

# Math 524: Linear Algebra

## Notes #1 — Vector Spaces

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## Student Learning Targets, and Objectives

### Target Properties of the Complex Numbers, $\mathbb{C}$

Objective Know the definitions of, and be able to perform basic complex arithmetic (addition, multiplication, subtraction, division)

Objective Be able to apply the properties of commutativity, associativity, additive and multiplicative identities and inverses, as well as the distributive property.

### Target $\mathbb{R}^n$ and $\mathbb{C}^n$

Objective Be able to define  $\mathbb{R}^n$  and  $\mathbb{C}^n$  as lists of length  $n$ , and to abstract to general fields,  $\mathbb{F}^n$ .

Objective Be able to transfer the algebraic rules and properties from  $\mathbb{R}$  and  $\mathbb{C}$  ( $\mathbb{F}$ ), to  $\mathbb{F}^n$ .

## Student Learning Targets, and Objectives

### Target Vector Spaces

Objective Be able to define a vector space in terms of its necessary operations, and properties.

Objective Be able to understand the notation  $\mathbb{F}^S$ , and show that it is a vector space.

Objective Be able to formally show the uniqueness of the additive identity and inverse.

### Target Subspaces

Objective Be able to apply the subspace conditions in order to show that a subset of a Vector space is (or is not) a Subspace

### Target Sums and Direct Sums of Subspaces

Objective Be able to apply the definitions to identify whether a sum of subspaces is a direct sum, or not.

## Introduction

We will follow the notation, and structure of Axler's *Linear Algebra Done Right*.

The first couple of lectures will fairly quickly cover material (mostly) familiar from [MATH 254] (or alternatives).

The goal is to shake off some mental “dust,” and build a foundation of common notation and language.

Note that some new material will be “folded” into these lectures.

Time-Target:  $3 \times 75$ -minute lectures.

Math 254  $\rightsquigarrow$  Math 524

One fairly significant difference between [MATH 254] and [MATH 524] is that we will state most of our results in terms of complex numbers  $z \in \mathbb{C}$  rather than real numbers  $x \in \mathbb{R}$ . When there are differences behaviour/properties over  $\mathbb{C}$  and  $\mathbb{R}$ , we carefully explore those.

$z = x + yi$ , where  $x, y \in \mathbb{R}$ ; and we view the real numbers as a special case of the complex numbers (where  $y = 0$ ).

The added bonus is that we get *more general* results, which are “future-proofed” (for cases where we need complex numbers).

Additionally, [MATH 524] provides a *much more formal* and complete discussion of linear algebra.

## Complex Numbers

Hopefully you have not forgotten all your encounters with complex numbers.

We quickly review / introduce the essentials of complex arithmetic that we need.

The complex numbers solve the “core problem” of assigning a value to  $\sqrt{-1}$ .

Following Euler<sup>(1777)</sup>:  $i = \sqrt{-1}$ ,  $i^2 = -1$ .

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**Note:** Mathematicians tend to use  $i = \sqrt{-1}$ , whereas (electrical) engineers prefer  $j = \sqrt{-1}$  ( $i$  being reserved for electrical current).

## Complex Numbers :: Formal Definition

## Definition (Complex Numbers)

- A **complex number**  $z$  is an ordered pair  $(a, b)$  where  $a, b \in \mathbb{R}$ ; usually we write  $z = a + bi$ .
- The set of all complex numbers is denoted by  $\mathbb{C}$ :

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$$

- Rules for addition and multiplication ( $a, b, c, d \in \mathbb{R}$ )
  - $(a + bi) + (c + di) = (a + c) + (b + d)i$
  - $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$



## Complex Numbers :: Properties

Proofs by direct computation

There are no surprises when it comes to the properties of complex numbers (they are “inherited” from the real numbers + definition of complex addition/multiplication):

## Properties (Complex Numbers)

Let  $u, v, w \in \mathbb{C}$ , then

- **commutativity**:  $u + v = v + u$ , and  $uv = vu$ ;
- **associativity**:  $(u + v) + w = u + (v + w)$ , and  $(uv)w = u(vw)$ ;
- **0** is the **additive identity** and **1** the **multiplicative identity**:  
$$u + 0 = 0 + u = u, \quad v1 = 1v = 1v = v$$
- $u$  has an **additive inverse**, i.e.  $\exists!v: u + v = 0$ , ( $v$  is unique)
- $u \neq 0$  has a **multiplicative inverse**, i.e.  $\exists!v: uv = 1$ , ( $v$  is unique)
- the **distributive property** holds:

$$u(v + w) = uv + uw$$

$\{\text{Inverse}(+), \text{Inverse}(\cdot)\} \rightsquigarrow \{\text{Subtraction}, \text{Division}\}$

Definition (Subtraction and Division)

Let  $u, v \in \mathbb{C}$ ,

- Let  $(-u)$  be the unique **additive inverse** of  $u$ ,

$$u + (-u) = 0$$

- We define **subtraction** using the **additive inverse**:

$$u - v = u + (-v)$$

- Likewise for  $u \neq 0$ , let  $(1/u)$  denote the unique **multiplicative inverse** of  $u$ ,

$$u(1/u) = 1$$

- We define **division** using the **multiplicative inverse**:

$$u/v = u(1/v)$$

Real and/or Complex?  $\rightsquigarrow \mathbb{F}$

$x \in \mathbb{R}$  and  $z \in \mathbb{C}$  are **scalars** (single numbers).

Throughout our discussion we will use the notation  $y \in \mathbb{F}$ , where  $\mathbb{F}$  can be either  $\mathbb{C}$  or  $\mathbb{R}$  (in such a case, the results are true for both complex and real entries).

Why  $\mathbb{F}$ ??? Both  $\mathbb{R}$  and  $\mathbb{C}$  are *fields*:

Definition (Field (Thanks “Aunt Wiki”))

In mathematics, a **field** is a **set** on which addition, subtraction, multiplication, and division are defined, and behave as the corresponding operations on rational and real numbers do. A field is thus a fundamental algebraic structure, which is widely used in (abstract) algebra [MATH 320, MATH 520], number theory [MATH 522] and many other areas of mathematics.

[https://en.wikipedia.org/wiki/Field\\_\(mathematics\)](https://en.wikipedia.org/wiki/Field_(mathematics))



Lists ( $n$ -tuples)

## Definition (list, length)

Let  $n > 0$  be a positive integer ( $n \in \mathbb{Z}^+$ ). A **list** of **length**  $n$  is an ordered collection of  $n$  elements. Here, we write them separated by commas and surrounded by parenthesis<sup>‡</sup>:

$$(x_1, x_2, \dots, x_n)$$

Two lists are equal if and only if they have the same lengths, and the same elements in the same order.

<sup>‡</sup> computer scientists can think of it as some form of “container class.” Python uses (...) for immutable “tuples” and [...] for “lists”...

In this class (almost) all our lists have *finite length*.

The empty list — () — is a list of length 0.

$\mathbb{F}^n$  $(\mathbb{R}^n, \mathbb{C}^n)$ 

Definition ( $\mathbb{F}^n$ )

$\mathbb{F}^n$  is the set of all lists of length  $n$  of elements of  $\mathbb{F}$ :

$$\mathbb{F}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{F}\}$$

For  $(x_1, \dots, x_n) \in \mathbb{F}^n$  we say that  $x_j$  is the  $j^{\text{th}}$  **coordinate** of  $(x_1, \dots, x_n)$ .

When  $\mathbb{F} = \mathbb{R}$ , this matches our [MATH 254] definitions of  $\mathbb{R}^n$ .

Addition in  $\mathbb{F}^n$ Definition (Addition in  $\mathbb{F}^n$ )Addition in  $\mathbb{F}^n$  is defined element-by-element:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

Property (Addition is Commutative in  $\mathbb{F}^n$ )If  $x, y \in \mathbb{F}^n$ , then  $x + y = y + x$ .

Proof: Commutativity of Addition in  $\mathbb{F}^n$ Proof (Commutativity of Addition in  $\mathbb{F}^n$ )Let  $x, y \in \mathbb{F}^n$ . Then  $x = (x_1, \dots, x_n)$ , and  $y = (y_1, \dots, y_n)$ , so**why?**

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$$\begin{aligned}x + y &= (x_1, \dots, x_n) + (y_1, \dots, y_n) \\ &= (x_1 + y_1, \dots, x_n + y_n) && \text{definition of } (\mathbb{F}^n + \mathbb{F}^n) \\ &= (y_1 + x_1, \dots, y_n + x_n) && (\mathbb{F} + \mathbb{F}) \text{ is commutative} \\ &= (y_1, \dots, y_n) + (x_1, \dots, x_n) && \text{definition of } (\mathbb{F}^n + \mathbb{F}^n) \\ &= y + x\end{aligned}$$

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**Method:** Direct computation, definitions, and properties of  $\mathbb{F}$ .

“0”, “1”

Definition (The Zero-Element)

Let  $0 \in \mathbb{F}^n$  denote the list of length  $n$  whose coordinates are all 0:

$$0 = (0, \dots, 0)$$

So... is “0”  $0 \in \mathbb{F}$  or  $0 \in \mathbb{F}^n$ ???

It is “obvious from context” or “0 is the additive-identity object in the current context.”

The same thing will apply to “1”, it is always the “multiplicative-identity object in the current context.”



## Lists, $n$ -tuples, and vectors

They're all the “same” thing... it's just a matter of perspective.

## Additive Inverse, and Scalar Multiplication

Definition (Additive inverse in  $\mathbb{F}^n$ )

For  $x \in \mathbb{F}^n$  the **additive inverse** of  $x$ ,  $(-x)$  is the vector  $(-x) \in \mathbb{F}^n$  such that

$$x + (-x) = 0$$

that is, if  $x = (x_1, \dots, x_n)$ , then  $(-x) = (-x_1, \dots, -x_n)$ .

Definition (Scalar multiplication in  $\mathbb{F}^n$ )

The product of a number  $\alpha \in \mathbb{F}$  and a vector  $v \in \mathbb{F}^n$  is computed by multiplying each coordinate of the vector by  $\alpha$ :

$$\alpha v = \alpha(v_1, \dots, v_n) = (\alpha v_1, \dots, \alpha v_n).$$

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e.g. 1A- $\{1, 4, \mathbf{7}, 8, 9\}$

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**1A-1:** Suppose  $a, b \in \mathbb{R}$ , not both 0. Find  $c, d \in \mathbb{R}$  such that

$$1/(a + bi) = c + di$$

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**“Trick” — Multiply by 1**

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We multiply by a conveniently complicated way to write “1”:

$$\left[ \frac{a - bi}{a - bi} \right] \frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$$

We can identify  $c = \frac{a}{a^2 + b^2}$ , and  $d = \frac{-b}{a^2 + b^2}$ ; both of which are well-defined since  $a$  and  $b$  not both being 0  $\Rightarrow (a^2 + b^2) > 0$ .

\*

**Verification**

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Using the definition of multiplication of complex numbers, we can show that the expression we derived above is indeed the multiplicative inverse of any non-zero complex number  $(a + bi)$

$$(a + bi) \left( \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2} i \right) = \frac{a^2 + b^2}{a^2 + b^2} = 1.$$

□

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**1A-4:** Show that  $\alpha + \beta = \beta + \alpha \forall \alpha, \beta \in \mathbb{C}$ 

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**Direct Computation**

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Since  $\alpha, \beta \in \mathbb{C}$ , we can represent  $\alpha = a + bi$  and  $\beta = c + di$  where  $a, b, c, d \in \mathbb{R}$ ; then

$$\begin{aligned}\alpha + \beta &= (a + bi) + (c + di) \\ &= (a + c) + (b + d)i \\ &= (c + a) + (d + b)i \\ &= (c + di) + (a + bi) \\ &= \beta + \alpha\end{aligned}$$

representation of complex numbers  
definition of addition on  $\mathbb{C}$   
commutativity of addition on  $\mathbb{R}$   
definition of addition on  $\mathbb{C}$   
representation of complex numbers

## Live Math :: Solved Examples

## 1A-7

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**1A-7:** Show that for every  $\alpha \in \mathbb{C}$ , there exists a unique  $\beta \in \mathbb{C}$  such that  $\alpha + \beta = 0$

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## Existence

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Suppose  $\alpha = (a + bi)$ , where  $a, b \in \mathbb{R}$ . Let  $\beta = (-a - bi)$  — here we are using the unique additive inverses of  $a, b \in \mathbb{R}$ . Then, using the definition of complex addition:

$$\alpha + \beta = (a + bi) + (-a - bi) = (a - a) + (b - b)i = 0 + 0i = 0$$

\*

## Uniqueness

\*

Now, suppose  $\gamma \in \mathbb{C}$  such that  $\alpha + \gamma = 0$ . We add  $\beta$  on both sides of the equality:

$$\underbrace{\alpha + \beta}_{0} + \gamma = \beta, \text{ which shows that } \gamma = \beta.$$

## Live Math :: Solved Examples

## 1A-8

**1A-8:** Show that for every  $\alpha \in \mathbb{C} \setminus \{0\}$ , there exists a unique  $\beta \in \mathbb{C}$  such that  $\alpha\beta = 1$

\*

**Existence**

\*

$\alpha = (a + bi)$ ;  $a, b \in \mathbb{R}$  such that  $(a^2 + b^2) > 0$ . Inspired by 1A-1, we let

$$\beta = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i$$

Now

$$(a + bi) \left( \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i \right) = \frac{a^2 + b^2}{a^2 + b^2} = 1$$

which establishes existence.

\*

**Uniqueness**

\*

Now, suppose  $\gamma \in \mathbb{C}$  such that  $\alpha\gamma = 1$ . We multiply by  $\beta$  on both sides of the equality:

$$\underbrace{\beta\alpha}_{1} \gamma = \beta, \text{ which shows that } \gamma = \beta.$$



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**1A-9:** Show that  $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta, \forall \alpha, \beta, \lambda \in \mathbb{C}$ .

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**Direct Computation**

✱

With  $\alpha = (a + bi)$ ;  $\beta = (c + di)$ ;  $\lambda = (x + yi)$ ; with  $a, b, c, d, x, y \in \mathbb{R}$ , we use the definitions of addition and multiplication on  $\mathbb{C}$ :

$$\begin{aligned}\lambda(\alpha + \beta) &= (x + yi)((a + bi) + (c + di)) \\ &= (x + yi)((a + c) + (b + d)i) \\ &= (x(a + c) - y(b + d)) + (x(b + d) + y(a + c))i \\ &= (xa + xc - yb - yd) + (xb + xd + ya + yc)i \\ &= ((xa - yb) + (xc - yd)) + ((xb + ya) + (xd + yc))i \\ &= ((xa - yb) + (xb + ya))i + ((xc - yd) + (xd + yc))i \\ &= \lambda\alpha + \lambda\beta\end{aligned}$$

## Introduction: Vector Spaces

We define Vector Spaces in a more general way than we did in [MATH 254].

We need the following building blocks:

Definition (addition, scalar multiplication)

- **addition** on a set  $V$  is a function that assigns an element  $u + v \in V$  for all  $u, v \in V$ .
- **scalar multiplication** on a set  $V$  is a function that assigns an element  $\alpha v \in V$  for all  $\alpha \in \mathbb{F}$  and each  $v \in V$

## Definition: Vector Spaces

### Definition (Vector space)

A **vector space** is a set  $V$  along with addition and scalar multiplication {sometimes: " $(V, +, \times)$ " } such that the following properties hold:

- **commutativity** (of addition) ::  $u + v = v + u, \quad \forall u, v \in V$
- **associativity** (of addition) ::  $(u + v) + w = u + (v + w), \quad \forall u, v, w \in V$
- **additive identity** (exists) ::  $\exists 0 \in V : v + 0 = v \quad \forall v \in V$
- **additive inverse** (exists) ::  $\forall v \in V \exists w \in V : v + w = 0$
- **multiplicative identity** (exists) ::  $1v = v \quad \forall v \in V$
- **distributive properties**,  $\forall a, b \in \mathbb{F}$ , and  $\forall u, v \in V$ :
  - $a(u + v) = au + av$
  - $(a + b)u = au + bu$

Elements of a vector space are called **vectors** or **points**.

A vector space over  $(\mathbb{R} / \mathbb{C})$  is called a **(real / complex) vector space**.

Notation  $\mathbb{F}^S$ Notation ( $\mathbb{F}^S$ ... yes, this is a vector space!)

- If  $S$  is a set, then  $\mathbb{F}^S$  denotes the set of functions from  $S$  to  $\mathbb{F}$
- For  $f, g \in \mathbb{F}^S$ , the **sum**  $f + g \in \mathbb{F}^S$  is the function defined by
$$(f + g)(x) = f(x) + g(x), \quad \forall x \in S$$
- For  $\alpha \in \mathbb{F}$  and  $f \in \mathbb{F}^S$ , the **product**  $\alpha f \in \mathbb{F}^S$  is the function defined by

$$(\alpha f)(x) = \alpha f(x), \quad \forall x \in S$$

- The additive identity is the trivial function  $0 : S \rightarrow \mathbb{F}$  defined by
$$0(x) = 0, \quad \forall x \in S$$
- For  $f \in \mathbb{F}^S$ , the additive inverse of  $f$  is the function  $-f : S \rightarrow \mathbb{F}$  defined by

$$(-f)(x) = -f(x), \quad \forall x \in S$$

## Things to Prove

Property (Unique Additive Identity)

A vector space has a unique additive identity.

Property (Unique Additive Inverse)

Every element in a vector space has a unique additive inverse.

## Proof :: Uniqueness of the Additive Identity

**Method:** Assume  $\exists 2$ , show they are the same; using the properties.

Proof (Additive Identity is Unique)

Suppose  $0$  and  $0'$  are both additive identities for some vector space  $V$ . Then

$$0' \stackrel{(1)}{=} 0' + 0 \stackrel{(2)}{=} 0 + 0' \stackrel{(3)}{=} 0$$

where we used

(1) that  $0$  is an additive identity, then

(2) commutativity, and then

(3) that  $0'$  is also an additive identity.

Thus we have  $0' = 0$ .

## Proof :: Uniqueness of the Additive Inverse

**Method:** Assume  $\exists 2$ , show they are the same; using the properties.

Proof (Additive Inverse is Unique)

Suppose  $V$  is a vector space. Let  $v \in V$ , and suppose both  $w$  and  $w'$  are additive inverses of  $v$ . Then

$$w \stackrel{(1)}{=} w + 0 \stackrel{(2)}{=} w + (v + w') \stackrel{(3)}{=} (w + v) + w' \stackrel{(4)}{=} 0 + w' \stackrel{(5)}{=} w'$$

where we used

- (1) the additive identity;
- (2)  $w'$  is an additive inverse of  $v$ ;
- (3) associativity;
- (4)  $w$  is an additive inverse;
- (5) the additive identity.

Thus we have  $w = w'$ .

Notation:  $-v$ ,  $w - v$

Notation ( $-v$ ,  $w - v$ ) (additive inverse, subtraction)

Let  $v, w \in V$ , then

- $-v$  denotes the additive inverse of  $v$ ,
- $w - v$  is defined to be  $w + (-v)$

Convention:  $V$  — Going Forward —

Unless otherwise specified,  $V$  denotes the vector space over  $\mathbb{F}$



## More Theorem–Proofs to Ponder

Theorem (The Number 0 Times a Vector)

$$0v = 0 \quad \forall v \in V$$

Theorem (A Number Times the Zero-Vector)

$$a0 = 0 \quad \forall a \in \mathbb{F}$$

Theorem (The Number  $(-1)$  Times a Vector)

$$(-1)v = -v \quad \forall v \in V$$

## Proofs...

Proof (The Number 0 Times a Vector)

For  $v \in V$ , we have

$$0v = (0 + 0)v = 0v + 0v$$

then add  $-0v$  (the additive inverse of  $0v$ ) on both sides

$$\underbrace{0v - 0v}_0 = \underbrace{0v + 0v - 0v}_{0v}$$

and we have  $0 = 0v$ .

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**Method:** Direct computation, definitions, and properties of  $\mathbb{F}$ .

## Proofs...

## Proof (A Number Times the Zero-Vector)

For  $a \in \mathbb{F}$ , we have

$$a0 = a(0 + 0) = a0 + a0$$

as in the previous proof, we add the inverse of  $a0$  to both sides...

$$\underbrace{a0 - a0}_0 = \underbrace{a0 + a0 - a0}_{a0}$$

and we have  $0 = a0$ .

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**Method:** Direct computation, definitions, and properties of  $\mathbb{F}$ .

## Proofs...

Proof (The Number  $(-1)$  Times a Vector)

For  $v \in V$ , we have

$$v + (-1)v = 1v + (-1)v + (1 + (-1))v = 0v = 0$$

therefore  $(-1)v$  must be the additive inverse of  $v$ ;  $(-1)v = -v$ .

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**Method:** Direct computation, definitions, and properties of  $\mathbb{F}$ .

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e.g. 1B-**{5}**

## Live Math :: Solved Examples

## 1B-5

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**1B-5:** Show that in the definition of a vector space, the additive inverse condition can be replaced with the condition that  $0v = 0 \forall v \in V$ . Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of  $V$ .

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Suppose  $0v = 0 \forall v \in V$ , then for  $v \in V$ :

$$\begin{aligned} 0 &= 0v = (1 + (-1))v \\ &= 1v + (-1)v \\ &= v + (-1)v \end{aligned}$$

which makes  $(-1)v$  an additive inverse of  $v \rightsquigarrow$  the additive inverse condition is satisfied.

We used the additive inverse of  $1 \in \mathbb{R}$ , and the distributive property of  $V$ .

## Subspace :: Definition

## Definition ([Linear] Subspace)

A subset  $U$  of  $V$  is called a **subspace** of  $V$  if  $U$  also is a vector space (“inheriting” the addition and scalar multiplication from  $V$ ).

Some “obvious examples” of subspaces of  $\mathbb{F}^4$ :

- $\{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \mathbb{F}\}$
- $\{(x_1, x_2, x_3, 0) : x_1, x_2, x_3 \in \mathbb{F}\}$
- $\{(x_1, 0, 0, x_4) : x_1, x_4 \in \mathbb{F}\}$
- $\{(0, x_2, 0, 0) : x_2 \in \mathbb{F}\}$

## Subspace :: Conditions

### Conditions for a Subspace

A subset  $U$  of  $V$  is a subspace of  $V$  if and only if  $U$  satisfies:

- 1  $U$  has an **additive identity**

$$0 \in U$$

- 2  $U$  is **closed under addition**

$$u, w \in U \Rightarrow u + w \in U$$

- 3  $U$  is **closed under scalar multiplication**

$$a \in \mathbb{F} \text{ and } u \in U \Rightarrow au \in U$$



## Proof — Subspace :: Conditions

## Proof (Conditions for a Subspace)

$\Rightarrow$  If  $U$  is a subspace of  $V$ , then  $U$  satisfies the three conditions (**By DEFINITION**, since it is a vector space).

$\Leftarrow$  Conversely; if  $U$  satisfies the three conditions.

- (1) The additive identity condition ensures that the additive identity of  $V$  is in  $U$ ;
- (2) additive closure of  $U$  means that addition is well-defined on  $U$ ;
- (3) closure of  $U$  under scalar multiplication means that scalar multiplication is well-defined on  $U$ .

Now, if  $u \in U$ , then  $-u \stackrel{(3)}{\equiv} (-1)u$  also  $\in U$  (so, every element in  $U$  has an additive inverse in  $U$ ). Associativity and Commutativity holds in  $U$  since they hold in the larger space  $V$ . Therefore,  $U$  is a vector space; and since  $U$  is a subset of  $V$  it is a subspace of  $V$ .

## Subspaces :: Examples

- 1  $V(\alpha, b) = \{ (x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = \alpha x_4 + b \}$  is a subspace of  $\mathbb{F}^4$   $\forall \alpha \in \mathbb{F}$ , and  $b = 0$ ; if  $b \neq 0$ , then  $(0, 0, 0, 0) \notin V$ . (additive identity)
- 2  $C([-\pi, \pi])$  (the set of continuous functions on  $[-\pi, \pi]$ ) is a subspace of  $\mathbb{R}^{[-\pi, \pi]}$ .
- 3 The set of differentiable real-valued functions on  $\mathbb{R}$  is a subspace of  $\mathbb{R}^{\mathbb{R}}$ .
- 4 The set of differentiable real-valued functions  $f$  on the interval  $(-\pi, \pi)$  such that  $f'(0) = \beta$  is a subspace of  $\mathbb{R}^{(-\pi, \pi)}$  if and only if  $\beta = 0$ . (additive closure)
- 5 The set of all sequences of complex numbers is a subspace of  $\mathbb{C}^{\infty}$

(2)-(3)-(4) show that a huge amount of calculus is built on top of linear structures; and a better understanding of linear algebra can improve and formalize our understanding of calculus.

## Sums of Subspaces :: Definition

## Definition (Sum of Subsets)

Suppose  $U_1, \dots, U_m$  are subsets of  $V$ .

The sum of  $U_1, \dots, U_m$ , denoted

$$U_1 + \cdots + U_m,$$

is the set of all possible sums of elements of  $U_1, \dots, U_m$ .

More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}.$$

## Sums of Subspaces :: Examples

## Captain Obvious' Example: Sums of Subspaces

Suppose  $U$  is the set of all elements of  $\mathbb{F}^n$  whose second-to- $n^{\text{th}}$  coordinates equal 0, and  $W$  is the set of all elements of  $\mathbb{F}^n$  whose first and third-to- $n^{\text{th}}$  coordinates equal 0:

$$\begin{aligned}U &= \{(x, 0, 0, \dots, 0) \in \mathbb{F}^n : x \in \mathbb{F}\} \quad \text{and} \\W &= \{(0, y, 0, \dots, 0) \in \mathbb{F}^n : y \in \mathbb{F}\}\end{aligned}$$

Then

$$U + W = \{(x, y, 0, \dots, 0) \in \mathbb{F}^n : x, y \in \mathbb{F}\}$$



## Sums of Subspaces :: Examples

Example: Sums of Subspaces

Suppose

$$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}, \quad \text{and}$$

$$W = \{(a, a, a, b) \in \mathbb{F}^4 : a, b \in \mathbb{F}\}.$$

Then

$$U + W = \{(\alpha, \alpha, \beta, \gamma) \in \mathbb{F}^4 : \alpha, \beta, \gamma \in \mathbb{F}\}$$

 $u \in U$  and  $w \in W \Rightarrow (u + w) \in U + W$  —

$$\alpha = x + a, \quad \beta = y + a, \quad \gamma = y + b$$

 $\forall z \in U + W \exists u \in U$  and  $w \in W : z = u + w$  —
Given any  $\alpha, \beta, \gamma, x \in \mathbb{F}^n$ , simply let

$$a = \underbrace{\alpha - x}, \quad y = \underbrace{\beta - a}, \quad b = \gamma - y$$

## Sum of Subspaces

The sum of subspaces is a subspace, and is the smallest subspace containing all the summands.

Theorem (Sum of subspaces is the smallest containing subspace)  
*Suppose  $U_1, \dots, U_m$  are subspaces of  $V$ . Then  $U_1 + \dots + U_m$  is the smallest subspace of  $V$  containing  $U_1, \dots, U_m$ .*

## Proof — Sum of Subspaces

Proof (Sum of Subspaces is the Smallest Containing Subspace)

$0 \in U_1 + \cdots + U_m$ , and the closure under addition and scalar multiplication on  $U_1 + \cdots + U_m$  are both fairly straight-forward.

**Thus  $U_1 + \cdots + U_m$  is a subspace of  $V$ .**

$U_1, \dots, U_m$  are all contained in  $U_1 + \cdots + U_m$ : — let  $u_k \in U_k$  and consider sums  $u_1 + \cdots + u_m$  where all except one of the  $u_k$ 's are 0.

Conversely, every subspace of  $V$  containing  $U_1, \dots, U_m$  contains  $U_1 + \cdots + U_m$  (subspaces contain all finite sums of their elements).

**Thus  $U_1 + \cdots + U_m$  is the smallest subspace of  $V$  containing  $U_1, \dots, U_m$ .**

## Direct Sums

### Definition (Direct Sum)

Suppose  $U_1, \dots, U_m$  are subspaces of  $V$

- The sum  $U_1 + \dots + U_m$  is called a **direct sum** if each element of  $U_1 + \dots + U_m$  can be written in only one way (**UNIQUELY**) as a sum  $u_1 + \dots + u_m$ , where each  $u_j \in U_j$ .
- If  $U_1 + \dots + U_m$  is a direct sum, then  $U_1 \oplus \dots \oplus U_m$  denotes  $U_1 + \dots + U_m$ , with the  $\oplus$  notation serving as an indication that this is a direct sum.

Note that the spaces in the previous example do not form a direct sum

$$\begin{aligned}U &= \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}, \quad \text{and} \\W &= \{(a, a, a, b) \in \mathbb{F}^4 : a, b \in \mathbb{F}\}.\end{aligned}$$

since there are multiple ways to write any vector  $\vec{v} \in U + W$ .



## Example :: Direct Sum

## Example :: Direct Sum

Let  $U_k$  be the subspace of  $\mathbb{F}^n$  of the form

$$U_k = \{(0, \dots, 0, u_k, 0, \dots, 0) \in \mathbb{F}^n, u_k \in \mathbb{F}\}$$

*i.e.* only the  $k^{\text{th}}$  coordinate is allowed to be non-zero.

Then  $\mathbb{F}^n = U_1 \oplus \dots \oplus U_n$ .

With

$$W_k = \bigoplus_{j=1}^k U_j = U_1 \oplus \dots \oplus U_k$$

then

$$W_k = \{(w_1, \dots, w_k, 0, \dots, 0) \in \mathbb{F}^n : w_j \in \mathbb{F}, j = 1, \dots, k\}, \quad k = 1, \dots, n$$

## Example :: Not a Direct Sum

## Example :: Not a Direct Sum

Let

$$U_1 = \{(x, y, 0) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$$

$$U_2 = \{(0, 0, z) \in \mathbb{F}^3 : z \in \mathbb{F}\}$$

$$U_3 = \{(0, \beta, \beta) \in \mathbb{F}^3 : \beta \in \mathbb{F}\}$$

Then  $\mathbb{F}^3 = U_1 + U_2 + U_3$ ; also  $0 \in U_1 \cap U_2 \cap U_3$ , but  $\forall \alpha \in \mathbb{F}$ :

$$u_1 = (0, \alpha, 0) \in U_1$$

$$u_2 = (0, 0, \alpha) \in U_2$$

$$u_3 = (0, -\alpha, -\alpha) \in U_3$$

so that  $u_1 + u_2 + u_3 = 0$ . Since we can write  $0 \in \mathbb{F}^3$  in more than one way,  $U_1 + U_2 + U_3$  is not a direct sum.

Note:  $\mathbb{F}^3 = U_1 \oplus U_2$ .

Question: Are there more direct sums?

## Condition for a direct sum; Direct sum of two subspaces

Theorem (Condition for a direct sum)

*Suppose  $U_1, \dots, U_m$  are subspaces of  $V$ . Then  $U_1 + \dots + U_m$  is a direct sum if and only if the only way to write  $0$  as a sum  $u_1 + \dots + u_m$ , where each  $u_j \in U_j$ , is by taking each  $u_j = 0$ .*

Theorem (Direct sum of **two** subspaces)

*Suppose  $U$  and  $W$  are subspaces of  $V$ . Then  $U \oplus W$  is a direct sum if and only if  $U \cap W = \{0\}$ .*

## Proof — Condition for a Direct Sum

## Proof (Condition for a Direct Sum)

First suppose  $U_1 + \cdots + U_m$  is a direct sum. Then the only way to write 0 as a sum  $u_1 + \cdots + u_m$ , where each  $u_j \in U_j$ , is by taking each  $u_j = 0$ . (By uniqueness)

Now suppose that the only way to write 0 as a sum  $u_1 + \cdots + u_m$ , where each  $u_j \in U_j$ , is by taking each  $u_j = 0$ . To show that  $U_1 + \cdots + U_m$  is a direct sum, let  $v \in U_1 + \cdots + U_m$ .

We can write  $v = u_1 + \cdots + u_m$ , for some  $u_j \in U_j$ , ( $j = 1, \dots, m$ ).

To show that this representation is unique, suppose we also have  $v = v_1 + \cdots + v_m$  where  $v_1 \in U_1, \dots, v_m \in U_m$ . Subtracting these two equations, we have

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m).$$

Because  $(u_j - v_j) \in U_j$ , the equation above implies that each  $(u_j - v_j) = 0$ . Thus  $u_j = v_j$ , ( $j = 1, \dots, m$ ), as desired.

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**Method:** Assume  $\exists!$ , show they are the same (using the properties).

## Proof — Direct Sum of Two Subspaces

Proof (Direct Sum of two Subspaces)

First suppose that  $U + W$  is a direct sum. If  $v \in U \cap W$ , then  $0 = v + (-v)$ , where  $v \in U$  and  $(-v) \in W$ .

By the unique representation of 0 as the sum of a vector in  $U$  and a vector in  $W$ , we have  $v = 0$ . Thus  $U \cap W = \{0\}$ , completing the proof in one direction.

To prove the other direction, now suppose  $U \cap W = \{0\}$ . To prove that  $U + W$  is a direct sum, suppose  $u \in U$ ,  $w \in W$ , and  $0 = u + w$ :

We need only show that  $u + w = 0$  (by the previous theorem). The equation above implies that  $u = -w \in W$ . Thus  $u \in U \cap W$ . Hence  $u = 0$ , which by the equation above implies that  $w = 0$ , completing the proof.

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e.g.  $1\mathbb{C}-\{1, \mathbf{5}\}$

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**1C-5:** Is  $\mathbb{R}^2$  a subspace of the complex vector space  $\mathbb{C}^2$ ?

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**No:** For  $\mathbb{R}^2$  to be a subspace, it must be closed under the operations (addition, and scalar multiplication) “inherited” from  $\mathbb{C}^2$ .

- Addition is not a problem since  $\forall u, v \in \mathbb{R}^2, (u + v) \in \mathbb{R}^2$ .
- However, whereas  $\mathbb{C}^2$  is closed under scaling by  $\alpha \in \mathbb{C}$ ,  $\mathbb{R}^2$  is not; in particular  $\forall u \in \mathbb{R}^2: u \neq 0, iu \notin \mathbb{R}^2$ .

## Suggested Problems

**1.A**—1, 4, **5**, **6**, 7, 8, 9

**1.B**—1, 3, 5

**1.C**—1, 5, **10**, **20**



## Assigned Homework

## HW#1, Due Date in Canvas/Gradescope

**1.A**—5, 6

**1.B**—1, 3

**1.C**—10, 20

**Note:** Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually “scheduled.”)

Upload homework to [www.Gradescope.com](http://www.Gradescope.com)

# Supplements

⟨PLACEHOLDER⟩