1	2	3	4	5	6	7	8	9
RedID								

Math 254, Spring 2017
Final — Practice Test

Tools: Brain/Pen/Pencil/Eraser/Paper.
Rules: This is just a test-test; see below:

v 2017.5.11.0

I, ____________________________, understand that this is a practice exam; which is provided for adults and entertainment purposes only. Any resemblence to a real test is purely intentional, but not guaranteed.

____
Signature (REQUIRED to continue (yeah, that will be enforced?!?))

Suggestions:

- Present your solutions using standard notation in an easy-to-read format. It is your job to yourself you did the problem correctly...
- ***Your answers MUST logically follow from your calculations in order to be considered! ("Miracle solutions" $\Rightarrow$ zero points.)***
- If you are a computer science major, perform all calculations in the hexadecimal basis.

Note:

- There is a high probablility that the actual test will have slightly fewer questions.
- **Hint:** The first 3 problems are extra highly recommended.

Problem	Pts Possible	Pts Scored
1	**000**	
⋮	**000**	
25	**000**	
Total	**000**	

1. Consider the matrices*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$E = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \quad F = \begin{bmatrix} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}, \quad G = \begin{bmatrix} 9 & 1 & 0 & 0 & 0 \\ 0 & 9 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 \end{bmatrix}, \quad H = \begin{bmatrix} 11 & 1 & 0 & 0 & 0 \\ 0 & 11 & 1 & 0 & 0 \\ 0 & 0 & 12 & 1 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}$$

For each matrix:

(a) What are the eigenvalues, with *algebraic multiplicities*?

(b) For each matrix, and corresponding eigenvalue(s), find the Eigenspace(s); indicate the dimension of the Eigenspace(s) (and thus the geometric multiplicity of the associated eigenvalue)

(c) What matrices are invertible / non-invertible? Why/Why not?

(d) Do you see any patterns here?

(e) For the Eigenspaces, state the dimension of the orthogonal complements $E_\lambda^\perp$; then find bases for $E_\lambda^\perp$

* Not every matrix can be diagonalized, but every matrix can (via a similarity transform) be put into this form, known as the *Jordan Normal Form*, or *Jordan Canonical Form*.

2. Let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

(a) i. Find an orthonormal basis for the subspace $V = \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$?
   ii. $\dim(V) =$

(b) Project the vector $\vec{x}$ onto:
   i. The vector $\vec{v}_1$
   ii. The subspace V, giving $\vec{x}^{\|}$

(c) Next, reflect the vector $\vec{x}$ in
   i. (25 pts.) The origin (the point $(0, 0, 0)$, or vector $\vec{0}$)
   ii. (25 pts.) The vector $\vec{v}_1$
   iii. (25 pts.) The subspace V

(d) Consider $V^{\perp}$, the orthonormal complement of V
   i. $\dim(V^{\perp}) =$
   ii. The following problems may be messy:
      A. Find $V^{\perp}$
      B. Find an orthonomal basis for $V^{\perp}$
      C. Project $\vec{x}$ onto $V^{\perp}$
      D. Reflect $\vec{x}$ in $V^{\perp}$

3. Consider the matrix, $A \in \mathbb{R}^{r \times c}$:

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and the linear transformation $T(\vec{x}) = A\vec{x}$.

(a) What can you say about the relations between

i. $\dim(\mathrm{im}(A))$, and $\dim(\ker(A))$

ii. $\dim(\mathrm{im}(A))$, and $\dim(\mathrm{im}(A)^\perp)$

iii. $\dim(\ker(A))$, and $\dim(\ker(A)^\perp)$

The following sketch may be useful (or useless):

$$\begin{bmatrix} \mathbb{R}^m \\ \\ \ker(A)^\perp \\ \ker(A) \end{bmatrix} \overset{T(\vec{x})}{\longrightarrow} \begin{bmatrix} \mathbb{R}^n \\ \mathrm{im}(A)^\perp \\ \mathrm{im}(A) \\ \vec{0} \end{bmatrix}$$

(b) Find Bases for

i. $\mathrm{im}(A)$.

ii. $\ker(A)$.

iii. $\mathrm{im}(A)^\perp$.

iv. $\ker(A)^\perp$.

(c) Find Orthonormal Bases for (and forever curse Jørgen Pedersen Gram and Erhard Schmidt)

i. $\mathrm{im}(A)$.

ii. $\ker(A)$.

iii. $\mathrm{im}(A)^\perp$.

iv. $\ker(A)^\perp$.

(d) Use the QR-factorization to relate the answers $\{$(b).i–iv$\}$ and $\{$(c).i–iv$\}$.

4. (§3.1) Find vectors that span the *kernel* of

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}$$

5. (§3.1) Find vectors that span the *image* of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

6. (§3.1, §3.2) *(i)* Are the column vectors in

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

linearly independent? Why/Why Not? — Give *(ii)* a "mechanical" (computational) answer, and *(iii)* a "theoretical" answer which does not require any computations.

7. (§2.4, §3.2, §3.3) *(i)* Determine whether the given vectors are linearly independent:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

(ii) If we use these vectors as the columns in a matrix A, will the matrix be invertible? *(iii)* Why/Why not? *(iv)* If the inverse exists, compute it.

8. (§3.2) Consider the 5×4 matrix

$$A = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix},$$

we are told that the vector

$$\vec{n}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

is in the *kernel* of A. *(i)* Find a linear relation between the vectors; and *(ii)* write $\vec{v}_4$ as a linear combination of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$.

9. (§3.3) Find the reduced row echelon form of A; then find a basis for $\mathrm{im}(A)$, and a basis $\ker(A)$, where

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 1 \end{bmatrix}.$$

10. (§3.3) Determine whether the following vectors form a basis of $\mathbb{R}^4$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix}.$$

11. (§3.3) *(i)* Find a basis of the subspace, V of $\mathbb{R}^4$ defined by the equation

$$2x_1 - x_2 + 2x_3 + 4x_4 = 0.$$

then *(ii)* find a basis for the orthogonal complement $V^\perp$.

12. (§3.4) Determine whether the vector $\vec{x}$ is in $V = \text{span}(\vec{v}_1, \dots, \vec{v}_m)$. If $\vec{x} \in V$, find the coordinates of $\vec{x}$ with respect to the basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$ of V, and write the coordinate vector $[\vec{x}]_{\mathfrak{B}}$:

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

13. (§3.4) Find the matrix B of the linear transformation $T(\vec{x}) = A\vec{x}$, with respect to the basis $\mathfrak{B} = (\vec{v}_1, \dots, \vec{v}_m)$.

$$A = \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & = 2 & 4 \end{bmatrix}; \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

14. (§2.2, §3.1, §3.2, §3.3) Describe *(i)* the *image*, $\text{im}(A)$; and *(ii)* the *kernel*, $\ker(A)$ of the transformation $T(\vec{x}) = A\vec{x}$ geometrically, where

$$T(\vec{x}) = \left\{ \begin{array}{c} \text{Orthogonal projection onto the plane} \\ \{x + 2y + 3z = 0\} \text{ in } \mathbb{R}^3 \end{array} \right\}.$$

what is *(iii)* $\dim(\text{im}(A))$, and *(iv)* $\dim(\ker(A))$? Then find a basis for *(v)* $\text{im}(A)$ and *(vi)* $\ker(A)$

15. (§5.1) Find the orthogonal projections of $\vec{e}_1$ onto the subspaces W, and $W^\perp$ of $\mathbb{R}^4$, where

$$W = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right), \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

16. (§5.2) Perform the Gram-Schmidt process on the sequence of vectors given:

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 18 \\ 0 \\ 0 \end{bmatrix}.$$

17. (§5.2) Find an orthonormal basis for the image of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 2 & -2 & 0 \end{bmatrix}.$$

18. (§5.3) Are the given matrices Orthogonal?

$$A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$$

19. (§5.3)

    (a) Consider an $n \times m$ matrix A such that $A^T A = I_m$. Is is necessarily true that $AA^T = I_n$? (Explain!)

    (b) Consider an $n \times n$ matrix A such that $A^T A = I_n$. Is is necessarily true that $AA^T = I_n$? (Explain!)

20. (§6.1) Use the determinant to decide whether the matrices are invertible:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix}$$

21. (§6.1) For which values of k, ℓ are the matrices invertible?

$$A = \begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} \ell & 3 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

22. (§6.1–§6.2) Given the value of $\det(A)$, what is the value of

    (a) $\det(A^T)$

    (b) $\det(A^{-1})$

    (c) $\det(A^2)$

    (d) $\det(A^T A^{-1})$

23. (§6.2) Ponder the matrix

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

    compute the determinant using

    (a) The Combinatorial "Pattern" Method

    (b) Row Reductions (Gaussian Elimination)

    (c) The Laplace co-factor Method

24. (§6.2) Given the values of $\det(A)$ and $\det(B)$ is the value of $\det(AB)$ always defined? (Explain) If/when it is defined, what is it?

25. (§6.2) Explain how the Row Reduction operations impact the value of the determinant:

    (a) Row swap

    (b) Row scaling

    (c) Row addition

    (d) In the light of 9.(a) what does (a)–(c) imply for column operations?