	Outline
Math 524: Linear Algebra Notes #3.1 — Linear Maps	 Student Learning Targets, and Objectives SLOs: Linear Maps
Peter Blomgren (blomgren@sdsu.edu) Department of Mathematics and Statistics Dynamical Systems Group Computational Sciences Research Center San Diego State University San Diego, CA 92182-7720 http://terminus.sdsu.edu/ Fall 2021 (Revised: December 7, 2021)	 2 Linear Maps, <i>i</i> The Vector Space of Linear Maps Null Spaces and Ranges 3 Linear Maps, <i>ii</i> Matrices 4 Problems, Homework, and Supplements Suggested Problems Assigned Homework Supplements
Peter Blomgren (blomgren@sdsu.edu) 3.1. Linear Maps — (1/59)	Peter Blomgren (blomgren@sdsu.edu) 3.1. Linear Maps — (2/59)
Student Learning Targets, and Objectives SLOs: Linear Maps	Linear Maps, iThe Vector Space of Linear MapsLinear Maps, iiNull Spaces and Ranges
Student Learning Targets, and Objectives	Introduction
 Target Fundamental Theorem of Linear Maps Objective Know how to apply FTLM to relate the dimensions of the range- and null-spaces of a linear map in a vector space Target The Matrix of a Linear Map with Respect to Given Bases Objective Know how to identify the matrix of a given Linear Map, given bases for the domain and range spaces. 	 "So far our attention has focused on vector spaces. No one gets excited about vector spaces. The interesting part of linear algebra is the subject to which we now turn — linear maps." — Sheldon Axler Notation If denotes either of the fields C or R U, V and W are vector spaces over F
Peter Blomgren (blomgren@sdsu.edu) 3.1. Linear Maps — (3/59)	Time-Target: 3×75-minute lectures.

Linear Maps

Definition (Linear Map)

A **linear map** from V to W is a function $T : V \mapsto W$ with the following properties:

• additivity (for vectors)

$$I(u+v) = I(u) + I(v), \ \forall u, v \in V$$

• homogeneity [of degree 1] (of scalar multiplication) $T(\lambda u) = \lambda T(u), \ \forall u \in V, \ \forall \lambda \in \mathbb{F}$

Language:

Linear Map, Linear Mapping, Linear Transform, Linear Transformation... many names for the same operation.

identity: ("one")

The **identity map**, denoted *I*, is the function on some vector space that takes each element to itself.

 $I \in \mathcal{L}(V, V)$ is defined by $Iv \equiv I(v) = v$.

differentiation:

Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{F}), \mathcal{P}(\mathbb{F}))$ be defined by $Dp \equiv D(p) = p'$. This function is a linear map, since (f+g)' = f'+g', and $(\lambda f)' = \lambda f'$ for differentiable functions f, g, and $\lambda \in \mathbb{F}$

multiplication by z^q :

Let $T \in \mathcal{L}(\mathcal{P}(\mathbb{F}), \mathcal{P}(\mathbb{F}))$ be defined by $Tp \equiv T(p) = z^q p(z)$, for $z \in \mathbb{F}$.

Linear Maps, i The Linear Maps, ii Nul

Notation and Examples

Notation (The Set of Linear Maps — $\mathcal{L}(V, W)$)

The set of all linear maps from V to W is denoted by $\mathcal{L}(V, W)$.

0, zero:

Let the symbol 0 denote the function that takes each element of some vector space to the additive identity of another vector space.

 $0 \in \mathcal{L}(V, W)$ is defined by $0v \equiv 0(v) = 0$.

The 0 on the left side of the equation above is a function in $\mathcal{L}(V, W)$, whereas the 0 on the right side is the additive identity in W. As usual, the meaning of "0" is "obvious from context."

So far, we have 4 "zeros": $\in \mathbb{F}, V, W, \mathcal{L}(V, W)$...

Note: $\mathcal{L}(V, W) \subset W^V$ (the space of all functions $f : V \mapsto W$).



Examples

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integration:

Let
$$\mathcal{T} \in \mathcal{L}(\mathcal{P}(\mathbb{R}),\mathbb{R})$$
 be defined by

$$Tp \equiv T(p) = \int_0^1 p(x) \, dx$$

is a linear map since $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$, and $\int (\lambda f(x)) dx = \lambda \int f(x) dx$, for integrable functions f(x), g(x)and $\lambda \in \mathbb{R}$.

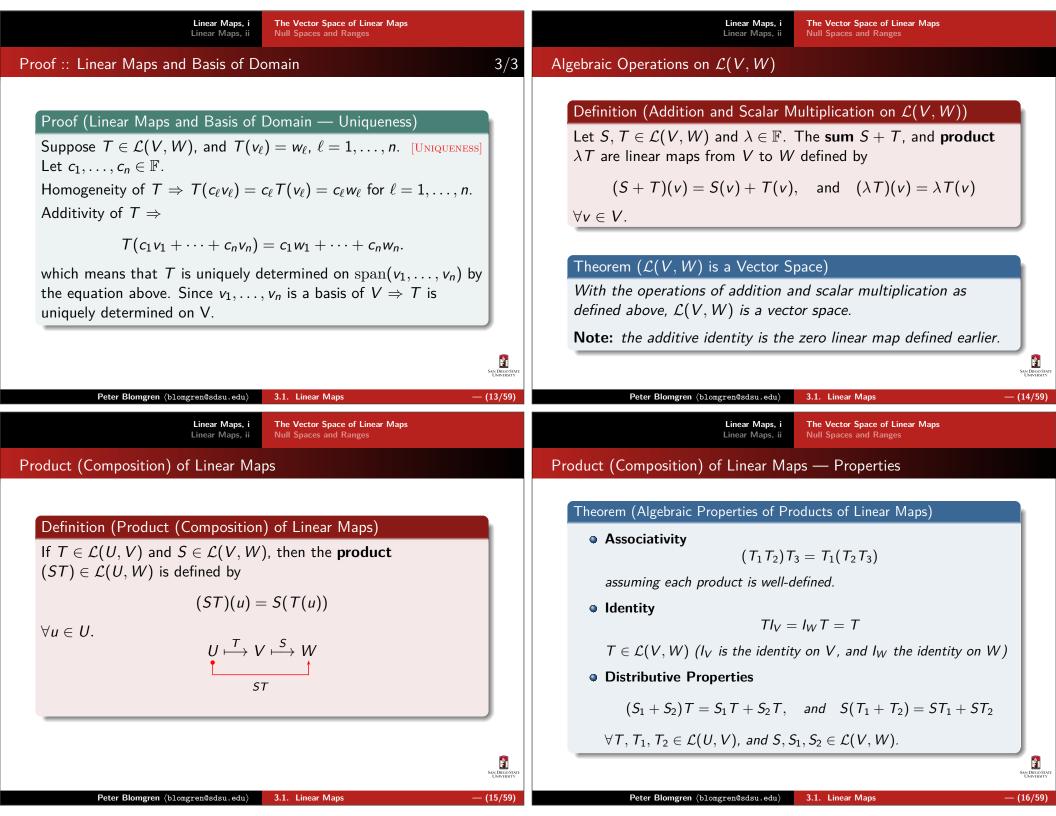
backward shift:

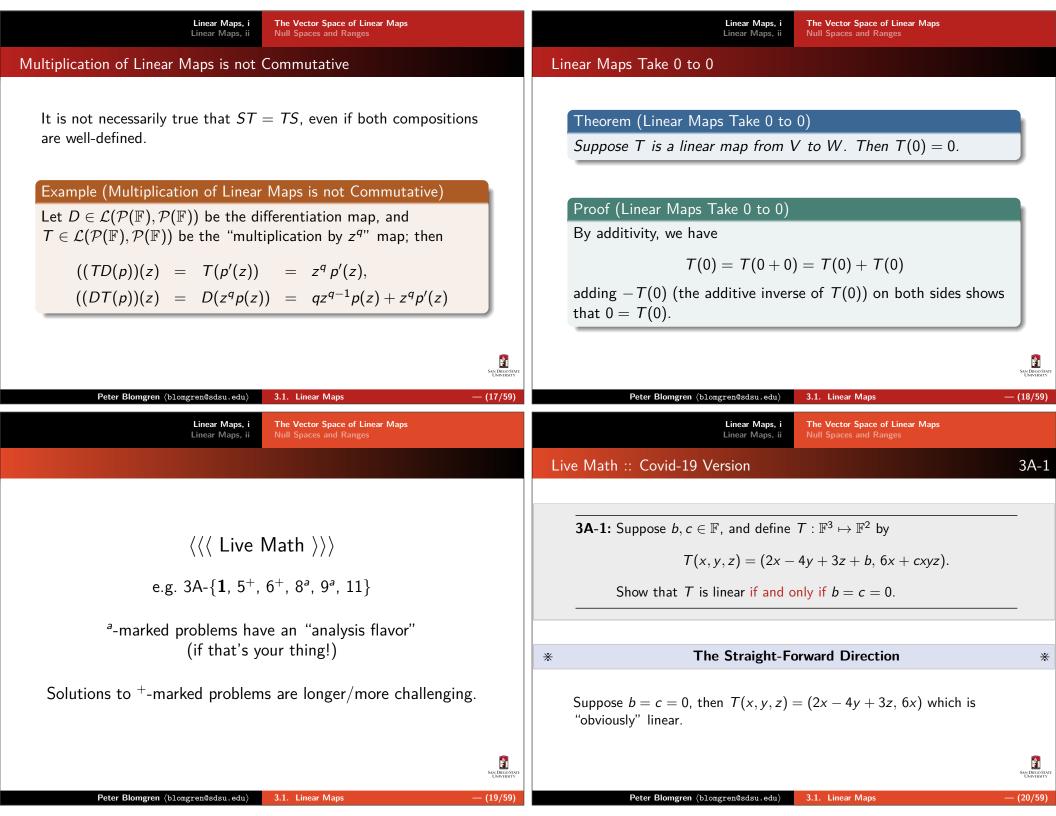
 \mathbb{F}^{∞} is the (infinite dimensional) vector space of all sequences of elements of \mathbb{F} . Let $\mathcal{T} \in \mathcal{L}(\mathbb{F}^{\infty}, \mathbb{F}^{\infty})$ be defined by

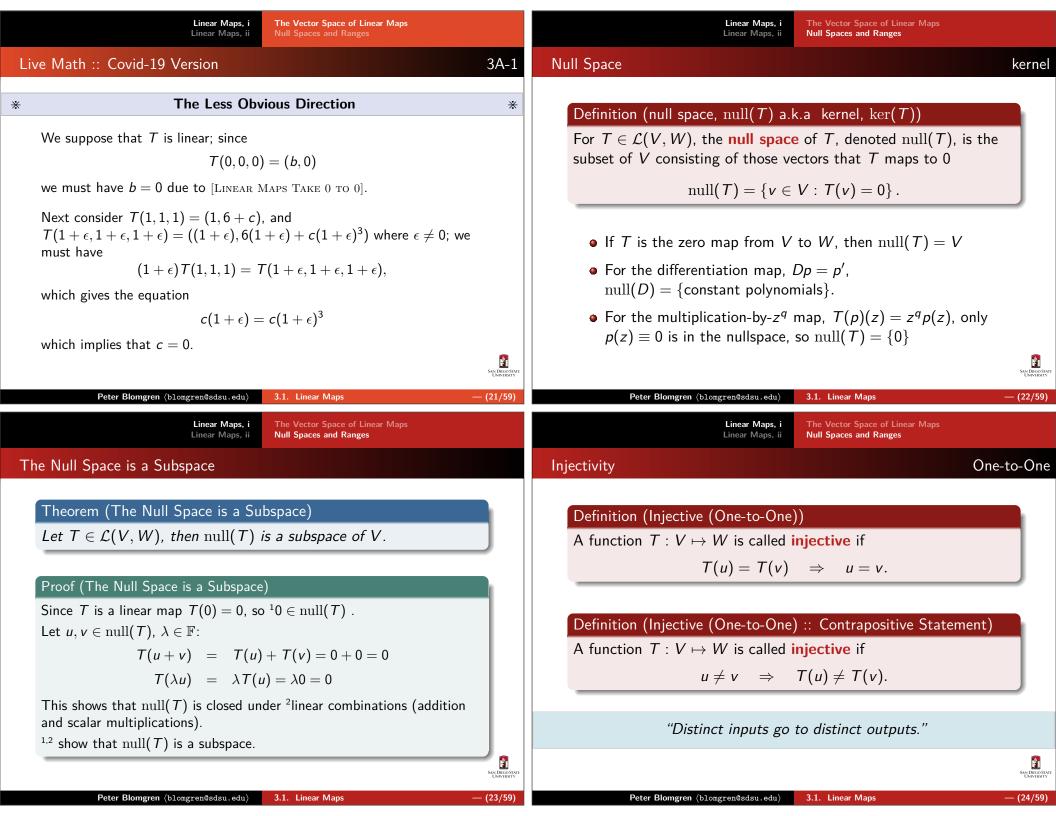
$$T(z_1, z_2, z_3, \dots) = (z_2, z_3, z_4, \dots)$$

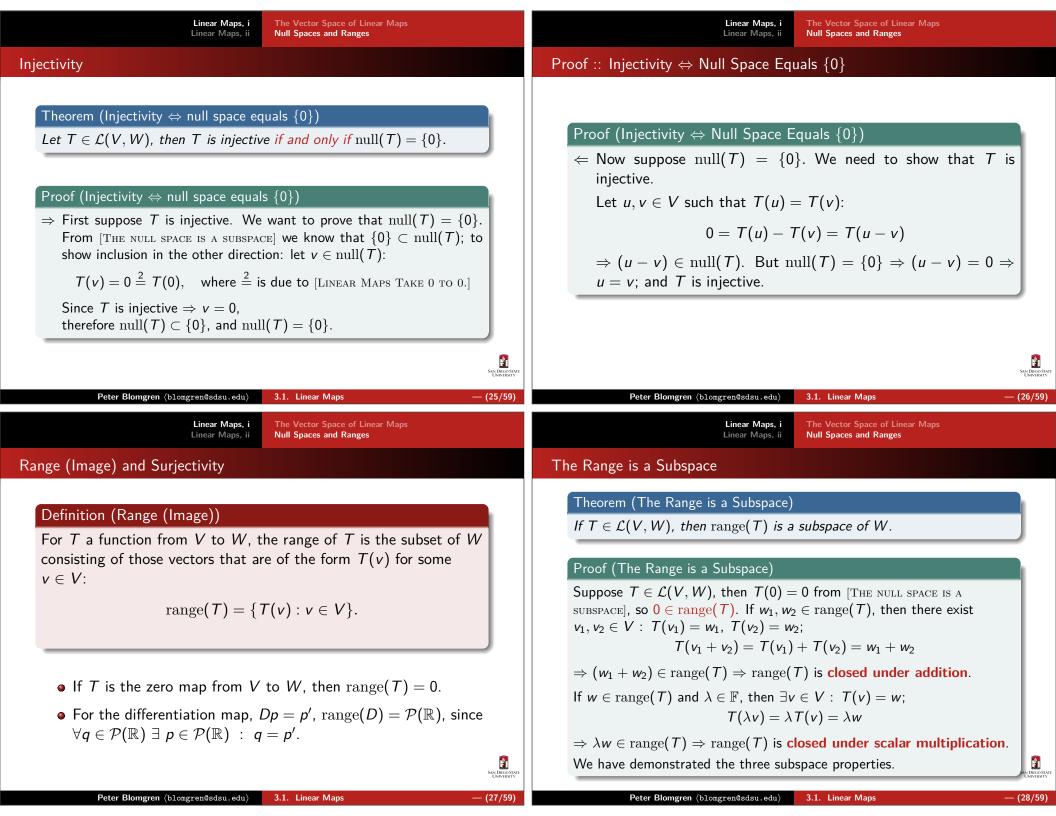
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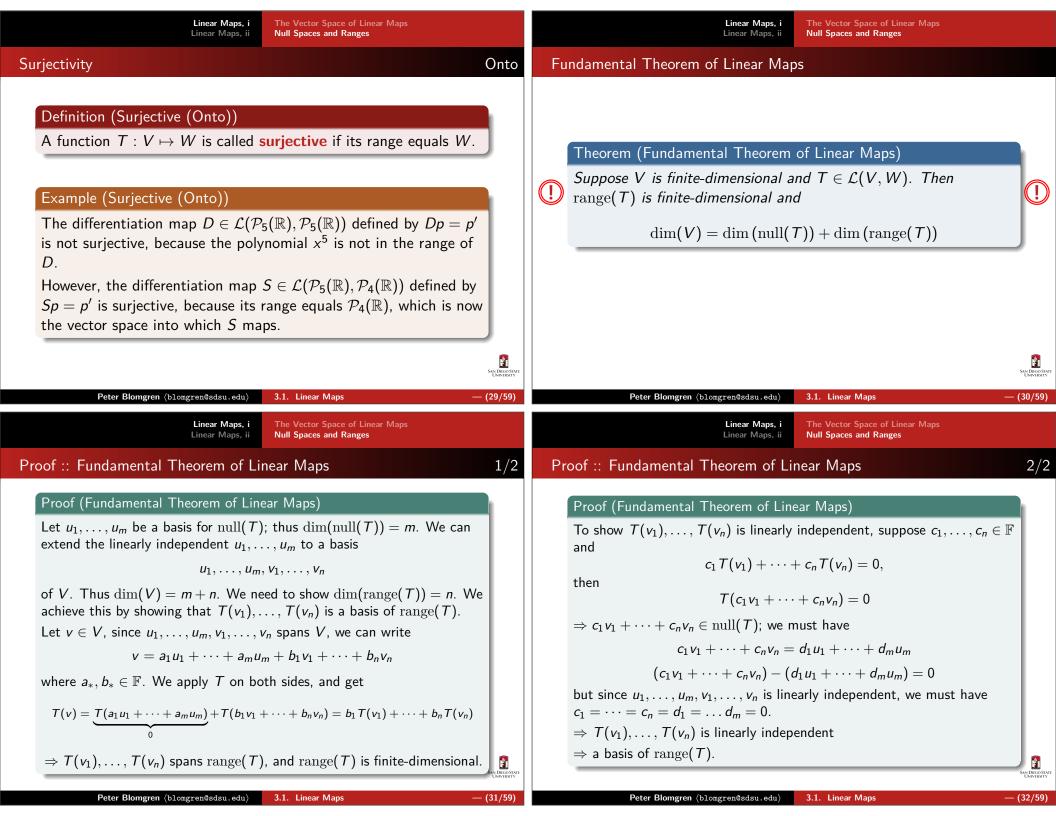
Linear Maps, iThe Vector Space of Linear MapsLinear Maps, iiNull Spaces and Ranges
Linear Maps and Basis of Domain
Theorem (Linear Maps and Basis of Domain)Suppose v_1, \ldots, v_n is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \mapsto W$ ($\exists ! T \in \mathcal{L}(V, W)$) such that $T(v_\ell) = w_\ell$, $\ell = 1, \ldots, n$.us example; $+ a_{m,n} x_n$)
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- (9/59) Peter Blomgren (blomgren@sdsu.edu) 3.1. Linear Maps - (10/59)
Linear Maps, iThe Vector Space of Linear MapsLinear Maps, iiNull Spaces and Ranges
1/3 Proof :: Linear Maps and Basis of Domain 2/3
[EXISTENCE] e equation ement of V b_nv_n , then w_n) Proof (Linear Maps and Basis of Domain — Existence) Similarly, $\forall \lambda \in \mathbb{F}$, and $v \in V$, with $v = c_1v_1 + \dots + c_nv_n$, we have $T(\lambda v) = T(\lambda c_1v_1 + \dots + \lambda c_nv_n)$ $= \lambda c_1w_1 + \dots + \lambda c_nw_n$ $= \lambda T(v)$ This shows that we have a linear map from V to W. Next, uniqueness \rightarrow
+ [] e e m b









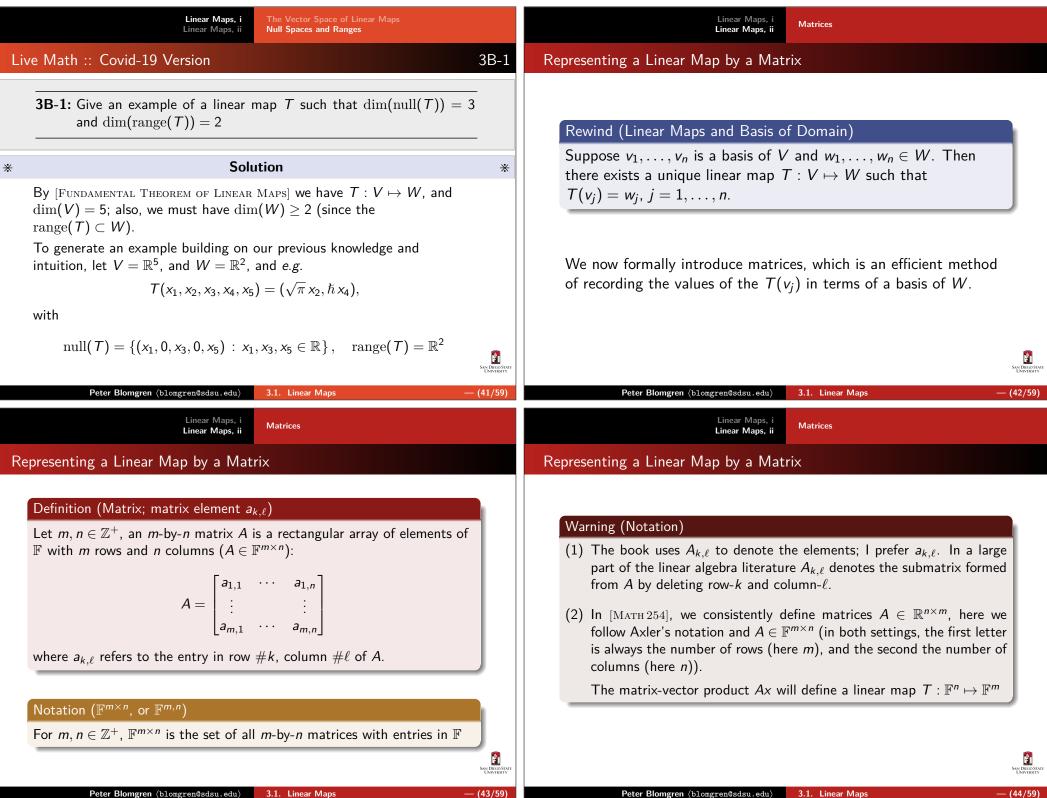


Linear Maps, i The Vector Space of Linear Maps Linear Maps, ii Null Spaces and Ranges	Linear Maps, iThe Vector Space of Linear MapsLinear Maps, iiNull Spaces and Ranges
A Map to a Smaller Dimensional Space is not Injective	Proof :: A Map to a Smaller Dimensional Space is not Injective
Theorem (A Map to a Smaller Dimensional Space is not Injective) Suppose V and W are finite-dimensional vector spaces such that $\dim(V) > \dim(W)$. Then no linear map from V to W is injective (One-to-One).	Proof (A Map to a Smaller Dimensional Space is not Injective)Let $T \in \mathcal{L}(V, W)$, then $\dim(\operatorname{null}(T)) = \dim(V) - \dim(\operatorname{range}(T))$ $\geq \dim(V) - \dim(W)$ ≥ 0 where the equality above comes from [FUNDAMENTAL THEOREM OF
No linear map from a finite-dimensional vector space to a "smaller" vector space can be injective.	LINEAR MAPS]. The inequality above states that $\dim(\operatorname{null}(T)) > 0$. This means that $\operatorname{null}(T)$ contains vectors other than 0. Thus T is not injective by [INJECTIVITY \Leftrightarrow NULL SPACE EQUALS {0}]
Peter Blomgren (blomgren@sdsu.edu) 3.1. Linear Maps — (33/59)	Peter Blomgren (blomgren@sdsu.edu) 3.1. Linear Maps — (34/59)
Linear Maps, i Linear Maps, ii Null Spaces and Ranges	Linear Maps, i Linear Maps, ii Null Spaces and Ranges
A Map to a Larger Dimensional Space is not Surjective	Proof :: A Map to a Larger Dimensional Space is not Surjective
Theorem (A Map to a Larger Dimensional Space is not Surjective) Suppose V and W are finite-dimensional vector spaces such that $\dim(V) < \dim(W)$. Then no linear map from V to W is surjective (onto).	Proof (A Map to a Larger Dimensional Space is not Surjective)Let $T \in \mathcal{L}(V, W)$, then $\dim(\operatorname{range}(T)) = \dim(V) - \dim(\operatorname{null}(T))$ $\leq \dim(V)$ $< \dim(W)$
Comment No linear map from a finite-dimensional vector space to a "bigger" vector space can be surjective.	where the equality above comes from [FUNDAMENTAL THEOREM OF LINEAR MAPS]. The inequality above states that $\dim(\operatorname{range}(T)) < \dim(W)$. This means that $\operatorname{range}(T)$ cannot equal W . Thus T is not surjective.
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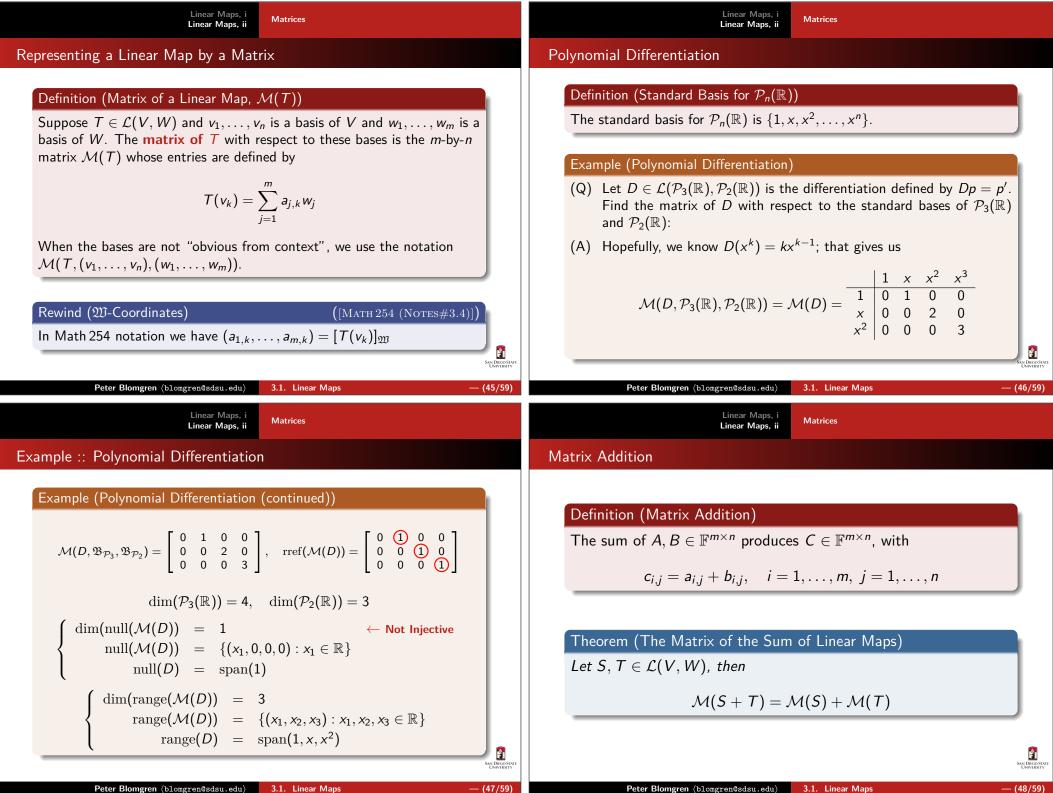
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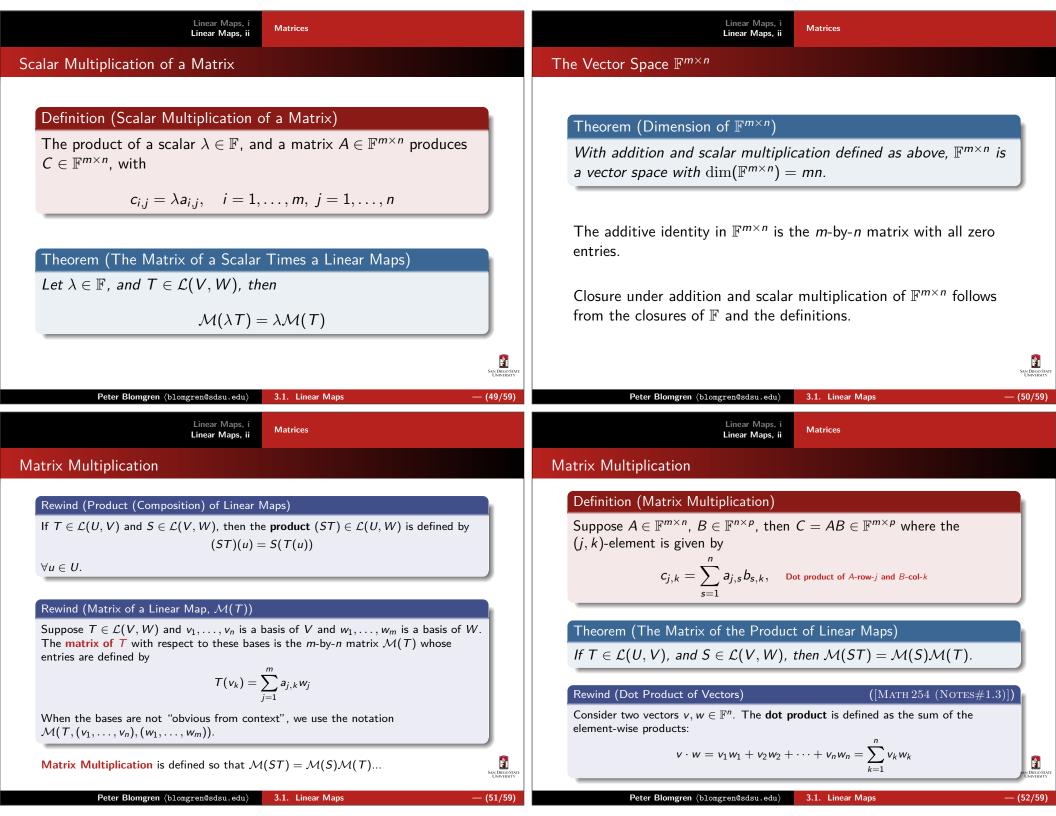
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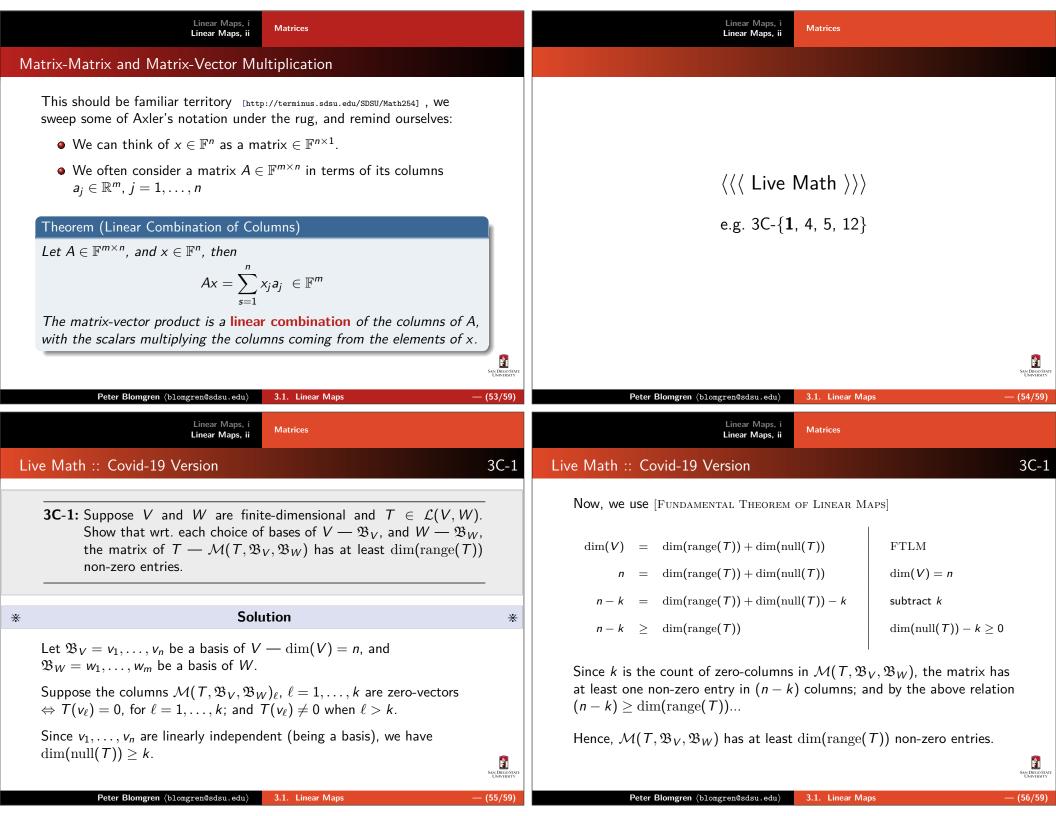
Linear Maps, i The Vector Space of Linear Maps Linear Maps, ii Null Spaces and Ranges	Linear Maps, i The Vector Space of Linear Maps Linear Maps, ii Null Spaces and Ranges
From Linear Maps to Linear Equations	Homogeneous System of Linear Equations
 Shortly, we will formally define how we specify the Linear Map T : Fⁿ → F^m by a matrix; for now, we appeal to previous knowledge and the promise of a formal definition, and consider the map in matrix-vector notation T(x) = Ax, and ponder the questions whether linear systems have solutions: Ax = 0 (Homogeneous System of Linear Equations) We always have one solution since 0 ∈ null(T), but if dim(null(T)) ≥ 1 we will have more solutions. Ax = b (Inhomogeneous System of Linear Equations) When b ∈ range(T), we definitely have a solution; and if we can guarantee range(T) = F^m, we would have solutions ∀b ∈ F^m; but when dim(range(T)) < m, there will be some b ∈ F^m for which we have no solutions. 	Theorem (Homogeneous System of Linear Equations) A homogeneous system of linear equations with more variables than equations has nonzero solutions. <i>{T</i> : 𝑘 ⁿ → 𝑘 ^m , n > m} Proof (Homogeneous System of Linear Equations) T : 𝑘 ⁿ → 𝑘 ^m is a linear map from 𝑘 ⁿ to 𝑘 ^m , and we have a homogeneous system of <i>m</i> linear equations with <i>n</i> variables x ₁ ,, x _n . From [A MAP TO A SMALLER DIMENSIONAL SPACE IS NOT INJECTIVE] WE SEE that <i>T</i> is not injective (one-to-one) if <i>n</i> > <i>m</i> .
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Linear Maps, i The Vector Space of Linear Maps	
Linear Maps, ii Null Spaces and Ranges	Linear Maps, i The Vector Space of Linear Maps Linear Maps, ii Null Spaces and Ranges
Linear Maps, ii Null Spaces and Ranges	
Linear Maps, ii Null Spaces and Ranges Inhomogeneous System of Linear Equations Theorem (Inhomogeneous System of Linear Equations) An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.	Linear Maps, ii Null Spaces and Ranges $\langle \langle \langle \langle \text{Live Math } \rangle \rangle \rangle$



(44/59)







Problems, Homework, and Supplements Suggested Problems Assigned Homework Supplements	Problems, Homework, and Supplements Supplements
Suggested Problems	Assigned Homework HW#3.1, Due Date in Canvas/Gradescope
3.A —1, 4, 5 ⁺ , 6 ⁺ , 8 ^a , 9 ^a , 11, 14 3.B —1, 2, 5, 6, 9, 17 ⁺ , 18 ⁺ , 31 3.C —1, 2–3–4–5, 12	3.A —4, 14 3.B —5, 6, 9 3.C —2, 3
^a -marked problems have an "analysis flavor" (if that's your thing!) Solutions to ⁺ -marked problems are longer/more challenging.	Note: Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually "scheduled.") Upload homework to www.Gradescope.com
Peter Blomgren (blomgren@sdsu.edu) 3.1. Linear Maps - (57/59)	Peter Blomgren (blomgren@sdsu.edu) 3.1. Linear Maps (58/59)
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