Math 524: Linear Algebra

Notes #2 — Finite Dimensional Vector-Spaces

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Outline

Student Learning Targets, and Objectives

Target Span
- Objective: Know how to build a finite-dimensional vector space using spanning vectors.

Target Linear Independence
- Objective: Know how to determine whether a set of vectors is linearly independent, and how to remove linearly dependent vectors from a set to generate a linearly independent set.

Target Bases
- Objective: Be able to reduce a spanning list to a basis of a vector space
- Objective: Be able to extend a linearly indepenent list to a basis of a vector space

Target Dimension
- Objective: Know how to determine the dimension of a subspace

Introduction

Previously, we discussed vector spaces; and we even included one brief mention of $\mathbb{C}^\infty$.

However in **Linear Algebra** the main focus is on finite-dimensional vector spaces (which we will formally introduce shortly).

The study of infinite-dimensional vector spaces mainly fall under the umbrella of

Functional Analysis $\approx$ *Linear Algebra* + *Real Analysis*

see e.g. Hilbert Spaces, Banach Spaces.

The Road to Infinity...

Time-Target: 3×75-minute lectures.

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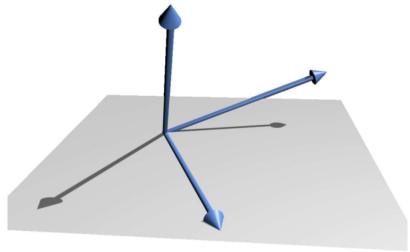
Span and Linear Independence



Linearly independent vectors in $\mathbb{R}^3$ — 3D Visualization

Image Credit: Creative Commons Attribution-Share Alike 4.0 International License.
https://commons.wikimedia.org/wiki/File:Vec-indep.png

Linear Combination :: Definition

Definition (Linear Combination)

A **linear combination** of a set $\{v_1, \ldots, v_m\}$ vectors $v_k \in V$ is a vector of the form

$$w = \sum_{k=1}^{m} a_k v_k,$$

with $a_k \in \mathbb{F}$.

Rewind (Notation)

V is a vector space.

Linear Combination :: Examples

We follow the Axler's notation, writing vectors as lists:

Example (Linear Combination)

The vector $(5, 10, 12 + 2i, 30) \in \mathbb{F}^4$ is a linear combination of the vectors $(1, 2, i, 3)$, and $(1, 2, 4, 8)$ since

$$(5, 10, 12 + 2i, 30) = 2(1, 2, i, 3) + 3(1, 2, 4, 8)$$

Example (Not Linear Combination)

The vector $(1, 1, 1) \in \mathbb{F}^3$ is not a linear combination of the vectors $(1, 0, 0)$, and $(1, 1, 0)$ since $\forall a_1, a_2 \in \mathbb{F}$:

$$(1, 1, 1) \neq a_1(1, 0, 0) + a_2(1, 1, 0).$$

Or, if you prefer: $\nexists a_1, a_2 \in \mathbb{F}$: $(1, 1, 1) = a_1(1, 0, 0) + a_2(1, 1, 0)$

Span

Definition (Span)

The set of all linear combinations of a list of vectors $v_1, \ldots, v_m \in V$ is called the **span** of $v_1, \ldots, v_m$, denoted $\mathrm{span}(v_1, \ldots, v_m)$. In other words,

$$\mathrm{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbb{F}\}.$$

We **define** the span of the empty list $()$ to be $\{0\}$.

Span :: Examples

Revisiting the previous examples

Example (Span)

$$\underbrace{(5, 10, 12+2i, 30)}_{w} \in \operatorname{span}\left(\underbrace{(1,2,i,3)}_{v_1}, \underbrace{(1,2,4,8)}_{v_2}\right)$$

since there is a linear combination so that $w = a_1 v_1 + a_2 v_2$.

Example (Not Linear Combination)

$$\underbrace{(1,1,1)}_{w} \notin \operatorname{span}\left(\underbrace{(1,0,0)}_{v_1}, \underbrace{(1,1,0)}_{v_2}\right)$$

since there is no linear combination so that $w = a_1 v_1 + a_2 v_2$.

Rewind :: Span and Linear Independence

Connecting with Previous Classes

On vectors in $\mathbb{C}^n$ and $\mathbb{R}^n$ we can directly re-use row-reductions from [MATH 254] to *Reduced Row Echelon Form* to determine existence of linear combinations:

Rewind (Span and Linear Independence)

$$\operatorname{rref}\left(\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 2 & 10 \\ i & 4 & 12+2i \\ 3 & 8 & 30 \end{array}\right]\right) = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow a_1 = 2,\ a_2 = 3.$$

$$\operatorname{rref}\left(\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]\right) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \hline 0 & 0 & 1 \end{array}\right] \Rightarrow$$ No Solutions; not in the span

Span is the Smallest Containing Subspace

Theorem (Span is the Smallest Containing Subspace)

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

We notice that this statment is similar to

Rewind (Sum of Subspaces is the Smallest Containing Subspace [NOTES#1])

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

and, not surprisingly, the proof is also similar...

Proof :: Span is the Smallest Containing Subspace 1/2

Proof (Span is the Smallest Containing Subspace)

Let $v_1, \ldots, v_m \in V$.

(1) Show that $W = \operatorname{span}(v_1, \ldots, v_m)$ is a subspace of V:

- $0 \in W$ since $0 = \sum_{k=1}^{m} 0 v_k$
- W is closed under addition since $\sum_{k=1}^{m} a_k v_k + \sum_{k=1}^{m} b_k v_k = \sum_{k=1}^{m} (a_k + b_k) v_k$
- W is closed under scalar multiplication since $\forall \lambda \in \mathbb{F}$: $\lambda \left(\sum_{k=1}^{m} a_k v_k\right) = \sum_{k=1}^{m} (\lambda a_k) v_k$

Therefore W is a subspace of V.

Next, we show that it is the *smallest* subspace...

Proof :: Span is the Smallest Containing Subspace 2/2

Proof (Span is the Smallest Containing Subspace)

Each $v_\ell \in W = \mathrm{span}(v_1, \ldots, v_m)$, since $v_\ell = \sum_{k=1}^{m} \delta_{\ell k} v_k$, where

$$\delta_{\ell k} = \begin{cases} 1 & \text{if} \quad \ell = k \\ 0 & \text{if} \quad \ell \neq k \end{cases}$$

Conversely, **every** subspace $\widehat{W}$ of V which contains each of the vectors $v_1, \ldots, v_m$ must contain $\mathrm{span}(v_1, \ldots, v_m)$ (since subspaces are closed under addition and scalar multiplication).

Therefore W is the smallest subspace of V.

Note: $\delta_{\ell k}$ is known as the **Kronecker delta**; (Leopold Kronecker, 1823–1891). It is a very convenient notation for generating coefficients that are either zero or one, with predicable patterns.

Challenge :: Alternative Proof

Challenge (Alternative Proof)

Can you formulate a different proof, which directly uses the SUM-OF-SUBSPACES result from [NOTES#1]?

Finite-Dimensional Vector Space :: Formal Definition

Definition (Language: "Spans")

If $V = \mathrm{span}(v_1, \ldots, v_m)$, then we say that the *set* of vectors $\{v_1, \ldots, v_m\}$, or if you prefer the *list* of vectors $v_1, \ldots, v_m$ **spans the vector space V.**

Definition (Finite-Dimensional Vector Space)

A vector space is called **finite-dimensional** if some list [FINITE LENGTH, n, BY DEFINITION] of vectors in it spans the space.

Definition (Infinite-Dimensional Vector Space)

A vector space is called **infinite-dimensional** if it is not finite-dimensional.

Polynomial Detour

Polynomials will have many uses for us going forward, so let's introduce some (familiar?) definitions:

Definition (Polynomial, $\mathcal{P}(\mathbb{F})$)

- A function $p : \mathbb{F} \to \mathbb{F}$ is called a polynomial with coefficients on $\mathbb{F}$ if there exists $a_0, \ldots, a_m \in \mathbb{F}$ such that $\forall z \in \mathbb{F}$
$$p(z) = \sum_{k=0}^{m} a_k z^k.$$
- $\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in $\mathbb{F}$.

Polynomial Detour $\mathcal{P}(\mathbb{F}) \subset \mathbb{F}^{\mathbb{F}}$

With the usual definitions of addition and scalar multiplication, $\mathcal{P}(\mathbb{F})$ is a subspace of $\mathbb{F}^{\mathbb{F}}$

Definition (Degree of a Polynomial)

- A polynomial $p \in \mathcal{P}(\mathbb{F})$ is said to have **degree** m if there exist scalars $a_0, \ldots, a_m \in \mathbb{F}$, with $a_m \neq 0$ such that

$$p(z) = \sum_{k=0}^{m} a_k z^k.$$

$\forall z \in \mathbb{F}$. If p has degree m, we write $\deg(p) = m$.

- We **define** the degree of the zero-polynomial $p(z) \equiv 0$ to be $-\infty$.

Polynomial Detour

Definition ($\mathcal{P}_m(\mathbb{F})$)

For a non-negative integer m, $\mathcal{P}_m(\mathbb{F})$ denotes the set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m.

Note: $\mathcal{P}_m(\mathbb{F}) = \mathrm{span}\,(1, z, \ldots, z^m)$.

Note: $\mathcal{P}_m(\mathbb{F})$ is a finite-dimensional vector space for each non-negative integer m.

Note: $\mathcal{P}(\mathbb{F})$ is infinite-dimensional.

Linear Independence

Let $v_1, \ldots, v_m \in V$, and $v \in \mathrm{span}(v_1, \ldots, v_m)$. By our definitions we must have $a_1, \ldots, a_m \in \mathbb{F}$ so that $v = \sum_{k=1}^{m} a_k v_k$.

Question: Are the scalars $a_1, \ldots, a_m \in \mathbb{F}$ unique?

If they are not, then we can find $b_1, \ldots, b_m \in \mathbb{F}$ so that $v = \sum_{k=1}^{m} b_k v_k$, and

$$0 = (v - v) = \sum_{k=1}^{m} (a_k - b_k) v_k$$

Clearly $a_k = b_k$ $(k = 1, \ldots, m)$ provides one possibility.

The case where that is the only linear combination which gives 0 is extremely important; we call that *linear independence*...

Linear Independence :: Definition

Definition (Linear Independence)

- A list $v_1, \ldots, v_m \in V$ is called **linearly independent** if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ so that $0 = \sum_{k=1}^{m} a_k v_k$ is $a_k = 0$ $(k = 1, \ldots, m)$.
- The empty list is also linearly independent **by definition.**

Definition (Linearly Dependent)

- A list of vectors $\in V$ is called **linearly dependent** if it is not linearly independent.
- A list $v_1, \ldots, v_m \in V$ is **linearly dependent** if there exists $a_1, \ldots, a_m \in \mathbb{F}$, not all zeros, such that $0 = \sum_{k=1}^{m} a_k v_k$.

Linear Independence/Dependence :: Examples

- ⊕ A single vector $v \in V$ is linearly independent if and only if $v \neq 0$.
- ⊕ $u, v \in V$ are linearly independent if and only if neither is a scalar multiple of the other.
- ⊕ The **"Standard Coordinate Vectors"** $e_k = (\delta_{1k}, \dots, \delta_{mk}) \in \mathbb{F}^m$, $k = 1, \dots, m$ are linearly independent in $\mathbb{F}^m$.
- ⊕ The list $1, z, \dots, z^m$ is linearly independent in $\mathcal{P}(\mathbb{F})$ for each non-negative integer m.
- ⊖ If some vector in a list of vectors $\in V$ is a linear combination of the other vectors, then the list is linearly dependent.
- ⊖ Every list of vectors $\in V$ containing the 0-vector is linearly dependent.

Linear Dependence

Theorem (Linear Dependence)

Suppose $v_1, \dots, v_m$ is a linearly dependent list $\in V$. Then there exists $\ell \in \{1, \dots, m\}$ such that the following hold:

(1) $v_\ell \in \mathrm{span}(v_1, \dots, v_m)$,

(2) *if the ℓ^{th} term is removed from $v_1, \dots, v_m$, the span of the remaining list equals $\mathrm{span}(v_1, \dots, v_m)$:*

$$\mathrm{span}(v_1, \dots, v_{\ell-1}, v_{\ell+1}, \dots v_m) = \mathrm{span}(v_1, \dots, v_m)$$

Proof :: Key Pieces

Sketch-Proof

(1) from the expression $\displaystyle\sum_{k=1}^{m} a_k v_k = 0$, we can explicitly solve for

$$v_\ell = -\sum_{k=1}^{\ell-1} \frac{a_k}{a_\ell} v_k.$$

(2) we can replace v_ℓ by this sum in the expression for $u = \displaystyle\sum_{k=1}^{m} c_k v_k \in V$, and thus have an expression for u using only $(m-1)$ terms.

Length of Linearly Independent List $\leq$ Length of Spanning List

Theorem (Length of Linearly Independent List $\leq$ Length of Spanning List)

! *In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.* !

Proof :: Length of Linearly Independent List $\leq$ Length of Spanning List 1/2

Proof (Length of Linearly Independent List $\leq$ Length of Spanning List)

Suppose $u_1, \dots, u_m$ is linearly independent in V. Suppose also that $w_1, \dots, w_n$ spans V. We need to prove that $m \leq n$. We do so through the multi-step process described below; in each step we add one of the u's and remove one of the w's.

Step 1 Let B be the list $w_1, \dots, w_m$, which spans V. Thus adjoining any vector in V to this list produces a linearly dependent list (because the newly adjoined vector can be written as a linear combination of the other vectors). In particular, the list $u_1, w_1, \dots, w_m$ is linearly dependent. Thus by the previous theorem, we can remove one of the w's so that the new list B (of length n) consisting of u_1 and the remaining w's spans V.

Proof :: Length of Linearly Independent List $\leq$ Length of Spanning List 2/2

Proof (Length of Linearly Independent List $\leq$ Length of Spanning List)

Step j The list B (of length n) from step $(j-1)$ spans V. Thus adjoining any vector to this list produces a linearly dependent list. In particular, the list of length $(n+1)$ obtained by adjoining u_j to B, placing it just after $u_1, \dots, u_j$, is linearly dependent. By the previous theorem, one of the vectors in this list is in the span of the previous ones, and because $u_1, \dots, u_j$ is linearly independent, this vector is one of the w's, not one of the u's. We can remove that w from B so that the new list B (of length n) consisting of $u_1, \dots, u_j$ and the remaining w's spans V.

After **Step m**, we have added all the u's and the process stops. At each step as we add a u to B, the previous theorem implies that there is some w to remove. Thus there are at least as many w's as u's.

Finite-Dimensional Subspaces

Theorem (Finite-dimensional subspaces)

Every subspace of a finite-dimensional vector space is finite-dimensional.

Rewind (Finite-Dimensional Vector Space)

A vector space is called **finite-dimensional** if some list [FINITE LENGTH, n, BY DEFINITION] of vectors in it spans the space.

Proof :: Finite-Dimensional Subspaces

Proof (Finite-Dimensional Subspaces)

Suppose V is finite-dimensional and U is a subspace of V. We need to prove that U is finite-dimensional:

Step 1 If $U = \{0\}$, then U is finite-dimensional and we are done; otherwise choose a nonzero vector $v_1 \in U$.

Step ℓ If $U = \mathrm{span}(v_1, \dots, v_{\ell-1})$, then U is finite-dimensional and we are done; otherwise choose a vector $v_\ell \in U$ such that $v_\ell \notin \mathrm{span}(v_1, \dots, v_{\ell-1})$.

After each step, we have constructed a list of vectors such that no vector in this list is in the span of the previous vectors. Thus after each step we have constructed a linearly independent list.

This linearly independent list cannot be longer than any spanning list of V. Thus the process must terminate, which means that U is finite-dimensional.

⟨⟨⟨ Live Math ⟩⟩⟩

e.g. 2A-{1, **3**}

Live Math :: Covid-19 Version 2A-3

2A-3: Find a number t such that

$$(3,1,4),\ (2,-3,5),\ (5,9,t)$$

is not linearly independent in $\mathbb{R}^3$.

⋇ Strategy: Leverage [Math 254] Knowledge ⋇

In the language and notation of [Math 254], we make the vectors columns in a matrix (even though we have not formally defined a matrix yet), and recall that the row-reduced-echelon-form (RREF) reveals whether the columns of a matrix are linearly (in)dependent. Even simpler, it is sufficient to forward-reduce to lower-triangular form and look for the (non)existence of free variables (zeros on the diagonal).

Live Math :: Covid-19 Version 2A-3

⋇ Computations (Row-Reductions) ⋇

$$M = \begin{bmatrix} 3 & 2 & 5 \\ 1 & -3 & 9 \\ 4 & 5 & t \end{bmatrix}$$

$$\{r_1 \leftrightarrow r_2\}: \begin{bmatrix} 1 & -3 & 9 \\ 3 & 2 & 5 \\ 4 & 5 & t \end{bmatrix}, \quad \{r_2 \leftarrow r_2 - 3r_1\}: \begin{bmatrix} 1 & -3 & 9 \\ 0 & 11 & -22 \\ 4 & 5 & t \end{bmatrix},$$

$$\{r_3 \leftarrow r_3 - 4r_1\}: \begin{bmatrix} 1 & -3 & 9 \\ 0 & 11 & -22 \\ 0 & 17 & t-36 \end{bmatrix}, \quad \{r_2 \leftarrow r_2/11\}: \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -2 \\ 0 & 17 & t-36 \end{bmatrix},$$

$$\{r_3 \leftarrow r_3 - 17r_2\}: \begin{bmatrix} 1 & -3 & 9 \\ 0 & 1 & -2 \\ 0 & 0 & t-2 \end{bmatrix},$$

We have a free variable and linear dependence if and only if $(t-2) = 0 \Leftrightarrow \mathbf{t} = \mathbf{2}$.

Bases

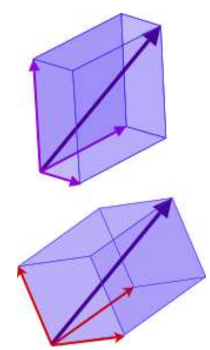


Figure: A vector (here in 3D, shown the purple arrow) can be represented in terms of two different bases (green and blue arrows), each basis vector is scalar-multiplied appropriately so they add to the vector.

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https://en.wikipedia.org/wiki/Basis_(linear_algebra)

Basis :: Definition

Definition (Basis)

A **basis** of V is a list of vector $\in V$ that is *linearly independent* **and** *spans* V.

Rewind (Linear Independence, Spans)

- A list $v_1, \dots, v_m \in V$ is called **linearly independent** if the only choice of $a_1, \dots, a_m \in \mathbb{F}$ so that $0 = \sum_{k=1}^{m} a_k v_k$ is $a_k = 0$ $(k = 1, \dots, m)$.
- If $V = \mathrm{span}(v_1, \dots, v_m)$, then we say that the set of vectors $\{v_1, \dots, v_m\}$, or if you want the list of vectors $v_1, \dots, v_m$ **spans the vector space** V**.**
- In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Bases :: Examples

Example (Bases)

- ⊕ $e_k = (\delta_{1k}, \dots, \delta_{mk}) \in \mathbb{F}^m$, $k = 1, \dots, m$ is a basis of $\mathbb{F}^m$, called the **standard basis**.
- ⊕ Any two linearly independent vectors $\in \mathbb{F}^2$ is a basis of $\mathbb{F}^2$.
- ⊕ $1, z, \dots, z^m$ is a basis for $\mathcal{P}_m(z)$
- ⊖ Two linearly independent vectors $\in \mathbb{F}^3$ is NOT a basis of $\mathbb{F}^3$, since they cannot span $\mathbb{F}^3$.
- ⊖ A list of linearly dependent vectors that span $\mathbb{F}^n$ is not a basis.

Application (Signal Processing :: Basis Pursuit)

`https://scholar.google.com/scholar?q=basis+pursuit`

See also, "frame" and "tight frame" (requires inner products, which we dont have... yet.)

Basis :: Criterion

Theorem (Criterion for Basis)

A list $v_1, \dots, v_n$ of vectors $\in V$ is a basis for V if and only if $\forall v \in V$ can be written uniquely in the form

$$v = \sum_{\ell=1}^{n} a_\ell v_\ell, \quad \text{where } a_1, \dots, a_n \in \mathbb{F}$$

Proof (Sketch Proof :: Criterion for Basis)

- Pick a basis, show uniqueness $\forall v \in V$ (just like the proof for linear independence)
- Assume uniqueness $\forall v \in V \Rightarrow$ the collection of vectors $v_1, \dots, v_n$ spans V; use $v = 0$, which forces $a_1 = \cdots = a_n = 0$, this shows linear independence $\rightsquigarrow$ a basis of V.

Spanning List Contains a Basis

Theorem (Spanning List Contains a Basis)

Every spanning list in a vector space can be reduced to a basis of the vector space.

Comment

A spanning list in a vector space may not be a basis because it is not linearly independent. The theorem says that given any spanning list, some (possibly none) of the vectors in it can be discarded so that the remaining list is linearly independent and still spans the vector space.

Proof :: Spanning List Contains a Basis

Proof (Spanning List Contains a Basis)

Suppose $v_1, \dots, v_n$ spans V. We want to remove linearly dependent vectors from $v_1, \dots, v_n$ so that the remaining set form a basis for V:

0 Start with $B = \{v_1, \dots, v_n\}$.

1 if $v_1 = 0$, delete it from B.

k if $v_k \in \mathrm{span}(v_1, \dots, v_{k-1})$, delete v_k from B.

Repeat until $k = n$. The final list B still spans V and contains only linearly independent vectors. $\Rightarrow$ We have a basis.

Rewind :: Spanning List Contains a Basis

Connecting with Previous Classes

Rewind (Spanning List Contains a Basis)

Given a set of spanning vectors in $\mathbb{R}^4$:

$$v_1 = (2,2,2,4),\ v_2 = (2,4,1,3),\ v_3 = (1,4,1,3),$$
$$v_4 = (2,2,4,1),\ v_5 = (1,3,3,4),\ v_6 = (1,2,2,3)$$

$$\mathrm{rref}\left(\begin{bmatrix} 2 & 2 & 1 & 2 & 1 & 1 \\ 2 & 4 & 4 & 2 & 3 & 2 \\ 2 & 1 & 1 & 4 & 3 & 2 \\ 4 & 3 & 3 & 1 & 4 & 3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 29/38 & 12/19 \\ 0 & 1 & 0 & 0 & -26/19 & -13/19 \\ 0 & 0 & 1 & 0 & 30/19 & 15/19 \\ 0 & 0 & 0 & 1 & 6/19 & 3/19 \end{bmatrix}$$

The columns with leading ones — $\{1,2,3,4\}$ tell us that $\{v_1, \dots, v_4\}$ form a basis for $\mathbb{R}^4$.

The fact that we have 4 leading ones confirms that we indeed have a spanning set of vectors.

Basis of Finite-Dimensional Vector Space

Theorem (Basis of Finite-Dimensional Vector Space)

Every finite-dimensional vector space has a basis.

Proof (Basis of Finite-Dimensional Vector Space)

By definition, a finite-dimensional vector space has a spanning list. The previous result tells us that each spanning list can be reduced to a basis.

Linearly Independent List Extends to a Basis

Theorem (Linearly Independent List Extends to a Basis)

Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Comment

We have shown that every spanning list can be reduced to a basis. The statement above is the "dual" that result; giving us a path in the opposite direction.

Proof :: Linearly Independent List Extends to a Basis

Proof (Linearly Independent List Extends to a Basis)

Suppose $u_1, \dots, u_m$ is linearly independent in a finite-dimensional vector space V. Let $w_1, \dots, w_n$ be a basis of V. Thus the list

$$u_1, \dots, u_m, w_1, \dots, w_n$$

spans V. Applying the steps from the proof for [Spanning List Contains Basis] to this list produces a list of the vectors $u_1, \dots, u_m$ (all of them since they are linearly independent), and some of the w-vectors. This list must be a basis since $w_1, \dots, w_n$ is a basis.

Rewind :: Linearly Independent List Extends to a Basis — Connecting with Previous Classes

Rewind (Linearly Independent List Extends to a Basis)

Given a set of linearly independent vectors in $\mathbb{R}^5$:

$$v_1 = (7,3,7,4,2),\ v_2 = (5,7,7,4,4),\ v_3 = (3,5,4,6,7)$$

and let $w_1, \dots, w_5$ be the standard basis for $\mathbb{R}^5$:

$$\mathrm{rref}\left(\begin{bmatrix} 7 & 5 & 3 & 1 & 0 & 0 & 0 & 0 \\ 3 & 7 & 5 & 0 & 1 & 0 & 0 & 0 \\ 7 & 7 & 4 & 0 & 0 & 1 & 0 & 0 \\ 4 & 4 & 6 & 0 & 0 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1/13 & 33/52 & -1/2 \\ 0 & 1 & 0 & 0 & 0 & 4/13 & -41/52 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & -2/13 & 7/26 & 0 \\ 0 & 0 & 0 & 1 & 0 & -7/13 & -17/13 & 1 \\ 0 & 0 & 0 & 0 & 1 & -15/13 & 59/26 & -2 \end{bmatrix}$$

The columns with leading ones — $\{1,2,3,4,5\}$ tell us that $\{v_1, \dots, v_3, w_1, \dots, w_2\}$ form a basis for $\mathbb{R}^5$.

Every Subspace of V is Part of a Direct Sum equal to V

Theorem (Every Subspace of V is Part of a Direct Sum equal to V)

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Comment: The proof is a direct application of the previous theorem...

Proof :: Every Subspace of V is Part of a Direct Sum equal to V 1/2

Proof (Every Subspace of V is Part of a Direct Sum equal to V)

Construction: Since V is finite-dimensional, so is U. There is a basis $u_1, \dots, u_m$ of U; $u_1, \dots, u_m$ is a linearly independent list in V. We can extend this list to a basis $u_1, \dots, u_m, w_1, \dots, w_n$ of V; Let $W = \mathrm{span}(w_1, \dots, w_n)$.

To show $V = U \oplus W$, we have to show

$$①\, V = U + W, \quad \text{and} \quad ②\, U \cap W = \{0\}.$$

① let $v \in V$. Since $u_1, \dots, u_m, w_1, \dots, w_n$ spans V, we can find $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that

$$v = \underbrace{a_1 u_1 + \cdots + a_m u_m}_{u \in U} + \underbrace{b_1 w_1 + \cdots + b_n w_n}_{w \in W}.$$

This shows $v \in U + W$, which shows $V = U + W$.

Proof :: Every Subspace of V is Part of a Direct Sum equal to V — 2/2

Proof (Every Subspace of V is Part of a Direct Sum equal to V)

2 let $v \in U \cap W$. Then $\exists a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$, so that

$$v = \underbrace{a_1 u_1 + \cdots + a_m u_m}_{\in U} = \underbrace{b_1 w_1 + \cdots + b_n w_n}_{\in W}$$

$$0 = (v - v) = a_1 u_1 + \cdots + a_m u_m - b_1 w_1 - \cdots - b_n w_n$$

Since $u_1, \dots, u_m, w_1, \dots, w_n$ is linearly independent, we must have

$$a_1 = \cdots = a_m = b_1 = \dots b_n = 0$$

which makes $v = 0$, and therefore $U \cap W = \{0\}$

Example: Every Subspace of V is Part of a Direct Sum equal to V

Revisiting with the "Rewind" Example (Linearly Independent List Extends to a Basis) —

Let

$$U = \operatorname{span}\left((7,3,7,4,2),\ (5,7,7,4,4),\ (3,5,4,6,7)\right)$$

the previous example shows that with

$$W = \operatorname{span}\left((1,0,0,0,0),\ (0,1,0,0,0)\right)$$

we have

$$\mathbb{R}^5 = U \oplus W$$

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e.g. 2B-{2, **8**}

Live Math :: Covid-19 Version — 2B-8

2B-8: Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V.

Solution :: Linear Independence

Suppose we have coeffiecients $a_1, \dots, a_m$, and $b_1, \dots, b_n$, so that the "joint" linear combination:

$$a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n = 0,$$

then (by well-defined algebra)

$$\underbrace{a_1 u_1 + \cdots + a_m u_m}_{\in U} = -\underbrace{(b_1 w_1 + \cdots + b_n w_n)}_{\in W}$$

Live Math :: Covid-19 Version 2B-8

Since $V = U \oplus W$, we have $U \cap W = \{0\}$, which makes

$$a_1 u_1 + \cdots + a_m u_m = 0$$
$$b_1 w_1 + \cdots + b_n w_n = 0$$

Both the $u_1, \dots, u_m$, and $w_1, \dots, w_n$ are linearly independent (one of the properties of being a **basis**); therefore

$$a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$$

it follows that the joint list $u_1, \dots, u_m, w_1, \dots, w_n$ is linearly independent.

⋇ Solution :: Spanning the Space ⋇

We need to show that $\mathrm{span}\,(u_1, \dots, u_m, w_1, \dots, w_n) = V$.

Live Math :: Covid-19 Version 2B-8

Suppose $v \in V$. Then (since $V = U \oplus W$), $\exists u \in U$ and $w \in W$: $v = u + w$.

Since $\mathrm{span}\,(u_1, \dots, u_m) = U$, and $\mathrm{span}\,(w_1, \dots, w_n) = W$, we can find $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ such that

$$u = a_1 u_1 + \cdots + a_m u_m, \quad w = b_1 w_1 + \cdots + b_n w_n$$

which gives

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n$$

which shows that $\mathrm{span}\,(u_1, \dots, u_m, w_1, \dots, w_n)$ spans V.

⋇ Solution :: Basis ⋇

Linearly Independent + **Spanning** $\rightsquigarrow$ **Basis**.

Dimension

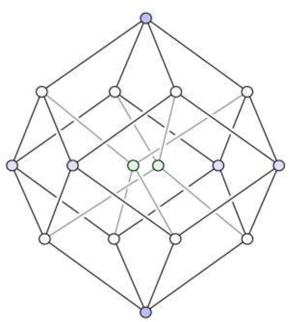

Figure: The 4D-hypercube, layered according to distance from one corner. As described in "Alice in Wonderland" by the Cheshire Cat, this vertex-first-shadow of the tesseract forms a rhombic dodecahedron. The two central vertices would coincide in an orthogonal projection from 4 to 3 dimensions, but here they were drawn slightly apart. .

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https://commons.wikimedia.org/wiki/File:Hypercubeorder.svg

Dimension

We have discussed finite-dimensional vector spaces, but not yet formally defined the *dimension* of a vector space; it is time to patch that hole.

There are no big surprises; the dimension of $\mathbb{F}^n$ is indeed n.

First, we note that the list of standard basis vectors $\{e_k = (\delta_{1k}, \dots, \delta_{nk}),\ k = 1, \dots, n\}$ of $\mathbb{F}^n$ has length n.

However, a finite-dimensional vector space in general has infinitely many different bases; so if we can show that all bases to have the same length, we can define the dimension as the length of the basis.

Basis Length Does Not Depend on Basis

Theorem (Basis Length Does Not Depend on Basis)

Any two bases of a finite-dimensional vector space have the same length.

Proof (Basis Length Does Not Depend on Basis)

Suppose V is finite-dimensional. Let B_1 and B_2 be two bases of V. Then B_1 is linearly independent in V and B_2 spans V, so the length of B_1 is at most the length of B_2 (by [LENGTH OF LINEARLY INDEPENDENT LIST ≤ LENGTH OF SPANNING LIST].)

Interchanging the roles of B_1 and B_2, we also see that the length of B_2 is at most the length of B_1. Thus the length of B_1 equals the length of B_2.

Dimension of a Finite-Dimensional Vector Space

Definition (Dimension, $\dim(V)$)

- The **dimension** of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V is denoted by $\dim(V)$.

Example (Dimensions)

- $\dim(\mathbb{F}^n) = n$.
- $\dim(\mathcal{P}_m(\mathbb{F})) = (m+1)$ since the basis $\{1, z, \ldots, z^m\}$ has $(m+1)$ basis vectors.

Dimension of a Subspace

Theorem (Dimension of a Subspace)

If V is finite-dimensional and U is a subspace of V, then $\dim(U) \leq \dim(V)$.

Proof (Dimension of a Subspace)

Suppose V is finite-dimensional and U is a subspace of V. Think of a basis of U as a linearly independent list in V, and think of a basis of V as a spanning list in V. Now use [LENGTH OF LINEARLY INDEPENDENT LIST ≤ LENGTH OF SPANNING LIST] to conclude that $\dim(U) \leq \dim(V)$.

Linearly Independent List of length $\dim(V)$ is a Basis

Theorem (Linearly Independent List of length $\dim(V)$ is a Basis)

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim(V)$ is a basis of V.

Comment: This means the second property "the list spans V" is automatically satisfied.

Proof (Linearly Independent List of length $\dim(V)$ is a Basis)

Suppose $\dim(V) = n$, and $v_1, \ldots, v_n$ is linearly independent in V. The list $v_1, \ldots, v_n$ can be extended to a basis of V (by [LINEARLY INDEPENDENT LIST EXTENDS TO A BASIS]). However, every basis of V has length n, so in this case the extension is the trivial one, meaning that no elements are adjoined to $v_1, \ldots, v_n$. In other words, $v_1, \ldots, v_n$ is a basis of V.

Spanning List of length $\dim(V)$ is a Basis

Theorem (Spanning List of Length $\dim(V)$ is a Basis)

Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim(V)$ is a basis of V.

Comment: This means the first property "the list is linearly independent" is automatically satisfied.

Proof (Spanning List of Length $\dim(V)$ is a Basis)

Suppose $\dim(V) = n$, and $v_1, \ldots, v_n$ spans V. The list $v_1, \ldots, v_n$ can be reduced to a basis of V (by [SPANNING LIST CONTAINS A BASIS]). However, every basis of V has length n, so in this case the reduction is the trivial one, meaning that no elements are deleted from $v_1, \ldots, v_n$. In other words, $v_1, \ldots, v_n$ is a basis of V.

Dimension of a Sum of Subspaces

We close out this discussion of Dimension by stating the result for subspaces:

Theorem (Dimension of a Sum of Subspaces)

If U_1 and U_2 are subspaces of a finite-dimensional vector space, then $\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2)$.

Proof :: Dimension of a Sum of Subspaces 1/3

Proof (Dimension of a Sum of Subspaces)

Let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$; thus $\dim(U_1 \cap U_2) = m$. $u_1, \ldots, u_m$ must be linearly independent, and can therefore be extended to a basis [LINEARLY INDEPENDENT LIST EXTENDS TO A BASIS] of U_1 and U_2 (independently):

$$\begin{array}{ll} \mathrm{basis}(U_1) = u_1, \ldots, u_m, v_1, \ldots, v_j & \dim(U_1) = m + j \\ \mathrm{basis}(U_2) = u_1, \ldots, u_m, w_1, \ldots, w_k & \dim(U_2) = m + k \end{array}$$

Showing that

$$u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$$

is a basis for $U_1 + U_2$ completes the proof; since we will have

$$\begin{aligned} \dim(U_1 + U_2) &= m + j + k \\ &= (m + j) + (m + k) - m \\ &= \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2). \end{aligned}$$

Proof :: Dimension of a Sum of Subspaces 2/3

Proof (Dimension of a Sum of Subspaces)

$\mathrm{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ contains U_1 and U_2, and $\mathrm{span}(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k) = U_1 + U_2$. We need to show that $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is linearly independent.

Consider ($a_\gamma, b_\delta, c_\zeta \in \mathbb{F}^n$; we need to show all are 0)

$$\sum_{\gamma=1}^{m} a_\gamma u_\gamma + \sum_{\delta=1}^{j} b_\delta v_\delta + \sum_{\zeta=1}^{k} c_\zeta w_\zeta = 0$$

Rearrange

$$\underbrace{\sum_{\zeta=1}^{k} c_\zeta w_\zeta}_{\in U_2} = \underbrace{\sum_{\gamma=1}^{m} -a_\gamma u_\gamma + \sum_{\delta=1}^{j} -b_\delta v_\delta}_{\in U_1}$$

$\Rightarrow \sum_{\zeta=1}^{k} c_\zeta w_\zeta \in U_1 \cap U_2$. $\Rightarrow \sum_{\zeta=1}^{k} c_\zeta w_\zeta = \sum_{\gamma=1}^{m} d_\gamma u_\gamma$

Proof :: Dimension of a Sum of Subspaces 3/3

Proof (Dimension of a Sum of Subspaces)

We have $\sum_{\zeta=1}^{k} c_\zeta w_\zeta = \sum_{\gamma=1}^{m} d_\gamma u_\gamma$, but $u_1, \dots, u_m, w_1, \dots, w_k$ is linearly independent, which forces $c_{1,\dots,k} = d_{1,\dots,m} = 0$.

$$\underbrace{\sum_{\zeta=1}^{k} c_\zeta w_\zeta}_{0} = \underbrace{\sum_{\gamma=1}^{m} -a_\gamma u_\gamma + \sum_{\delta=1}^{j} -b_\delta v_\delta}_{\in U_1}$$

but $u_1, \dots, u_m, v_1, \dots, v_j$ is linearly independent, which forces $a_{1,\dots,m} = b_{1,\dots,j} = 0$. Collecting all a, b, cs:

$$a_{1,\dots,m} = b_{1,\dots,j} = c_{1,\dots,k} = 0$$

which is what we needed.

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e.g. 2C-{**1**, 4, 12, 14, 17}

Live Math :: Covid-19 Version 2C-1

2C-1: Suppose V is finite-dimensional and U is a subspace of V such that $\dim(U) = \dim(V)$. Prove that $U = V$.

⋇ **Solution** ⋇

Let $u_1, \dots, u_n$ be a basis of U. Thus $n = \dim(U)$, and therefore $n = \dim(V)$. Thus $u_1, \dots, u_n$ is a linearly independent (since it is a basis) list of vectors in V, with length $\dim(V)$. Using [Linearly Independent List of length $\dim(V)$ is a Basis], $u_1, \dots, u_n$ must be a basis of V. $\forall v \in V$ can be written as a linear combination of $u_1, \dots, u_n$, and since $u_k \in U \Rightarrow U = V$.

Suggested Problems

2.A — 1, 3, **8**, **9**, **11**

2.B — 2, **3**, **5**, 8

2.C — 1, 4, **5**, **9**, 12, 14, 17 (some of these are quite challenging)

Assigned Homework — HW#2, Due Date in Canvas/Gradescope

2.A — 8, 9, 11

2.B — 3, 5

2.C — 5, 9

Note: Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually "scheduled.")

Upload homework to www.Gradescope.com

"Frame" — Just for Fun!

Definition (Frame — Generalization of bases to linearly dependent sets of vectors)

A **frame** of an inner product space is a generalization of a basis of a vector space to sets that may be linearly dependent. In the terminology of signal processing, a frame provides a redundant, stable way of representing a signal. Frames are used in error detection and correction and the design and analysis of filter banks and more generally in applied mathematics, computer science, and engineering.

[https://en.wikipedia.org/wiki/Frame_(linear_algebra)]

Useless Wiki-Knowledge

Q: *"Do our fields have anything to do with the Fields medal?"*

A: The Fields medal is named after John Charles Fields (1863 – 1932).

The term "Field" is due to work by (non-exhaustive list) Lagrange (1770), Vandermonde (1770), Ruffini (1799), Gauss (1801), Abel (1824), Galois (1832).

Dedekind (1871) introduced the word *"Körper"* (German — "Body" / "Corpus"), and Moore (1893) *"Field"* (English).