# Math 254 — Study Guide(s)

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## Introduction / Disclaimer

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## Midterm #1 — Material

### Linear Equations, Matrices, Vectors, Solutions of Linear systems, Matrix Algebra, Linear Transformations (w/Geometrical Interpretations), Matrix-Vector and Matrix-Matrix Multiplications, Inverse Linear Transformations, Inverse of a Matrix

- Be able to *Interpret* solutions of linear systems of 3 variables as intersections of planes.
- *Know* that linear systems can have `no`, `one`, or `infinitely many` solutions.
- Be able to *Perform* basic row-operations to determine all solutions of linear systems.
- Know basic language and concepts:
  - Matrices, vectors, and their components
  - Matrix types: square, diagonal, triangular, zero, identity
  - The collection of all $n$-vectors, denoted $\mathbb{R}^n$ is a vector space
- Vector addition
- Know the difference between the Coefficient matrix, and the Augmented matrix; their uses in the solution of linear systems
- Know how to use elimination to identify leading and non-leading (a.k.a. *free variables*), and when necessary introduce parameters to express *all* solutions of linear systems.
- Know what `Reduced--Row--Echelon--Form` (`RREF`) of a Matrix is, and how to achieve it using elementary row operations.

- Know what the **Rank** of a Matrix is; and its connection to total/leading/free variables and the number of solutions of a linear system
- Know the Fundamentals of:
  - Matrix-Vector algebra
  - Vector-Vector Dot Product / Inner Product
  - Matrix-Vector Product: Linear combinations
- Know the functional definitions of the fundamental vector algebra operations
- Know the two ways to compute dot (inner) products of vectors
- Be able to compute the length (norm) of a vector
- Know what Unit vectors are
- Understand Orthogonality of Vectors, and the relation to the dot product
- Understand how a Matrix[-Vector product] defines a *Linear Transformation*
- Understand the concept of *The Inverse of a Linear Transformation*
- Be able to compute *The Inverse of a (small-ish) Matrix*
- Understand the concept of, and identify *The Matrix of a Linear Transformation*
- Know the *Matrix Forms* for scaling, rotation, reflection, shear.
- Be the Inter-Galactic Grand Emperor of *Orthogonal Projections:* know the formula for projection onto a line, and the geometric interpretations.
- Be able to perform *Reflections Across a Line:* know the formula and relation to orthogonal projections onto a line
- Understand the Computational, and Linear Transformation Points-Of-View of Matrix Products
- Know that Matrix Multiplication is *Non-Commutative*
- Know that it is *not* always possible to multiply two matrices
- Know the connection between invertibility, the form of $\text{rref}(A)$, and the rank of $A$:
  $A \in \mathbb{R}^{n\times n}$ is invertible $\Leftrightarrow \text{rref}(A) = I_n \Leftrightarrow \text{rank}(A) = n$.
- Know the definition of, and be able to compute the kernel of a matrix: $\ker(A) = \{\, \vec{x} \,:\, A\vec{x} = \vec{0}\,\}$
- Given an invertible matrix $A$, perform row-operations to find the inverse:
  $$\left[\begin{array}{c|c} A & I_n \end{array}\right] \rightsquigarrow \left[\begin{array}{c|c} I_n & A^{-1} \end{array}\right]$$
- Be able to compute the determinant, $\det(A)$ and know its Geometrical Interpretation for $A \in \mathbb{R}^{2\times 2}$.

# Midterm #2 — Additional Material

## Rank, Dimension of Subspaces of $\mathbb{R}^n$, Image, and Kernel, Linear Independence, Bases, Coordinates

- Given a matrix $A \in \mathbb{R}^{n \times m}$, what is
  - $\text{rank}(A)$ (definition)
  - How do you determine $\text{rank}(A)$ (computation)
  - What are the limits on the values of $\text{rank}(A)$
  - What is the image of $A$, $\text{im}(A)$. (definition)
  - How do you determine a basis for $\text{im}(A)$ (computation)
  - What is the kernel of $A$, $\ker(A)$. (definition)
  - How do you determine a basis for $\ker(A)$ (computation)
  - What is the dimension of a subspace $V \subset \mathbb{R}^n$, $\dim(V)$?
  - What can you say about $\dim(\text{im}(A))$, and $\dim(\ker(A))$.
- Given a square matrix $A \in \mathbb{R}^{n \times n}$
  - When is $A$ invertible?
  - How does the invertibility of $A$ relate to $\text{im}(A)$, and $\ker(A)$.
  - How does the invertibility of $A$ relate to the linear independence (or dependence) of the columns of $A$?
- Consider subspaces $V$ of $\mathbb{R}^n$
  - What types of subspaces are there?
  - Is there a subspace which is finite?
  - What is the maximal number of linearly independent vectors you can find in $\mathbb{R}^n$.
- Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ associated with the orthogonal projection onto a subspace $V$ described by a set of equations $A\vec{v} = \vec{0}$.
  - What is the subspace $V$ (in relation to the matrix $A$).
  - What is the subspace $V$ (in relation to the linear transform $T$)
  - What is $\dim(\text{im}(T))$?
  - What is $\dim(\ker(T))$?
  - Find a basis for $\text{im}(T)$
  - Find a basis for $\ker(T)$
  - What is the subspace $V^\perp$? How does it relate to $V$ (geometrically), and $\text{im}(T)$ / $\ker(T)$?
- Given a collection of vectors, $(\vec{v}_1, \ldots, \vec{v}_m) \in \mathbb{R}^n$ — maybe the columns of a matrix(?)
  - How do you determine if they are linearly independent.

- If they are linearly dependent, how do you find the corresponding linear relations? and how are they related to the kernel?
- if $m < n$, what (if anything) can you say about the linear dependence of the vectors?
- if $m > n$, what (if anything) can you say about the linear dependence of the vectors?
- Under what circumstances do the vectors form a basis for a subspace $V$ of $\mathbb{R}^n$?
- Given a vector $\vec{x} \in V$, and a basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of $V$, determine the coordinates of $\vec{x}$ with respect to the basis, $[\vec{x}]_{\mathfrak{B}}$
- Given a basis $\mathfrak{B} = (\vec{v}_1, \ldots, \vec{v}_m)$ of $V$ (a subspace of $\mathbb{R}^n$), and the coordinates of $\vec{x}$ with respect to the basis, $[\vec{x}]_{\mathfrak{B}}$, find $\vec{x} \in \mathbb{R}^n$.
- Given an orthogonal projection onto a subspace $V$ of $\mathbb{R}^n$, what is a basis ($\mathfrak{B}$) which makes the description of the projection as simple as possible in $\mathfrak{B}$-coordinates? What does the transformation matrix look like in $\mathfrak{B}$-coordinates?
- Given a reflection across a subspace $V$ of $\mathbb{R}^n$, what is a basis ($\mathfrak{B}$) which makes the description of the projection as simple as possible in $\mathfrak{B}$-coordinates? What does the transformation matrix look like in $\mathfrak{B}$-coordinates?
- What do we hope to achieve by changing coordinates?

## Midterm #3 — Additional Material

**§5.1** Orthogonal Projections and Orthonormal Bases

- Understand the concept of *Orthonormality*: {{*Orthogonality + unit vectors*}}
- Be able to compute the Projection onto any subspace $V$ (with $\dim(V) > 1$), using an orthonormal basis.
- Be comfortable with the use of the *Orthogonal Complement*, $V^\perp$ of a subspace $V$. How do $V$ and $V^\perp$ relate to the projection $\text{proj}_V(\vec{x}) = A_P\vec{x}$ onto the subspace $V$. {{*Here $A_p$ denotes the matrix which describes the orthogonal projection.*}} It it common to have a description of $V \subset \mathbb{R}^m$ in the form of the solutions to a linear system $A\vec{x} = \vec{0}$. Think about how the "puzzle-pieces" $\ker(A)$, $V$, $V^\perp$, (and their associated bases) relate to the columns, rows, and kernel-space vectors of the matrix $A$. {{*What tools do we need to build the matrix $A_P$?*}}

**§5.2** Gram-Schmidt Process and $QR$ Factorization {{*This is 99.999993% likely to show up in some form on Midterm #3, and the Final.*}}

- Know the *The Gram-Schmidt Orthogonalization Process, (GSOP)*. Can anything go wrong in the process? When/why does the process break down?
- How can the GSOP be used to compute *The QR-factorization* of a matrix $A$. What vectors go in the $Q$-matrix? What type of entries go where in the $R$-matrix? Is there a special "structure" to the $R$-matrix?

**§5.3** Orthogonal Transformations and Orthogonal Matrices

- Know what Orthogonal Transformations are; and their relation to Orthonormal Bases.

- Know the Properties of Orthogonal Matrices.
- Be able to perform an Orthogonal Projection using Orthonormal Basis you have constructed. {{*This is where a lot of things come together... Sounds like a good final/midterm question is "hiding" here.*}}

**§6.1** Determinants

- Know that a Square Matrix has a Non-Zero Determinant if and only if it is Invertible
- Be familiar with the Connection between the determinant of a $3 \times 3$ matrix and the Cross Product (especially for Engineering / Physics students)
- Be familiar with Definition of the Determinant using the Combinatorial "Pattern" approach, and be able to use this definition to compute determinants of sparse matrices (*i.e.* matrices that have LOTS of zero-entries)

**§6.2** Properties of the Determinant

- Know the Impact of Row Divisions/Swaps/Additions on the value of the Determinant
- Be familiar with computation of the Determinant of Products, Powers, Transposes, and Inverses of matrices:
  $\det(AB)$, $\det(A^k)$, $\det(A^T)$, and $\det(A^{-1})$
- Be able to compute the determinant using
  * Combinatorial "Pattern" approach,
  * Row Reductions,
  * Laplace (co-factor) Expansion Method ["Traditional" Way].

**§6.3** Determinant: Geometrical View, Cramer's Rule

- Know that $|\det(A)| = 1 \Leftrightarrow$ Orthogonal Matrix; *e.g.* rotation or reflection.
- Be familiar with the Interpretation of the determinant as an $m$-Volume; and/or an expansion factor.
- Remember to "Forget" about Cramer's Rule: In general, it is a terrible (excessively computationally intensive) way to solve linear systems.

## Final — Additional Material