\lambda = -5 \Leftrightarrow (\lambda - 2)^2 - 4 = -5 \Leftrightarrow (\lambda - 2)^2 = -1 \Leftrightarrow \lambda_{\pm} = 2 \pm i$$

Eigenspaces:

$$E_{\lambda_+} = \ker(A - \lambda_+ I) = \ker \begin{pmatrix} -2 - i & 1 \\ -5 & 2 - i \end{pmatrix} = \ker \begin{pmatrix} -5 & 2 - i \\ -5 & 2 - i \end{pmatrix} = \ker \begin{pmatrix} -5 & 2 - i \\ 0 & 0 \end{pmatrix} = \text{span} \left(\begin{bmatrix} 1 \\ 2 + i \end{bmatrix} \right)$$

$$E_{\lambda_-} = \ker(A - \lambda_- I) = \ker \begin{pmatrix} -2 + i & 1 \\ -5 & 2 + i \end{pmatrix} = \ker \begin{pmatrix} -5 & 2 + i \\ -5 & 2 + i \end{pmatrix} = \ker \begin{pmatrix} -5 & 2 + i \\ 0 & 0 \end{pmatrix} = \text{span} \left(\begin{bmatrix} 1 \\ 2 - i \end{bmatrix} \right)$$



Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization

(7.5.15)

Complex Diagonalization: use the vectors from E_{2+i} and E_{2-i} as columns in S :

$$S = \begin{bmatrix} 1 & 1 \\ 2 + i & 2 - i \end{bmatrix}, S^{-1} = \frac{1}{2} \begin{bmatrix} 1 + 2i & -i \\ 1 - 2i & i \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 2 + i & \\ & 2 - i \end{bmatrix}$$

Real (2 × 2)-Block Diagonalization: Use $\text{Imag}(E_{2+i})$ and $\text{Real}(E_{2+i})$ as columns in S :

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, S^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Note: By convention we always use vectors from the eigenspace with POSITIVE imaginary part.

Complex Diagonalization vs. Real 2 × 2 Block Diagonalization

(7.5.17)

$$A_{(7.5.17)} = \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}, \det(A) = 25, \text{trace}(A) = 6$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 5 - \lambda & 4 \\ -5 & 1 - \lambda \end{pmatrix} = (5 - \lambda)(1 - \lambda) + 20 = \lambda^2 - 6\lambda + 25$$

Eigenvalues = Roots of $p_A(\lambda)$:

$$\lambda^2 - 6\lambda = -25 \Leftrightarrow (\lambda - 3)^2 - 9 = -25 \Leftrightarrow (\lambda - 3)^2 = -16 \Leftrightarrow \lambda_{\pm} = 3 \pm 4i$$

Eigenspaces:

$$E_{\lambda_+} = \ker(A - \lambda_+ I) = \ker \begin{pmatrix} 2 - 4i & 4 \\ -5 & -2 - 4i \end{pmatrix} = \ker \begin{pmatrix} 5 & 2 + 4i \\ 5 & 2 + 4i \end{pmatrix} = \ker \begin{pmatrix} 5 & 2 + 4i \\ 0 & 0 \end{pmatrix} = \text{span} \left(\begin{bmatrix} 1 \\ -1 + 2i \end{bmatrix} \right)$$

$$E_{\lambda_-} = \ker(A - \lambda_- I) = \ker \begin{pmatrix} 2 + 4i & 4 \\ -5 & -2 + 4i \end{pmatrix} = \ker \begin{pmatrix} 5 & 2 - 4i \\ 5 & 2 - 4i \end{pmatrix} = \ker \begin{pmatrix} 5 & 2 - 4i \\ 0 & 0 \end{pmatrix} = \text{span} \left(\begin{bmatrix} 1 \\ -1 - 2i \end{bmatrix} \right)$$



Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization

(7.5.17)

Complex Diagonalization: use the vectors from E_{3+4i} and E_{3-4i} as columns in S :

$$S = \begin{bmatrix} 2 & 2 \\ -1 + 2i & -1 - 2i \end{bmatrix}, S^{-1} = \frac{1}{8} \begin{bmatrix} 2 - i & -2i \\ 2 + i & 2i \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 3 + 4i & \\ & 3 - 4i \end{bmatrix}$$

Real (2 × 2)-Block Diagonalization: Use $\text{Imag}(E_{3+4i})$ and $\text{Real}(E_{3+4i})$ as columns in S :

$$S = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}, S^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

Note: By convention we always use vectors from the eigenspace with POSITIVE imaginary part.

Complex Diagonalization vs. Real (2 × 2) Block Diagonalization (7.5.21)

$$A_{(7.5.21)} = \begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}, \det(A) = 13, \text{trace}(A) = 4$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det \left(\begin{bmatrix} 11 - \lambda & -15 \\ 6 & -7 - \lambda \end{bmatrix} \right) = (11 - \lambda)(-7 - \lambda) + 90 = \lambda^2 - 4\lambda + 13$$

Eigenvalues = Roots of $p_A(\lambda)$:

$$\lambda^2 - 4\lambda = -13 \Leftrightarrow (\lambda - 2)^2 - 4 = -13 \Leftrightarrow (\lambda - 2)^2 = -9 \Leftrightarrow \lambda_{\pm} = 2 \pm 3i$$

Eigenspaces:

$$E_{\lambda_+} = \ker(A - \lambda_+ I) = \ker \left(3 \begin{bmatrix} 3 - i & -5 \\ 2 & -3 - i \end{bmatrix} \right) = \ker \left(- \begin{bmatrix} 3 - i & -5 \\ 3 - i & -5 \end{bmatrix} \right) = \ker \left(\begin{bmatrix} 3 - i & -5 \\ 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 3 - i \\ 0 \end{bmatrix} \right)$$

$$E_{\lambda_-} = \ker(A - \lambda_- I) = \ker \left(3 \begin{bmatrix} 3 + i & -5 \\ 2 & -3 + i \end{bmatrix} \right) = \ker \left(- \begin{bmatrix} 3 + i & -5 \\ 3 + i & -5 \end{bmatrix} \right) = \ker \left(\begin{bmatrix} 3 + i & -5 \\ 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 3 + i \\ 0 \end{bmatrix} \right)$$



Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.21)

Complex Diagonalization: use the vectors from E_{2+3i} and E_{2-3i} as columns in S :

$$S = \begin{bmatrix} 5 & 5 \\ 3 - i & 3 + i \end{bmatrix}, S^{-1} = \frac{1}{10} \begin{bmatrix} 1 - 3i & 5i \\ 1 + 3i & -5i \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 2 + 3i & \\ & 2 - 3i \end{bmatrix}$$

Real (2 × 2)-Block Diagonalization: Use $\text{Imag}(E_{2+3i})$ and $\text{Real}(E_{2+3i})$ as columns in S :

$$S = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}, S^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix}, S^{-1}AS = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

Note: By convention we always use vectors from the eigenspace with POSITIVE imaginary part.



Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.23)

$$A_{(7.5.23)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \det(A) = 1, \text{trace}(A) = 0$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det \left(\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} \right) = (-\lambda)^3 + 1^3$$

Eigenvalues = Roots of $p_A(\lambda) = (1 - \lambda)(\lambda^2 + \lambda + 1)$, $\lambda_1 = 1$:

$$\lambda^2 + \lambda = -1 \Leftrightarrow (\lambda + 1/2)^2 - 1/4 = -1 \Leftrightarrow (\lambda + 1/2)^2 = -3/4 \Leftrightarrow \lambda_{\pm} = \frac{-1 \pm \sqrt{3}i}{2}$$



Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.23)

Eigenspaces:

$$E_1 = \ker(A - \lambda_1 I) = \ker \left(\text{rref} \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \right) = \ker \left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$E_{\lambda_+} = \ker \left(\frac{1}{2} \text{rref} \begin{bmatrix} 1 - i\sqrt{3} & 0 & 2 \\ 2 & 1 - i\sqrt{3} & 0 \\ 0 & 2 & 1 - i\sqrt{3} \end{bmatrix} \right) = \ker \left(\frac{1}{2} \begin{bmatrix} 2 & 0 & 1 + i\sqrt{3} \\ 0 & 2 & 1 - i\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\frac{1}{2} \begin{bmatrix} -1 - i\sqrt{3} \\ -1 + i\sqrt{3} \\ 2 \end{bmatrix} \right)$$

$$E_{\lambda_-} = \ker \left(\frac{1}{2} \text{rref} \begin{bmatrix} 1 + i\sqrt{3} & 0 & 2 \\ 2 & 1 + i\sqrt{3} & 0 \\ 0 & 2 & 1 + i\sqrt{3} \end{bmatrix} \right) = \ker \left(\frac{1}{2} \begin{bmatrix} 2 & 0 & 1 - i\sqrt{3} \\ 0 & 2 & 1 + i\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\frac{1}{2} \begin{bmatrix} -1 + i\sqrt{3} \\ -1 - i\sqrt{3} \\ 2 \end{bmatrix} \right)$$

 \mathbb{C} : Diagonalization / \mathbb{R} : (2 × 2)-Block Diagonalization:

$$S_{\mathbb{C}} = \begin{bmatrix} 1 & -1 - i\sqrt{3} & -1 + i\sqrt{3} \\ 1 & -1 + i\sqrt{3} & -1 - i\sqrt{3} \\ 1 & 2 & 2 \end{bmatrix}, S_{\mathbb{R}} = \begin{bmatrix} 1 & -\sqrt{3} & -1 \\ 1 & \sqrt{3} & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$S_{\mathbb{C}}^{-1}AS_{\mathbb{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1 + i\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{-1 - i\sqrt{3}}{2} \end{bmatrix}, S_{\mathbb{R}}^{-1}AS_{\mathbb{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$$



Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization

(7.5.x¹)

$$A_{(7.5.x^1)} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \det(A) = 3^3, \text{trace}(A) = 9$$

Characteristic Polynomial:

$$\rho_A(\lambda) = \det(A - \lambda I) = \det \left(\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right) = (3-\lambda)^3$$

Eigenvalues = Roots of $\rho_A(\lambda) = 3$, with *algebraic multiplicity 3*.**Eigenspaces:**

$$E_3 = \ker(A - 3I) = \ker \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

 $\lambda = 3$ has *geometric multiplicity 3* \Rightarrow the matrix is diagonalizable. (It's already diagonal!)

Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization

(7.5.x²)

$$A_{(7.5.x^2)} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \det(A) = 3^3, \text{trace}(A) = 9$$

Characteristic Polynomial:

$$\rho_A(\lambda) = \det(A - \lambda I) = \det \left(\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right) = (3-\lambda)^3$$

Eigenvalues = Roots of $\rho_A(\lambda) = 3$, with *algebraic multiplicity 3*.**Eigenspaces:**

$$E_3 = \ker(A - 3I) = \ker \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

 $\lambda = 3$ has *geometric multiplicity 2* \Rightarrow the matrix is **not diagonalizable**.

This matrix is in “Jordan Form” (or “Jordan Canonical Form”, or “Jordan Normal Form”) — for details see [MATH 524 (NOTES#8)].



Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization

(7.5.x³)

$$A_{(7.5.x^3)} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \det(A) = 3^3, \text{trace}(A) = 9$$

Characteristic Polynomial:

$$\rho_A(\lambda) = \det(A - \lambda I) = \det \left(\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right) = (3-\lambda)^3$$

Eigenvalues = Roots of $\rho_A(\lambda) = 3$, with *algebraic multiplicity 3*.**Eigenspaces:**

$$E_3 = \ker(A - 3I) = \ker \left(\text{rref} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

 $\lambda = 3$ has *geometric multiplicity 1* \Rightarrow the matrix is **not diagonalizable**.

This matrix is in “Jordan Form” (or “Jordan Canonical Form”, or “Jordan Normal Form”) — for details see [MATH 524 (NOTES#8)].



Summary of Findings

Matrix	E-values	# E-vectors	Diagonalizable	Det(A)	Trace	Invertible
$A_{(7.5.13)}$	$\pm 2i$	2	yes	4	0	yes
$A_{(7.5.15)}$	$2 \pm i$	2	yes	5	4	yes
$A_{(7.5.17)}$	$3 \pm 4i$	2	yes	25	6	yes
$A_{(7.5.21)}$	$2 \pm 3i$	2	yes	13	4	yes
$A_{(7.5.23)}$	$1, \frac{-1 \pm i\sqrt{3}}{2}$	3	yes	1	0	yes
$A_{(7.5.x^1)}$	3, 3, 3	3	yes	27	9	yes
$A_{(7.5.x^2)}$	3, 3, 3	2	no	27	9	yes
$A_{(7.5.x^3)}$	3, 3, 3	1	no	27	9	yes

Diagonalizability is determined by the existence of an eigenbasis — sufficiently many linearly independent eigenvectors.**Invertibility** is determined by the eigenvalues — in particular whether 0 is an eigenvalue or not. *No zero eigenvalue \Leftrightarrow invertible.*

Diagonalizability and Invertibility

Example (Diagonalizable, Not Invertible)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda \in \{1, 1, 0\}$, $\det(A) = 0$, already diagonal.

Example (Not Diagonalizable, Not Invertible)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda \in \{1, 1, 0\}$, $\det(A) = 0$.

$$E_0 = \text{span} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right), \quad E_1 = \ker \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$



Diagonalizability and Invertibility

Example (Diagonalizable, Invertible)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda \in \{1, 1, 1\}$, $\det(A) = 1$, already diagonal.

Example (Not Diagonalizable, Invertible)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda \in \{1, 1, 1\}$, $\det(A) = 1$.

$$E_1 = \ker \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

