Math 524: Linear Algebra

Notes #1 — Vector Spaces

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Outline

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Student Learning Targets, and Objectives

Target Properties of the Complex Numbers, $\mathbb{C}$

- **Objective** Know the definitions of, and be able to perform basic complex arithmetic (addition, multiplication, subtraction, division)
- **Objective** Be able to apply the properties of commutativity, associativity, additive and multiplicative identities and inverses, as well as the distributive property.

Target $\mathbb{R}^n$ and $\mathbb{C}^n$

- **Objective** Be able to define $\mathbb{R}^n$ and $\mathbb{C}^n$ as lists of length n, and to abstract to general fields, $\mathbb{F}^n$.
- **Objective** Be able to transfer the algebraic rules and properties from $\mathbb{R}$ and $\mathbb{C}$ ($\mathbb{F}$), to $\mathbb{F}^n$.

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Student Learning Targets, and Objectives

Target Vector Spaces

- **Objective** Be able to define a vector space in terms of its necessary operations, and properties.
- **Objective** Be able to understand the notation $\mathbb{F}^S$, and show that it is a vector space.
- **Objective** Be able to formally show the uniqueness of the additive identity and inverse.

Target Subspaces

- **Objective** Be able to apply the subspace conditions in order to show that a subset of a Vector space is (or is not) a Subspace

Target Sums and Direct Sums of Subspaces

- **Objective** Be able to apply the definitions to identify whether a sum of subspaces is a direct sum, or not.

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Introduction

We will follow the notation, and structure of Axler's *Linear Algebra Done Right*.

The first couple of lectures will fairly quickly cover material (mostly) familiar from [MATH 254] (or alternatives).

The goal is to shake off some mental "dust," and build a foundation of common notation and language.

Note that some new matrial will be "folded" into these lectures.

Time-Target: 3×75-minute lectures.

Math 254 ⇝ Math 524

One fairly significant difference between [MATH 254] and [MATH 524] is that we will state most of our results in terms of complex numbers $z \in \mathbb{C}$ rather than real numbers $x \in \mathbb{R}$. When there are differences behaviour/properties over $\mathbb{C}$ and $\mathbb{R}$, we carefully explore those.

$z = x + yi$, where $x, y \in \mathbb{R}$; and we view the real numbers as a special case of the complex numbers (where $y = 0$).

The added bonus is that we get *more general* results, which are "future-proofed" (for cases where we need complex numbers).

Additionally, [MATH 524] provides a *much more formal* and complete discussion of linear algebra.

Complex Numbers

Hopefully you have not forgotten all your encounters with complex numbers.

We quickly review / introduce the essentials of complex arithmetic that we need.

The complex numbers solve the "core problem" of assigning a value to $\sqrt{-1}$.

Following Euler$^{(1777)}$: $i = \sqrt{-1}$, $i^2 = -1$.

Note: Mathematicians tend to use $i = \sqrt{-1}$, whereas (electrical) engineers prefer $j = \sqrt{-1}$ (i being reserved for electrical current).

Complex Numbers :: Formal Definition

Definition (Complex Numbers)

- A **complex number** z is an ordered pair (a, b) where $a, b \in \mathbb{R}$; usually we write $z = a + b\,i$.
- The set of all complex numbers is denoted by $\mathbb{C}$:
$$\mathbb{C} = \{a + b\,i \;:\; a, b \in \mathbb{R}\}$$
- Rules for addition and multiplication $(a, b, c, d \in \mathbb{R})$
  - $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Complex Numbers :: Properties

Proofs by direct computation

There are no surprises when it comes to the properties of complex numbers (they are "inherited" from the real numbers + definition of complex addition/multiplication):

Properties (Complex Numbers)

Let $u, v, w \in \mathbb{C}$, then

- **commutativity**: $u + v = v + u$, and $uv = vu$;
- **associativity**: $(u + v) + w = u + (v + w)$, and $(uv)w = u(vw)$;
- **0** is the **additive identity** and **1** the **multiplicative identity**:
$$u + 0 = 0 + u = u, \quad v\,1 = 1\,v = 1v = v$$
- u has an **additive inverse**, *i.e.* $\exists! v$: $u + v = 0$, (v is unique)
- $u \neq 0$ has a **multiplicative inverse**, *i.e.* $\exists! v$: $uv = 1$, (v is unique)
- the **distributive property** holds:
$$u(v + w) = uv + uw$$

{Inverse(+), Inverse(*)} $\rightsquigarrow$ { Subtraction, Division }

Definition (Subtraction and Division)

Let $u, v \in \mathbb{C}$,

- Let $(-u)$ be the unique **additive inverse** of u,
$$u + (-u) = 0$$
- We define **subtraction** using the **additive inverse**:
$$u - v = u + (-v)$$
- Likewise for $u \neq 0$, let $(1/u)$ denote the unique **multiplicative inverse** of u,
$$u\,(1/u) = 1$$
- We define **division** using the **multiplicative inverse**:
$$u/v = u\,(1/v)$$

Real and/or Complex? $\rightsquigarrow \mathbb{F}$

$x \in \mathbb{R}$ and $z \in \mathbb{C}$ are **scalars** (single numbers).

Throughout our discussion we will use the notation $y \in \mathbb{F}$, where $\mathbb{F}$ can be either $\mathbb{C}$ or $\mathbb{R}$ (in such a case, the results are true for both complex and real entries).

Why $\mathbb{F}$??? Both $\mathbb{R}$ and $\mathbb{C}$ are *fields*:

Definition (Field (Thanks "Aunt Wiki"))

In mathematics, a **field** is a **set** on which addition, subtraction, multiplication, and division are defined, and behave as the corresponding operations on rational and real numbers do. A field is thus a fundamental algebraic structure, which is widely used in (abstract) algebra [MATH 320, MATH 520], number theory [MATH 522] and many other areas of mathematics.

https://en.wikipedia.org/wiki/Field_(mathematics)

Lists (n-tuples)

Definition (list, length)

Let $n > 0$ be a positive integer ($n \in \mathbb{Z}^+$). A **list** of **length** n is an ordered collection of n elements. Here, we write them separated by commas and surrounded by parenthesis‡:

$$(x_1, x_2, \ldots, x_n)$$

Two lists are equal if and only if they have the same lengths, and the same elements in the same order.

‡ computer scientists can think of it as some form of "container class." Python uses `(...)` for immutable "tuples" and `[...]` for "lists"...

In this class (almost) all our lists have *finite length*.

The empty list — ( ) — is a list of length 0.

$\mathbb{F}^n$ $(\mathbb{R}^n, \mathbb{C}^n)$

Definition ($\mathbb{F}^n$)

$\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) \ : \ x_j \in \mathbb{F}\}$$

For $(x_1, \ldots, x_n) \in \mathbb{F}^n$ we say that x_j is the j^{th} **coordinate** of $(x_1, \ldots, x_n)$.

When $\mathbb{F} = \mathbb{R}$, this matches our [MATH 254] definitions of $\mathbb{R}^n$.

Addition in $\mathbb{F}^n$

Definition (Addition in $\mathbb{F}^n$)

Addition in $\mathbb{F}^n$ is defined element-by-element:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

Property (Addition is Commutative in $\mathbb{F}^n$)

If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.

Proof: Commutativity of Addition in $\mathbb{F}^n$

Proof (Commutativity of Addition in $\mathbb{F}^n$)

Let $x, y \in \mathbb{F}^n$. Then $x = (x_1, \ldots, x_n)$, and $y = (y_1, \ldots, y_n)$, so

			why?
$x + y$	$=$	$(x_1, \ldots, x_n) + (y_1, \ldots, y_n)$	
	$=$	$(x_1 + y_1, \ldots, x_n + y_n)$	definition of $(\mathbb{F}^n + \mathbb{F}^n)$
	$=$	$(y_1 + x_1, \ldots, y_n + x_n)$	$(\mathbb{F} + \mathbb{F})$ is commutative
	$=$	$(y_1, \ldots, y_n) + (x_1, \ldots, x_n)$	definition of $(\mathbb{F}^n + \mathbb{F}^n)$
	$=$	$y + x$	

Method: Direct computation, definitions, and properties of $\mathbb{F}$.

"0", "1"

Definition (The Zero-Element)

Let $0 \in \mathbb{F}^n$ denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0)$$

So... is "0" $0 \in \mathbb{F}$ or $0 \in \mathbb{F}^n$???

It is "obvious from context" or "0 is the additive-identity object in the current context."

The same thing will apply to "1", it is always the "multiplicative-identity object in the current context."

Lists, n-tuples, and vectors

They're all the "same" thing... it's just a matter of perspective.

Additive Inverse, and Scalar Multiplication

Definition (Additive inverse in $\mathbb{F}^n$)

For $x \in \mathbb{F}^n$ the **additive inverse** of x, $(-x)$ is the vector $(-x) \in \mathbb{F}^n$ such that

$$x + (-x) = 0$$

that is, if $x = (x_1, \ldots, x_n)$, then $(-x) = (-x_1, \ldots, -x_n)$.

Definition (Scalar multiplication in $\mathbb{F}^n$)

The product of a number $\alpha \in \mathbb{F}$ and a vector $v \in \mathbb{F}^n$ is computed by multiplying each coordinate of the vector by α:

$$\alpha v = \alpha(v_1, \ldots, v_n) = (\alpha v_1, \ldots, \alpha v_n).$$

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e.g. 1A-{1, 4, **7**, 8, 9}

Live Math :: Solved Examples

1A-1: Suppose $a, b \in \mathbb{R}$, not both 0. Find $c, d \in \mathbb{R}$ such that

$$1/(a + bi) = c + di$$

"Trick" — Multiply by 1

We multiply by a conveniently complicated way to write "1":

$$\left[\frac{a - bi}{a - bi}\right] \frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i$$

We can identify $c = \dfrac{a}{a^2 + b^2}$, and $d = \dfrac{-b}{a^2 + b^2}$; both of which are well-defined since a and b not both being $0 \Rightarrow (a^2 + b^2) > 0$.

Live Math :: Solved Examples — 1A-1, 2 of 2

Verification

Using the definition of multiplication of complex numbers, we can show that the expression we derived above is indeed the multiplicative inverse of any non-zero complex number $(a + bi)$

$$(a+bi)\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i\right) = \frac{a^2+b^2}{a^2+b^2} = 1.$$

□

Live Math :: Solved Examples — 1A-4

1A-4: Show that $\alpha + \beta = \beta + \alpha \; \forall \alpha, \beta \in \mathbb{C}$

Direct Computation

Since $\alpha, \beta \in \mathbb{C}$, we can represent $\alpha = a + bi$ and $\beta = c + di$ where $a, b, c, d \in \mathbb{R}$; then

$$\begin{aligned}
\alpha + \beta &= (a+bi) + (c+di) && \text{representation of complex numbers} \\
&= (a+c) + (b+d)i && \text{definition of addition on } \mathbb{C} \\
&= (c+a) + (d+b)i && \text{commutativity of addition on } \mathbb{R} \\
&= (c+di) + (a+bi) && \text{definition of addition on } \mathbb{C} \\
&= \beta + \alpha && \text{representation of complex numbers}
\end{aligned}$$

Live Math :: Solved Examples — 1A-7

1A-7: Show that for every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$

Existence

Suppose $\alpha = (a + bi)$, where $a, b \in \mathbb{R}$. Let $\beta = (-a - bi)$ — here we are using the unique additive inverses of $a, b \in \mathbb{R}$. Then, using the definition of complex addition:

$$\alpha + \beta = (a+bi) + (-a-bi) = (a-a) + (b-b)i = 0 + 0i = 0$$

Uniqueness

Now, suppose $\gamma \in \mathbb{C}$ such that $\alpha + \gamma = 0$. We add β on both sides of the equality: $\underbrace{\alpha + \beta}_{0} + \gamma = \beta$, which shows that $\gamma = \beta$.

Live Math :: Solved Examples — 1A-8

1A-8: Show that for every $\alpha \in \mathbb{C} \setminus \{0\}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$

Existence

$\alpha = (a + bi)$; $a, b \in \mathbb{R}$ such that $(a^2 + b^2) > 0$. Inspired by `1A-1`, we let

$$\beta = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$$

Now

$$(a+bi)\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i\right) = \frac{a^2+b^2}{a^2+b^2} = 1$$

which establishes existence.

Uniqueness

Now, suppose $\gamma \in \mathbb{C}$ such that $\alpha\gamma = 1$. We multiply by β on both sides of the equality: $\underbrace{\beta\alpha}_{1}\, \gamma = \beta$, which shows that $\gamma = \beta$.

Live Math :: Solved Examples 1A-9

1A-9: Show that $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$, $\forall \alpha, \beta, \lambda \in \mathbb{C}$.

Direct Computation

With $\alpha = (a+bi)$; $\beta = (c+di)$; $\lambda = (x+yi)$; with $a, b, c, d, x, y \in \mathbb{R}$, we use the definitions of addition and multiplication on $\mathbb{C}$:

$$\begin{aligned}
\lambda(\alpha+\beta) &= (x+yi)((a+bi)+(c+di)) \\
&= (x+yi)((a+c)+(b+d)i) \\
&= (x(a+c)-y(b+d)) + (x(b+d)+y(a+c))i \\
&= (xa+xc-yb-yd) + (xb+xd+ya+yc)i \\
&= ((xa-yb)+(xc-yd)) + ((xb+ya)+(xd+yc))i \\
&= ((xa-yb)+(xb+ya)i) + ((xc-yd)+(xd+yc)i) \\
&= \lambda\alpha + \lambda\beta
\end{aligned}$$

Introduction: Vector Spaces

We define Vector Spaces in a more general way than we did in [MATH 254].

We need the following building blocks:

Definition (addition, scalar multiplication)

- **addition** on a set V is a function that assigns an element $u+v \in V$ for all $u, v \in V$.
- **scalar multiplication** on a set V is a function that assigns an element $\alpha v \in V$ for all $\alpha \in \mathbb{F}$ and each $v \in V$

Definition: Vector Spaces

Definition (Vector space)

A **vector space** is a set V along with addition and scalar multiplication {sometimes: "$(V,+,\times)$"} such that the following properties hold:

- **commutativity** (of addition) :: $u+v = v+u$, $\forall u, v \in V$
- **associativity** (of addition) :: $(u+v)+w = u+(v+w)$, $\forall u, v, w \in V$
- **additive identity** (exists) :: $\exists 0 \in V \ : \ v+0 = v \ \forall v \in V$
- **additive inverse** (exists) :: $\forall v \in V \ \exists \ w \in V \ : \ v+w = 0$
- **multiplicative identity** (exists) :: $1v = v \ \forall v \in V$
- **distributive properties**, $\forall a, b \in \mathbb{F}$, and $\forall u, v \in V$:
  - $a(u+v) = au+av$
  - $(a+b)u = au+bu$

Elements of a vector space are called **vectors** or **points**.

A vector space over ($\mathbb{R}$ / $\mathbb{C}$) is called a (**real** / **complex**) **vector space**.

Notation $\mathbb{F}^S$

Notation ($\mathbb{F}^S$... yes, this is a vector space!)

- If S is a set, then $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$
- For $f, g \in \mathbb{F}^S$, the **sum** $f+g \in \mathbb{F}^S$ is the function defined by
$$(f+g)(x) = f(x)+g(x), \quad \forall x \in S$$
- For $\alpha \in \mathbb{F}$ and $f \in \mathbb{F}^S$, the **product** $\alpha f \in \mathbb{F}^S$ is the function defined by
$$(\alpha f)(x) = \alpha f(x), \quad \forall x \in S$$

- The additive identity is the trivial function $0 : S \to \mathbb{F}$ defined by
$$0(x) = 0, \quad \forall x \in S$$
- For $f \in \mathbb{F}^S$, the additive inverse of f is the function $-f : S \to \mathbb{F}$ defined by
$$(-f)(x) = -f(x), \quad \forall x \in S$$

Things to Prove

Property (Unique Additive Identity)
A vector space has a unique additive identity.

Property (Unique Additive Inverse)
Every element in a vector space has a unique additive inverse.

Proof :: Uniqueness of the Additive Identity

Method: Assume $\exists 2$, show they are the same; using the properties.

Proof (Additive Identity is Unique)
Suppose 0 and 0' are both additive identities for some vector space V. Then

$$0' \overset{(1)}{=} 0' + 0 \overset{(2)}{=} 0 + 0' \overset{(3)}{=} 0$$

where we used
(1) that 0 is an additive identity, then
(2) commutativity, and then
(3) that $0'$ is also an additive identity.

Thus we have $0' = 0$.

Proof :: Uniqueness of the Additive Inverse

Method: Assume $\exists 2$, show they are the same; using the properties.

Proof (Additive Inverse is Unique)
Suppose V is a vector space. Let $v \in V$, and suppose both w and w' are additive inverses of v. Then

$$w \overset{(1)}{=} w + 0 \overset{(2)}{=} w + (v + w') \overset{(3)}{=} (w + v) + w' \overset{(4)}{=} 0 + w' \overset{(5)}{=} w'$$

where we used
(1) the additive identity;
(2) w' is an additive inverse of v;
(3) associativity;
(4) w is an additive inverse;
(5) the additive identity.

Thus we have $w = w'$.

Notation: $-v$, $w - v$

Notation ($-v$, $w - v$) (additive inverse, subtraction)
Let $v, w \in V$, then
- $-v$ denotes the additive inverse of v,
- $w - v$ is defined to be $w + (-v)$

Convention: V — Going Forward —
Unless otherwise specified, V denotes the vector space over $\mathbb{F}$

More Theorem–Proofs to Ponder

Theorem (The Number 0 Times a Vector)

$0v = 0 \; \forall v \in V$

Theorem (A Number Times the Zero-Vector)

$a0 = 0 \; \forall a \in \mathbb{F}$

Theorem (The Number (-1) Times a Vector)

$(-1)v = -v \; \forall v \in V$

Proofs...

Proof (The Number 0 Times a Vector)

For $v \in V$, we have

$$0v = (0+0)v = 0v + 0v$$

then add $-0v$ (the additive inverse of $0v$) on both sides

$$\underbrace{0v - 0v}_{0} = \underbrace{0v + 0v - 0v}_{0v}$$

and we have $0 = 0v$.

Method: Direct computation, definitions, and properties of $\mathbb{F}$.

Proofs...

Proof (A Number Times the Zero-Vector)

For $a \in \mathbb{F}$, we have

$$a0 = a(0+0) = a0 + a0$$

as in the previous proof, we add the inverse of $a0$ to both sides...

$$\underbrace{a0 - a0}_{0} = \underbrace{a0 + a0 - a0}_{a0}$$

and we have $0 = a0$.

Method: Direct computation, definitions, and properties of $\mathbb{F}$.

Proofs...

Proof (The Number (-1) Times a Vector)

For $v \in V$, we have

$$v + (-1)v = 1v + (-1)v + (1 + (-1))v = 0v = 0$$

therefore $(-1)v$ must be the additive inverse of v; $(-1)v = -v$.

Method: Direct computation, definitions, and properties of $\mathbb{F}$.

⟨⟨⟨ Live Math ⟩⟩⟩

e.g. 1B-{**5**}

Live Math :: Solved Examples 1B-5

1B-5: Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that $0v = 0 \ \forall v \in V$. Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

Suppose $0v = 0 \ \forall v \in V$, then for $v \in V$:

$$\begin{aligned} 0 = 0v &= (1 + (-1))v \\ &= 1v + (-1)v \\ &= v + (-1)v \end{aligned}$$

which makes $(-1)v$ an additive inverse of $v \rightsquigarrow$ the additive inverse condition is satisfied.

We used the additive inverse of $1 \in \mathbb{R}$, and the distributive property of V.

Subspace :: Definition

Definition ([Linear] Subspace)

A subset U of V is called a **subspace** of V if U also is a vector space ("inheriting" the addition and scalar multiplication from V).

Some "obvious examples" of subspaces of $\mathbb{F}^4$:

- $\{(x_1, x_2, x_3, x_4) \ : \ x_1, x_2, x_3, x_4 \in \mathbb{F}\}$
- $\{(x_1, x_2, x_3, 0) \ : \ x_1, x_2, x_3 \in \mathbb{F}\}$
- $\{(x_1, 0, 0, x_4) \ : \ x_1, x_4 \in \mathbb{F}\}$
- $\{(0, x_2, 0, 0) \ : \ x_2 \in \mathbb{F}\}$

Subspace :: Conditions

Conditions for a Subspace

A subset U of V is a subspace of V if and only if U satisfies:

1. U has an **additive identity**
$$0 \in U$$
2. U is **closed under addition**
$$u, w \in U \ \Rightarrow \ u + w \in U$$
3. U is **closed under scalar multiplication**
$$a \in \mathbb{F} \text{ and } u \in U \Rightarrow au \in U$$

Proof — Subspace :: Conditions

Proof (Conditions for a Subspace)

$\Rightarrow$ If U is a subspace of V, then U satisfies the three conditions (BY DEFINITION, since it is a vector space).

$\Leftarrow$ Conversely; if U satifies the three conditions.

(1) The additive identity condition ensures that the additive identity of V is in U;

(2) additive closure of U means that addition is well-defined on U;

(3) closure of U under scalar multiplication means that scalar multiplication is well-defined on U.

Now, if $u \in U$, then $-u \overset{(3)}{\equiv} (-1)u$ also $\in U$ (so, every element in U has an additive inverse in U). Associativity and Commutativity holds in U since they hold in the larger space V. Therefore, U is a vector space; and since U is a subset of V it is a subspace of V.

Subspaces :: Examples

1. $V(\alpha, b) = \{ (x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \ : \ x_3 = \alpha x_4 + b \}$ is a subspace of $\mathbb{F}^4$ $\forall \alpha \in \mathbb{F}$, and $b = 0$; if $b \neq 0$, then $(0, 0, 0, 0) \notin \mathbb{F}^4$. (additive identity)
2. $C([-\pi, \pi])$ (the set of continuous functions on $[-\pi, \pi]$) is a subspace of $\mathbb{R}^{[-\pi,\pi]}$.
3. The set of differentiable real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
4. The set of differentiable real-valued functions f on the interval $(-\pi, \pi)$ such that $f'(0) = \beta$ is a subspace of $\mathbb{R}^{(-\pi,\pi)}$ if and only if $\beta = 0$. (additive closure)
5. The set of all sequences of complex numbers is a subspace of $\mathbb{C}^{\infty}$

(2)-(3)-(4) show that a huge amount of calculus is built on top of linear structures; and a better understanding of linear algebra can improve and formalize our understanding of calculus.

Sums of Subspaces :: Definition

Definition (Sum of Subsets)

Suppose $U_1, \ldots, U_m$ are subsets of V.

The sum of $U_1, \ldots, U_m$, denoted

$$U_1 + \cdots + U_m,$$

is the set of all possible sums of elements of $U_1, \ldots, U_m$.

More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \ : \ u_1 \in U_1, \ldots, u_m \in U_m\}.$$

Sums of Subspaces :: Examples

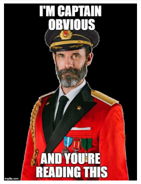
Captain Obvious' Example: Sums of Subspaces

Suppose U is the set of all elements of $\mathbb{F}^n$ whose second-to-n^{th} coordinates equal 0, and W is the set of all elements of $\mathbb{F}^n$ whose first and third-to-n^{th} coordinates equal 0:

$$\begin{aligned} U &= \{(x, 0, 0, \ldots, 0) \in \mathbb{F}^n : x \in \mathbb{F}\} \quad \text{and} \\ W &= \{(0, y, 0, \ldots, 0) \in \mathbb{F}^n : y \in \mathbb{F}\} \end{aligned}$$

Then

$$U + W = \{(x, y, 0, \ldots, 0) \in \mathbb{F}^n : x, y \in \mathbb{F}\}$$



Sums of Subspaces :: Examples

Example: Sums of Subspaces

Suppose

$$U = \{(x,x,y,y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}, \quad \text{and}$$
$$W = \{(a,a,a,b) \in \mathbb{F}^4 : a, b \in \mathbb{F}\}.$$

Then

$$U + W = \{(\alpha,\alpha,\beta,\gamma) \in \mathbb{F}^4 : \alpha,\beta,\gamma \in \mathbb{F}\}$$

$u \in U$ **and** $w \in W \Rightarrow (u+w) \in U+W$ —

$$\alpha = x + a, \quad \beta = y + a, \quad \gamma = y + b$$

$\forall z \in U+W \; \exists \; u \in U$ **and** $w \in W$ **:** $z = u + w$ —
Given any $\alpha, \beta, \gamma, x \in \mathbb{F}^n$, simply let

$$a = \underbrace{\alpha - x}, \quad y = \underbrace{\beta - a}, \quad b = \gamma - y$$

Sum of Subspaces

The sum of subspaces is a subspace, and is the smallest subspace containing all the summands.

Theorem (Sum of subspaces is the smallest containing subspace)

Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$.

Proof — Sum of Subspaces

Proof (Sum of Subspaces is the Smallest Containing Subspace)

$0 \in U_1 + \cdots + U_m$, and the closure under addition and scalar multiplication on $U_1 + \cdots + U_m$ are both fairly straight-forward.

Thus $U_1 + \cdots + U_m$ is a subspace of V.

$U_1, \dots, U_m$ are all contained in $U_1 + \cdots + U_m$: — let $u_k \in U_k$ and consider sums $u_1 + \cdots + u_m$ where all except one of the u_k's are 0.

Conversely, every subspace of V containing $U_1, \dots, U_m$ contains $U_1 + \cdots + U_m$ (subspaces contain all finite sums of their elements).

Thus $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$.

Direct Sums

Definition (Direct Sum)

Suppose $U_1, \dots, U_m$ are subspaces of V

- The sum $U_1 + \cdots + U_m$ is called a **direct sum** if each element of $U_1 + \cdots + U_m$ can be written in only one way (UNIQUELY) as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$.
- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

Note that the spaces in the previous example do not form a direct sum

$$U = \{(x,x,y,y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}, \quad \text{and}$$
$$W = \{(a,a,a,b) \in \mathbb{F}^4 : a, b \in \mathbb{F}\}.$$

since there are multiple ways to write any vector $\vec{v} \in U + W$.

Example :: Direct Sum

Example :: Direct Sum

Let U_k be the subspace of $\mathbb{F}^n$ of the form

$$U_k = \{(0, \dots, 0, u_k, 0, \dots, 0) \in \mathbb{F}^n, u_k \in \mathbb{F}\}$$

i.e. only the k^{th} coordinate is allowed to be non-zero.

Then $\mathbb{F}^n = U_1 \oplus \cdots \oplus U_n$.

With

$$W_k = \bigoplus_{j=1}^{k} U_j = U_1 \oplus \cdots \oplus U_k$$

then

$$W_k = \{(w_1, \dots, w_k, 0, \dots, 0) \in \mathbb{F}^n \,:\, w_j \in \mathbb{F}, j = 1, \dots, k\}, \;\; k = 1, \dots, n$$

Example :: Not a Direct Sum

Example :: Not a Direct Sum

Let

$$\begin{aligned} U_1 &= \{(x, y, 0) \in \mathbb{F}^3 \,:\, x, y \in \mathbb{F}\} \\ U_2 &= \{(0, 0, z) \in \mathbb{F}^3 \,:\, z \in \mathbb{F}\} \\ U_3 &= \{(0, \beta, \beta) \in \mathbb{F}^3 \,:\, \beta \in \mathbb{F}\} \end{aligned}$$

Then $\mathbb{F}^3 = U_1 + U_2 + U_3$; also $0 \in U_1 \cap U_2 \cap U_3$, but $\forall \alpha \in \mathbb{F}$:

$$\begin{aligned} u_1 &= (0, \alpha, 0) \in U_1 \\ u_2 &= (0, 0, \alpha) \in U_2 \\ u_3 &= (0, -\alpha, -\alpha) \in U_3 \end{aligned}$$

so that $u_1 + u_2 + u_3 = 0$. Since we can write $0 \in \mathbb{F}^3$ in more than one way, $U_1 + U_2 + U_3$ is not a direct sum.

Note: $\mathbb{F}^3 = U_1 \oplus U_2$. Question: Are there more direct sums?

Condition for a direct sum; Direct sum of two subspaces

Theorem (Condition for a direct sum)

Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$, is by taking each $u_j = 0$.

Theorem (Direct sum of two subspaces)

Suppose U and W are subspaces of V. Then $U \oplus W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof — Condition for a Direct Sum

Proof (Condition for a Direct Sum)

First suppose $U_1 + \cdots + U_m$ is a direct sum. Then the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$, is by taking each $u_j = 0$. (By uniqueness)

Now suppose that the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$, is by taking each $u_j = 0$. To show that $U_1 + \cdots + U_m$ is a direct sum, let $v \in U_1 + \cdots + U_m$.

We can write $v = u_1 + \cdots + u_m$, for some $u_j \in U_j$, $(j = 1, \dots, m)$.

To show that this representation is unique, suppose we also have $v = v_1 + \cdots + v_m$ where $v_1 \in U_1, \dots, v_m \in U_m$. Subtracting these two equations, we have

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m).$$

Because $(u_j - v_j) \in U_j$, the equation above implies that each $(u_j - v_j) = 0$. Thus $u_j = v_j$, $(j = 1, \dots, m)$, as desired.

Method: Assume $\exists 2$, show they are the same (using the properties).

Proof — Direct Sum of Two Subspaces

Proof (Direct Sum of two Subspaces)

First suppose that $U + W$ is a direct sum. If $v \in U \cap W$, then $0 = v + (-v)$, where $v \in U$ and $(-v) \in W$.

By the unique representation of 0 as the sum of a vector in U and a vector in W, we have $v = 0$. Thus $U \cap W = \{0\}$, completing the proof in one direction.

To prove the other direction, now suppose $U \cap W = \{0\}$. To prove that $U + W$ is a direct sum, suppose $u \in U$, $w \in W$, and $0 = u + w$:

We need only show that $u + w = 0$ (by the previous theorem). The equation above implies that $u = -w \in W$. Thus $u \in U \cap W$. Hence $u = 0$, which by the equation above implies that $w = 0$, completing the proof.

$\langle\langle\langle$ Live Math $\rangle\rangle\rangle$

e.g. 1C-{1, **5**}

Live Math :: Solved Examples 1C-5

1C-5: Is $\mathbb{R}^2$ a subspace of the complex vector space $\mathbb{C}^2$?

No: For $\mathbb{R}^2$ to be a subspace, is must be closed under the operations (addition, and scalar multiplication) "inherited" from $\mathbb{C}^2$.

- Addition is not a problems since $\forall u, v \in \mathbb{R}^2$, $(u + v) \in \mathbb{R}^2$.
- However, whereas $\mathbb{C}^2$ is closed under scaling by $\alpha \in \mathbb{C}$, $\mathbb{R}^2$ is not; in particular $\forall u \in \mathbb{R}^2$: $u \neq 0$, $iu \notin \mathbb{R}^2$.

Suggested Problems

1.A—1, 4, **5**, **6**, 7, 8, 9

1.B—**1**, **3**, 5

1.C—1, 5, **10**, **20**

Assigned Homework — HW#1, Due Date in Canvas/Gradescope

1.A—5, 6

1.B—1, 3

1.C—10, 20

Note: Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually "scheduled.")

Upload homework to www.Gradescope.com

Supplements

⟨Placeholder⟩