Math 254: Introduction to Linear Algebra

Notes #7.E — Examples

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Outline

1. Examples 7.5
   - (2×2)-Examples
   - (3×3)-Examples

Complex Diagonalization vs. Real (2 × 2) Block Diagonalization (7.5.13)

$$A_{\textbf{(7.5.13)}} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix},\ \det(A) = 4,\ \mathrm{trace}(A) = 0$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & -4 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 + 4$$

Eigenvalues = Roots of $p_A(\lambda) = \pm 2i$.

Eigenspaces:

$$E_{2i} = \ker(A - 2iI) = \ker\left(\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix}\right) = \ker\left(\begin{bmatrix} -2i & -4 \\ 2i & 4 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} -2i & -4 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 2i \\ 1 \end{bmatrix}\right)$$

$$E_{-2i} = \ker(A + 2iI) = \ker\left(\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 2i & -4 \\ 2i & -4 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 2i & -4 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} -2i \\ 1 \end{bmatrix}\right)$$

Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.13)

Complex Diagonalization: use the vectors from E_{2i} and E_{-2i} as columns in S:

$$S = \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix}, \quad S^{-1} = \frac{1}{4}\begin{bmatrix} -i & 2 \\ i & 2 \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 2i & \\ & -2i \end{bmatrix}$$

Real (2 × 2)-Block Diagonalization: Use $\mathrm{Imag}(E_{2i})$ and $\mathrm{Real}(E_{2i})$ *in that order* as columns in S:

$$S = \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ & 1 \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

Note: By convention we always use vectors from the eigenspace with POSITIVE imaginary part.

If we use $\mathrm{Imag}(E_{-2i})$ and $\mathrm{Real}(E_{-2i})$ as columns in S:

$$S = \begin{bmatrix} 2 & \\ & -1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ & -1 \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

the off-diagonal minus sign ends up in the wrong ("non-convention") location.

Complex Diagonalization vs. Real 2 × 2 Block Diagonalization (7.5.15)

$$A_{(7.5.15)} = \begin{bmatrix} 0 & 1 \\ -5 & 4 \end{bmatrix}, \ \det(A) = 5, \ \mathrm{trace}(A) = 4$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -5 & 4-\lambda \end{bmatrix}\right) = \lambda(\lambda-4)+5 = \lambda^2 - 4\lambda + 5$$

Eigenvalues = Roots of $p_A(\lambda)$:

$$\lambda^2 - 4\lambda = -5 \Leftrightarrow (\lambda-2)^2 - 4 = -5 \Leftrightarrow (\lambda-2)^2 = -1 \Leftrightarrow \lambda_\pm = 2 \pm \mathbf{i}$$

Eigenspaces:

$$E_{\lambda_+} = \ker(A - \lambda_+ I) = \ker\left(\begin{bmatrix} -2-i & 1 \\ -5 & 2-i \end{bmatrix}\right) = \ker\left(\begin{bmatrix} -5 & 2-i \\ -5 & 2-i \end{bmatrix}\right) = \ker\left(\begin{bmatrix} -5 & 2-i \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2+i \end{bmatrix}\right)$$

$$E_{\lambda_-} = \ker(A - \lambda_- I) = \ker\left(\begin{bmatrix} -2+i & 1 \\ -5 & 2+i \end{bmatrix}\right) = \ker\left(\begin{bmatrix} -5 & 2+i \\ -5 & 2+i \end{bmatrix}\right) = \ker\left(\begin{bmatrix} -5 & 2+i \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2-i \end{bmatrix}\right)$$

Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.15)

Complex Diagonalization: use the vectors from E_{2+i} and E_{2-i} as columns in S:

$$S = \begin{bmatrix} 1 & 1 \\ 2+i & 2-i \end{bmatrix}, \; S^{-1} = \frac{1}{2}\begin{bmatrix} 1+2i & -i \\ 1-2i & i \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 2+i & \\ & 2-i \end{bmatrix}$$

Real (2 × 2)-Block Diagonalization: Use $\mathrm{Imag}(E_{2+i})$ and $\mathrm{Real}(E_{2+i})$ as columns in S:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \; S^{-1} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

Note: By convention we always use vectors from the eigenspace with POSITIVE imaginary part.

Complex Diagonalization vs. Real 2 × 2 Block Diagonalization (7.5.17)

$$A_{(7.5.17)} = \begin{bmatrix} 5 & 4 \\ -5 & 1 \end{bmatrix}, \ \det(A) = 25, \ \mathrm{trace}(A) = 6$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 5-\lambda & 4 \\ -5 & 1-\lambda \end{bmatrix}\right) = (5-\lambda)(1-\lambda) + 20 = \lambda^2 - 6\lambda + 25$$

Eigenvalues = Roots of $p_A(\lambda)$:

$$\lambda^2 - 6\lambda = -25 \Leftrightarrow (\lambda - 3)^2 - 9 = -25 \Leftrightarrow (\lambda - 3)^2 = -16 \Leftrightarrow \lambda_\pm = \mathbf{3 \pm 4i}$$

Eigenspaces:

$$E_{\lambda_+} = \ker(A - \lambda_+ I) = \ker\left(\begin{bmatrix} 2-4i & 4 \\ -5 & -2-4i \end{bmatrix}\right) = \ker\left(-\begin{bmatrix} 5 & 2+4i \\ 5 & 2+4i \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 5 & 2+4i \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 2 \\ -1+2i \end{bmatrix}\right)$$

$$E_{\lambda_-} = \ker(A - \lambda_- I) = \ker\left(\begin{bmatrix} 2+4i & 4 \\ -5 & -2+4i \end{bmatrix}\right) = \ker\left(-\begin{bmatrix} 5 & 2-4i \\ 5 & 2-4i \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 5 & 2-4i \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 2 \\ -1-2i \end{bmatrix}\right)$$

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Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.17)

Complex Diagonalization: use the vectors from E_{3+4i} and E_{3-4i} as columns in S:

$$S = \begin{bmatrix} 2 & 2 \\ -1+2i & -1-2i \end{bmatrix}, \quad S^{-1} = \frac{1}{8}\begin{bmatrix} 2-i & -2i \\ 2+i & 2i \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 3+4i & \\ & 3-4i \end{bmatrix}$$

Real (2 × 2)-Block Diagonalization: Use $\mathrm{Imag}(E_{3+4i})$ and $\mathrm{Real}(E_{3+4i})$ as columns in S:

$$S = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}, \quad S^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$

Note: By convention we always use vectors from the eigenspace with POSITIVE imaginary part.

Complex Diagonalization vs. Real (2 × 2) Block Diagonalization (7.5.21)

$$A_{(7.5.21)} = \begin{bmatrix} 11 & -15 \\ 6 & -7 \end{bmatrix}, \ \det(A) = 13, \ \mathrm{trace}(A) = 4$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 11-\lambda & -15 \\ 6 & -7-\lambda \end{bmatrix}\right) = (11-\lambda)(-7-\lambda) + 90 = \lambda^2 - 4\lambda + 13$$

Eigenvalues = Roots of $p_A(\lambda)$:

$$\lambda^2 - 4\lambda = -13 \Leftrightarrow (\lambda-2)^2 - 4 = -13 \Leftrightarrow (\lambda-2)^2 = -9 \Leftrightarrow \lambda_\pm = 2 \pm 3i$$

Eigenspaces:

$$E_{\lambda_+} = \ker(A - \lambda_+ I) = \ker\left(3\begin{bmatrix} 3-i & -5 \\ 2 & -3-i \end{bmatrix}\right) = \ker\left(-\begin{bmatrix} 3-i & -5 \\ 3-i & -5 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 3-i & -5 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 5 \\ 3-i \end{bmatrix}\right)$$

$$E_{\lambda_-} = \ker(A - \lambda_- I) = \ker\left(3\begin{bmatrix} 3+i & -5 \\ 2 & -3+i \end{bmatrix}\right) = \ker\left(-\begin{bmatrix} 3+i & -5 \\ 3+i & -5 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 3+i & -5 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 5 \\ 3+i \end{bmatrix}\right)$$

Complex Diagonalization vs. Real (2×2)-Block Diagonalization (7.5.21)

Complex Diagonalization: use the vectors from E_{2+3i} and E_{2-3i} as columns in S:

$$S = \begin{bmatrix} 5 & 5 \\ 3-i & 3+i \end{bmatrix}, \quad S^{-1} = \frac{1}{10}\begin{bmatrix} 1-3i & 5i \\ 1+3i & -5i \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 2+3i & \\ & 2-3i \end{bmatrix}$$

Real (2×2)-Block Diagonalization: Use $\mathrm{Imag}(E_{2+3i})$ and $\mathrm{Real}(E_{2+3i})$ as columns in S:

$$S = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}, \quad S^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -5 \\ 1 & 0 \end{bmatrix}, \quad S^{-1}AS = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

Note: By convention we always use vectors from the eigenspace with POSITIVE imaginary part.

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Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.23)

$$A_{(7.5.23)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ \det(A) = 1, \ \mathrm{trace}(A) = 0$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}\right) = (-\lambda)^3 + 1^3$$

Eigenvalues = Roots of $p_A(\lambda) = (1-\lambda)(\lambda^2+\lambda+1)$, $\lambda_1 = 1$:

$$\lambda^2+\lambda = -1 \Leftrightarrow (\lambda+1/2)^2 - 1/4 = -1 \Leftrightarrow (\lambda+1/2)^2 = -3/4 \Leftrightarrow \lambda_\pm = \frac{-1 \pm \sqrt{3}i}{2}$$

Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.23)

Eigenspaces:

$$E_1 = \ker(A - \lambda_1 I) = \ker\left(\text{rref}\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\right) = \ker\left(\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$E_{\lambda_+} = \ker\left(\frac{1}{2}\text{rref}\begin{bmatrix} 1 - i\sqrt{3} & 0 & 2 \\ 2 & 1 - i\sqrt{3} & 0 \\ 0 & 2 & 1 - i\sqrt{3} \end{bmatrix}\right) = \ker\left(\frac{1}{2}\begin{bmatrix} 2 & 0 & 1 + i\sqrt{3} \\ 0 & 2 & 1 - i\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\frac{1}{2}\begin{bmatrix} -1 - i\sqrt{3} \\ -1 + i\sqrt{3} \\ 2 \end{bmatrix}\right)$$

$$E_{\lambda_-} = \ker\left(\frac{1}{2}\text{rref}\begin{bmatrix} 1 + i\sqrt{3} & 0 & 2 \\ 2 & 1 + i\sqrt{3} & 0 \\ 0 & 2 & 1 + i\sqrt{3} \end{bmatrix}\right) = \ker\left(\frac{1}{2}\begin{bmatrix} 2 & 0 & 1 - i\sqrt{3} \\ 0 & 2 & 1 + i\sqrt{3} \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\frac{1}{2}\begin{bmatrix} -1 + i\sqrt{3} \\ -1 - i\sqrt{3} \\ 2 \end{bmatrix}\right)$$

$\mathbb{C}$:Diagonalization / $\mathbb{R}$:(2 × 2)-Block Diagonalization:

$$S_{\mathbb{C}} = \begin{bmatrix} 1 & -1 - i\sqrt{3} & -1 + i\sqrt{3} \\ 1 & -1 + i\sqrt{3} & -1 - i\sqrt{3} \\ 1 & 2 & 2 \end{bmatrix}, \quad S_{\mathbb{R}} = \begin{bmatrix} 1 & -\sqrt{3} & -1 \\ 1 & \sqrt{3} & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$S_{\mathbb{C}}^{-1} A S_{\mathbb{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 0 & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}, \quad S_{\mathbb{R}}^{-1} A S_{\mathbb{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{bmatrix}$$

Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.x[1])

$$A_{(7.5.x^1)} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \ \det(A) = 3^3, \ \mathrm{trace}(A) = 9$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)^3$$

Eigenvalues = Roots of $p_A(\lambda) = 3$, with *algebraic multiplicity 3*.

Eigenspaces:

$$E_3 = \ker(A - 3I) = \ker\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$\lambda = 3$ has *geometric multiplicity 3* $\Rightarrow$ the matrix is diagonalizable.
(It's already diagonal!)

Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.x²)

$$A_{(7.5.x^2)} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \ \det(A) = 3^3, \ \mathrm{trace}(A) = 9$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)^3$$

Eigenvalues = Roots of $p_A(\lambda) = 3$, with *algebraic multiplicity 3*.

Eigenspaces:

$$E_3 = \ker(A - 3I) = \ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

$\lambda = 3$ has *geometric multiplicity 2* ⇒ the matrix is **not diagonalizable**.

This matrix is in "Jordan Form" (or "Jordan Canonical Form", or "Jordan Normal Form") — for details see [MATH 524 (NOTES#8)].

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Complex Diagonalization vs. Real (2 × 2)-Block Diagonalization (7.5.x³)

$$A_{(7.5.x^3)} = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \; \det(A) = 3^3, \; \mathrm{trace}(A) = 9$$

Characteristic Polynomial:

$$p_A(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 1 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)^3$$

Eigenvalues = Roots of $p_A(\lambda) = 3$, with *algebraic multiplicity 3*.

Eigenspaces:

$$E_3 = \ker(A - 3I) = \ker\left(\mathrm{rref}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

$\lambda = 3$ has *geometric multiplicity 1* $\Rightarrow$ the matrix is **not diagonalizable**.

This matrix is in “Jordan Form” (or “Jordan Canonical Form”, or “Jordan Normal Form”) — for details see [MATH 524 (NOTES#8)].

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Summary of Findings

Matrix	E-values	# E-vectors	Diagonalizable	Det(A)	Trace	Invertible
$A_{(7.5.13)}$	$\pm 2i$	2	yes	4	0	yes
$A_{(7.5.15)}$	$2 \pm i$	2	yes	5	4	yes
$A_{(7.5.17)}$	$3 \pm 4i$	2	yes	25	6	yes
$A_{(7.5.21)}$	$2 \pm 3i$	2	yes	13	4	yes
$A_{(7.5.23)}$	$1,\ \frac{-1 \pm i\sqrt{3}}{2}$	3	yes	1	0	yes
$A_{(7.5.x^1)}$	3, 3, 3	3	yes	27	9	yes
$A_{(7.5.x^2)}$	3, 3, 3	2	**no**	27	9	yes
$A_{(7.5.x^3)}$	3, 3, 3	1	**no**	27	9	yes

Diagonalizabilty is determined by the existence of an eigenbasis — sufficiently many linearly independent eigenvectors.

Invertibility is determined by the eigenvalues — in particular whether 0 is an eigenvalue or not. *No zero eigenvalue ⇔ invertible.*

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Diagonalizabilty and Invertibility

Example (Diagonalizable, Not Invertible)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda \in \{1, 1, 0\}$, $\det(A) = 0$, already diagonal.

Example (Not Diagonalizable, Not Invertible)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda \in \{1, 1, 0\}$, $\det(A) = 0$.

$$E_0 = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right), \quad E_1 = \ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

Diagonalizabilty and Invertibility

Example (Diagonalizable, Invertible)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda \in \{1, 1, 1\}$, $\det(A) = 1$, already diagonal.

Example (Not Diagonalizable, Invertible)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\lambda \in \{1, 1, 1\}$, $\det(A) = 1$.

$$E_1 = \ker\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

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