Math 524: Linear Algebra

Notes #3.1 — Linear Maps

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Outline

Student Learning Targets, and Objectives

Target Fundamental Theorem of Linear Maps

Objective Know how to apply FTLM to relate the dimensions of the range- and null-spaces of a linear map in a vector space

Target The Matrix of a Linear Map with Respect to Given Bases

Objective Know how to identify the matrix of a given Linear Map, given bases for the domain and range spaces.

Introduction

> *"So far our attention has focused on vector spaces. No one gets excited about vector spaces. The interesting part of linear algebra is the subject to which we now turn — linear maps."*
>
> — Sheldon Axler

Notation

- $\mathbb{F}$ denotes either of the fields $\mathbb{C}$ or $\mathbb{R}$
- U, V and W are vector spaces over $\mathbb{F}$

Time-Target: 3×75-minute lectures.

Linear Maps

Definition (Linear Map)

A **linear map** from V to W is a function $T : V \mapsto W$ with the following properties:

- **additivity** (for vectors)
  $T(u+v) = T(u) + T(v), \ \forall u, v \in V$
- **homogeneity** [of degree 1] (of scalar multiplication)
  $T(\lambda u) = \lambda T(u), \ \forall u \in V, \ \forall \lambda \in \mathbb{F}$

Language:

Linear Map, Linear Mapping, Linear Transform, Linear Transformation... many names for the same operation.

Notation and Examples

Notation (The Set of Linear Maps — $\mathcal{L}(V, W)$)

The set of all linear maps from V to W is denoted by $\mathcal{L}(V, W)$.

0, zero:

Let the symbol 0 denote the function that takes each element of some vector space to the additive identity of another vector space.

$0 \in \mathcal{L}(V, W)$ is defined by $0v \equiv 0(v) = 0$.

The 0 on the left side of the equation above is a function in $\mathcal{L}(V, W)$, whereas the 0 on the right side is the additive identity in W. As usual, the meaning of "0" is "obvious from context."

So far, we have 4 "zeros": $\in \mathbb{F}, V, W, \mathcal{L}(V, W)$...

Note: $\mathcal{L}(V, W) \subset W^V$ (the space of all functions $f : V \mapsto W$).

Examples — *Hello, Calculus!*

Where have you been, and why didn't you stay there?

identity: ("one")

The **identity map**, denoted I, is the function on some vector space that takes each element to itself.

$I \in \mathcal{L}(V, V)$ is defined by $Iv \equiv I(v) = v$.

differentiation:

Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{F}), \mathcal{P}(\mathbb{F}))$ be defined by $Dp \equiv D(p) = p'$.

This function is a linear map, since $(f+g)' = f' + g'$, and $(\lambda f)' = \lambda f'$ for differentiable functions f, g, and $\lambda \in \mathbb{F}$

multiplication by z^q:

Let $T \in \mathcal{L}(\mathcal{P}(\mathbb{F}), \mathcal{P}(\mathbb{F}))$ be defined by $Tp \equiv T(p) = z^q p(z)$, for $z \in \mathbb{F}$.

Examples

integration:

Let $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ be defined by

$$Tp \equiv T(p) = \int_0^1 p(x)\, dx$$

is a linear map since $\displaystyle\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$, and $\displaystyle\int (\lambda f(x))\, dx = \lambda \int f(x)\, dx$, for integrable functions $f(x), g(x)$ and $\lambda \in \mathbb{R}$.

backward shift:

$\mathbb{F}^\infty$ is the (infinite dimensional) vector space of all sequences of elements of $\mathbb{F}$. Let $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty)$ be defined by

$$T(z_1, z_2, z_3, \dots) = (z_2, z_3, z_4, \dots)$$

Examples

$\mathbb{R}^3 \mapsto \mathbb{R}^2$:

Define $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ by

$$T(x,y,z) = (x - y + z,\ \pi x + e^{\pi} y + z)$$

$\mathbb{F}^n \mapsto \mathbb{F}^m$:

This is our (hopefully) familiar generalization of the previous example; here $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ is defined by

$$T(x_1, \dots, x_n) = (a_{1,1}x_1 + \cdots + a_{1,n}x_n,\ \dots\ , a_{m,1}x_1 + \cdots + a_{m,n}x_n)$$

every linear map $\mathbb{F}^n \mapsto \mathbb{F}^m$ can be written in this form.

Linear Maps and Basis of Domain

Theorem (Linear Maps and Basis of Domain)

Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T : V \mapsto W$ ($\exists! T \in \mathcal{L}(V, W)$) such that $T(v_\ell) = w_\ell$, $\ell = 1, \dots, n$.

Proof :: Linear Maps and Basis of Domain 1/3

Proof (Linear Maps and Basis of Domain — Existence)

Define $T : V \mapsto W$ by [EXISTENCE]

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

where $c_1, \dots, c_n \in \mathbb{F}$. The list $v_1, \dots, v_n$ is a basis of V, so the equation above does indeed define a function T from V to W (each element of V can be uniquely written in the form $c_1 v_1 + \cdots + c_n v_n$).

For each ℓ, let $c_i = \delta_{i\ell}$, this shows $T(v_\ell) = w_\ell$.

If $u, v \in V$, with $u = a_1 v_1 + \cdots + a_n v_n$, and $v = b_1 v_1 + \cdots + b_n v_n$, then

$$\begin{aligned} T(u+v) &= T((a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n) \\ &= (a_1+b_1)w_1 + \cdots + (a_n+b_n)w_n \\ &= (a_1 w_1 + \cdots + a_n w_n) + (b_1 w_1 + \cdots + b_n w_n) \\ &= T(u) + T(v) \end{aligned}$$

Proof :: Linear Maps and Basis of Domain 2/3

Proof (Linear Maps and Basis of Domain — Existence)

Similarly, $\forall \lambda \in \mathbb{F}$, and $v \in V$, with $v = c_1 v_1 + \cdots + c_n v_n$, we have

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1 v_1 + \cdots + \lambda c_n v_n) \\ &= \lambda c_1 w_1 + \cdots + \lambda c_n w_n \\ &= \lambda (c_1 w_1 + \cdots + c_n w_n) \\ &= \lambda T(v) \end{aligned}$$

This shows that we have a linear map from V to W.

Next, uniqueness $\rightarrow$

Proof :: Linear Maps and Basis of Domain

Proof (Linear Maps and Basis of Domain — Uniqueness)

Suppose $T \in \mathcal{L}(V, W)$, and $T(v_\ell) = w_\ell$, $\ell = 1, \dots, n$. [UNIQUENESS]
Let $c_1, \dots, c_n \in \mathbb{F}$.

Homogeneity of $T \Rightarrow T(c_\ell v_\ell) = c_\ell T(v_\ell) = c_\ell w_\ell$ for $\ell = 1, \dots, n$.

Additivity of $T \Rightarrow$

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n.$$

which means that T is uniquely determined on $\mathrm{span}(v_1, \dots, v_n)$ by the equation above. Since $v_1, \dots, v_n$ is a basis of $V \Rightarrow T$ is uniquely determined on V.

Linear Maps, i
Linear Maps, ii
The Vector Space of Linear Maps
Null Spaces and Ranges

Algebraic Operations on $\mathcal{L}(V, W)$

Definition (Addition and Scalar Multiplication on $\mathcal{L}(V, W)$)

Let $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. The **sum** $S + T$, and **product** λT are linear maps from V to W defined by

$$(S + T)(v) = S(v) + T(v), \quad \text{and} \quad (\lambda T)(v) = \lambda T(v)$$

$\forall v \in V$.

Theorem ($\mathcal{L}(V, W)$ is a Vector Space)

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

Note: *the additive identity is the zero linear map defined earlier.*

Linear Maps, i
Linear Maps, ii
The Vector Space of Linear Maps
Null Spaces and Ranges

Product (Composition) of Linear Maps

Definition (Product (Composition) of Linear Maps)

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the **product** $(ST) \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(T(u))$$

$\forall u \in U$.

$$\begin{tikzcd} U \arrow[r, mapsto, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, mapsto, "S"] & W \end{tikzcd}$$

Linear Maps, i
Linear Maps, ii
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Product (Composition) of Linear Maps — Properties

Theorem (Algebraic Properties of Products of Linear Maps)

- **Associativity**
  $$(T_1 T_2) T_3 = T_1 (T_2 T_3)$$
  assuming each product is well-defined.
- **Identity**
  $$T I_V = I_W T = T$$
  $T \in \mathcal{L}(V, W)$ (I_V is the identity on V, and I_W the identity on W)
- **Distributive Properties**
  $$(S_1 + S_2)T = S_1 T + S_2 T, \quad \text{and} \quad S(T_1 + T_2) = ST_1 + ST_2$$
  $\forall T, T_1, T_2 \in \mathcal{L}(U, V)$, and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

Multiplication of Linear Maps is not Commutative

It is not necessarily true that $ST = TS$, even if both compositions are well-defined.

Example (Multiplication of Linear Maps is not Commutative)

Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{F}), \mathcal{P}(\mathbb{F}))$ be the differentiation map, and $T \in \mathcal{L}(\mathcal{P}(\mathbb{F}), \mathcal{P}(\mathbb{F}))$ be the "multiplication by z^q" map; then

$$\begin{aligned} ((TD(p))(z) &= T(p'(z)) &&= z^q\, p'(z), \\ ((DT(p))(z) &= D(z^q p(z)) &&= qz^{q-1}p(z) + z^q p'(z) \end{aligned}$$

Linear Maps Take 0 to 0

Theorem (Linear Maps Take 0 to 0)

Suppose T is a linear map from V to W. Then $T(0) = 0$.

Proof (Linear Maps Take 0 to 0)

By additivity, we have

$$T(0) = T(0+0) = T(0) + T(0)$$

adding $-T(0)$ (the additive inverse of $T(0)$) on both sides shows that $0 = T(0)$.

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e.g. 3A-{**1**, 5^+, 6^+, 8^a, 9^a, 11}

^{a}-marked problems have an "analysis flavor" (if that's your thing!)

Solutions to $^+$-marked problems are longer/more challenging.

Live Math :: Covid-19 Version 3A-1

3A-1: Suppose $b, c \in \mathbb{F}$, and define $T : \mathbb{F}^3 \mapsto \mathbb{F}^2$ by

$$T(x,y,z) = (2x - 4y + 3z + b,\ 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

The Straight-Forward Direction

Suppose $b = c = 0$, then $T(x,y,z) = (2x - 4y + 3z,\ 6x)$ which is "obviously" linear.

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The Less Obvious Direction

We suppose that T is linear; since

$$T(0,0,0) = (b,0)$$

we must have $b = 0$ due to [Linear Maps Take 0 to 0].

Next consider $T(1,1,1) = (1, 6+c)$, and $T(1+\epsilon, 1+\epsilon, 1+\epsilon) = ((1+\epsilon), 6(1+\epsilon) + c(1+\epsilon)^3)$ where $\epsilon \neq 0$; we must have

$$(1+\epsilon)T(1,1,1) = T(1+\epsilon, 1+\epsilon, 1+\epsilon),$$

which gives the equation

$$c(1+\epsilon) = c(1+\epsilon)^3$$

which implies that $c = 0$.

Null Space — kernel

Definition (null space, $\mathrm{null}(T)$ a.k.a kernel, $\ker(T)$)

For $T \in \mathcal{L}(V,W)$, the **null space** of T, denoted $\mathrm{null}(T)$, is the subset of V consisting of those vectors that T maps to 0

$$\mathrm{null}(T) = \{v \in V : T(v) = 0\}.$$

- If T is the zero map from V to W, then $\mathrm{null}(T) = V$
- For the differentiation map, $Dp = p'$, $\mathrm{null}(D) = \{\text{constant polynomials}\}$.
- For the multiplication-by-z^q map, $T(p)(z) = z^q p(z)$, only $p(z) \equiv 0$ is in the nullspace, so $\mathrm{null}(T) = \{0\}$

The Null Space is a Subspace

Theorem (The Null Space is a Subspace)

Let $T \in \mathcal{L}(V,W)$, then $\mathrm{null}(T)$ is a subspace of V.

Proof (The Null Space is a Subspace)

Since T is a linear map $T(0) = 0$, so [1]$0 \in \mathrm{null}(T)$.
Let $u, v \in \mathrm{null}(T)$, $\lambda \in \mathbb{F}$:

$$\begin{aligned} T(u+v) &= T(u) + T(v) = 0 + 0 = 0 \\ T(\lambda u) &= \lambda T(u) = \lambda 0 = 0 \end{aligned}$$

This shows that $\mathrm{null}(T)$ is closed under [2]linear combinations (addition and scalar multiplications).
[1,2] show that $\mathrm{null}(T)$ is a subspace.

Injectivity — One-to-One

Definition (Injective (One-to-One))

A function $T : V \mapsto W$ is called **injective** if

$$T(u) = T(v) \quad \Rightarrow \quad u = v.$$

Definition (Injective (One-to-One) :: Contrapositive Statement)

A function $T : V \mapsto W$ is called **injective** if

$$u \neq v \quad \Rightarrow \quad T(u) \neq T(v).$$

"Distinct inputs go to distinct outputs."

Injectivity

Theorem (Injectivity $\Leftrightarrow$ null space equals $\{0\}$)

Let $T \in \mathcal{L}(V,W)$, then T is injective if and only if $\mathrm{null}(T) = \{0\}$.

Proof (Injectivity $\Leftrightarrow$ null space equals $\{0\}$)

$\Rightarrow$ First suppose T is injective. We want to prove that $\mathrm{null}(T) = \{0\}$. From [THE NULL SPACE IS A SUBSPACE] we know that $\{0\} \subset \mathrm{null}(T)$; to show inclusion in the other direction: let $v \in \mathrm{null}(T)$:

$$T(v) = 0 \stackrel{2}{=} T(0), \quad \text{where } \stackrel{2}{=} \text{ is due to [LINEAR MAPS TAKE 0 TO 0.]}$$

Since T is injective $\Rightarrow v = 0$,
therefore $\mathrm{null}(T) \subset \{0\}$, and $\mathrm{null}(T) = \{0\}$.

Proof :: Injectivity $\Leftrightarrow$ Null Space Equals $\{0\}$

Proof (Injectivity $\Leftrightarrow$ Null Space Equals $\{0\}$)

$\Leftarrow$ Now suppose $\mathrm{null}(T) = \{0\}$. We need to show that T is injective.

Let $u, v \in V$ such that $T(u) = T(v)$:

$$0 = T(u) - T(v) = T(u - v)$$

$\Rightarrow (u - v) \in \mathrm{null}(T)$. But $\mathrm{null}(T) = \{0\} \Rightarrow (u - v) = 0 \Rightarrow u = v$; and T is injective.

Range (Image) and Surjectivity

Definition (Range (Image))

For T a function from V to W, the range of T is the subset of W consisting of those vectors that are of the form $T(v)$ for some $v \in V$:

$$\mathrm{range}(T) = \{T(v) : v \in V\}.$$

- If T is the zero map from V to W, then $\mathrm{range}(T) = 0$.
- For the differentiation map, $Dp = p'$, $\mathrm{range}(D) = \mathcal{P}(\mathbb{R})$, since $\forall q \in \mathcal{P}(\mathbb{R}) \; \exists \; p \in \mathcal{P}(\mathbb{R}) \; : \; q = p'$.

The Range is a Subspace

Theorem (The Range is a Subspace)

If $T \in \mathcal{L}(V,W)$, then $\mathrm{range}(T)$ *is a subspace of W.*

Proof (The Range is a Subspace)

Suppose $T \in \mathcal{L}(V,W)$, then $T(0) = 0$ from [THE NULL SPACE IS A SUBSPACE], so $0 \in \mathrm{range}(T)$. If $w_1, w_2 \in \mathrm{range}(T)$, then there exist $v_1, v_2 \in V : T(v_1) = w_1, \; T(v_2) = w_2$;

$$T(v_1 + v_2) = T(v_1) + T(v_2) = w_1 + w_2$$

$\Rightarrow (w_1 + w_2) \in \mathrm{range}(T) \Rightarrow \mathrm{range}(T)$ is **closed under addition**.

If $w \in \mathrm{range}(T)$ and $\lambda \in \mathbb{F}$, then $\exists v \in V : T(v) = w$;

$$T(\lambda v) = \lambda T(v) = \lambda w$$

$\Rightarrow \lambda w \in \mathrm{range}(T) \Rightarrow \mathrm{range}(T)$ is **closed under scalar multiplication**.

We have demonstrated the three subspace properties.

Surjectivity — Onto

Definition (Surjective (Onto))

A function $T : V \mapsto W$ is called **surjective** if its range equals W.

Example (Surjective (Onto))

The differentiation map $D \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_5(\mathbb{R}))$ defined by $Dp = p'$ is not surjective, because the polynomial x^5 is not in the range of D.

However, the differentiation map $S \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_4(\mathbb{R}))$ defined by $Sp = p'$ is surjective, because its range equals $\mathcal{P}_4(\mathbb{R})$, which is now the vector space into which S maps.

Fundamental Theorem of Linear Maps

Theorem (Fundamental Theorem of Linear Maps)

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\text{range}(T)$ is finite-dimensional and

$$\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$$

Proof :: Fundamental Theorem of Linear Maps — 1/2

Proof (Fundamental Theorem of Linear Maps)

Let $u_1, \dots, u_m$ be a basis for $\text{null}(T)$; thus $\dim(\text{null}(T)) = m$. We can extend the linearly independent $u_1, \dots, u_m$ to a basis

$$u_1, \dots, u_m, v_1, \dots, v_n$$

of V. Thus $\dim(V) = m + n$. We need to show $\dim(\text{range}(T)) = n$. We achieve this by showing that $T(v_1), \dots, T(v_n)$ is a basis of $\text{range}(T)$.

Let $v \in V$, since $u_1, \dots, u_m, v_1, \dots, v_n$ spans V, we can write

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

where $a_*, b_* \in \mathbb{F}$. We apply T on both sides, and get

$$T(v) = \underbrace{T(a_1 u_1 + \cdots + a_m u_m)}_{0} + T(b_1 v_1 + \cdots + b_n v_n) = b_1 T(v_1) + \cdots + b_n T(v_n)$$

$\Rightarrow T(v_1), \dots, T(v_n)$ spans $\text{range}(T)$, and $\text{range}(T)$ is finite-dimensional.

Proof :: Fundamental Theorem of Linear Maps — 2/2

Proof (Fundamental Theorem of Linear Maps)

To show $T(v_1), \dots, T(v_n)$ is linearly independent, suppose $c_1, \dots, c_n \in \mathbb{F}$ and

$$c_1 T(v_1) + \cdots + c_n T(v_n) = 0,$$

then

$$T(c_1 v_1 + \cdots + c_n v_n) = 0$$

$\Rightarrow c_1 v_1 + \cdots + c_n v_n \in \text{null}(T)$; we must have

$$c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m$$
$$(c_1 v_1 + \cdots + c_n v_n) - (d_1 u_1 + \cdots + d_m u_m) = 0$$

but since $u_1, \dots, u_m, v_1, \dots, v_n$ is linearly independent, we must have $c_1 = \cdots = c_n = d_1 = \dots d_m = 0$.

$\Rightarrow T(v_1), \dots, T(v_n)$ is linearly independent

$\Rightarrow$ a basis of $\text{range}(T)$.

A Map to a Smaller Dimensional Space is not Injective

Theorem (A Map to a Smaller Dimensional Space is not Injective)

Suppose V and W are finite-dimensional vector spaces such that $\dim(V) > \dim(W)$. Then no linear map from V to W is injective (One-to-One).

Comment

No linear map from a finite-dimensional vector space to a "smaller" vector space can be injective.

Proof :: A Map to a Smaller Dimensional Space is not Injective

Proof (A Map to a Smaller Dimensional Space is not Injective)

Let $T \in \mathcal{L}(V, W)$, then

$$\begin{aligned} \dim(\mathrm{null}(T)) &= \dim(V) - \dim(\mathrm{range}(T)) \\ &\geq \dim(V) - \dim(W) \\ &> 0 \end{aligned}$$

where the equality above comes from [Fundamental Theorem of Linear Maps]. The inequality above states that $\dim(\mathrm{null}(T)) > 0$. This means that $\mathrm{null}(T)$ contains vectors other than 0. Thus T is not injective by [Injectivity $\Leftrightarrow$ Null Space Equals $\{0\}$]

A Map to a Larger Dimensional Space is not Surjective

Theorem (A Map to a Larger Dimensional Space is not Surjective)

Suppose V and W are finite-dimensional vector spaces such that $\dim(V) < \dim(W)$. Then no linear map from V to W is surjective (onto).

Comment

No linear map from a finite-dimensional vector space to a "bigger" vector space can be surjective.

Proof :: A Map to a Larger Dimensional Space is not Surjective

Proof (A Map to a Larger Dimensional Space is not Surjective)

Let $T \in \mathcal{L}(V, W)$, then

$$\begin{aligned} \dim(\mathrm{range}(T)) &= \dim(V) - \dim(\mathrm{null}(T)) \\ &\leq \dim(V) \\ &< \dim(W) \end{aligned}$$

where the equality above comes from [Fundamental Theorem of Linear Maps]. The inequality above states that $\dim(\mathrm{range}(T)) < \dim(W)$. This means that $\mathrm{range}(T)$ cannot equal W. Thus T is not surjective.

From Linear Maps to Linear Equations

Shortly, we will formally define how we specify the Linear Map $T : \mathbb{F}^n \mapsto \mathbb{F}^m$ by a matrix; for now, we appeal to previous knowledge and the promise of a formal definition, and consider the map in matrix-vector notation

$$T(x) = Ax,$$

and ponder the questions whether linear systems have solutions:

- $Ax = 0$ (Homogeneous System of Linear Equations)
  - We always have one solution since $0 \in \mathrm{null}(T)$, but if $\dim(\mathrm{null}(T)) \geq 1$ we will have more solutions.
- $Ax = b$ (Inhomogeneous System of Linear Equations)
  - When $b \in \mathrm{range}(T)$, we definitely have a solution; and if we can guarantee $\mathrm{range}(T) = \mathbb{F}^m$, we would have solutions $\forall b \in \mathbb{F}^m$; but when $\dim(\mathrm{range}(T)) < m$, there will be some $b \in \mathbb{F}^m$ for which we have no solutions.

Homogeneous System of Linear Equations

Theorem (Homogeneous System of Linear Equations)

A homogeneous system of linear equations with more variables than equations has nonzero solutions.

$$\{T : \mathbb{F}^n \mapsto \mathbb{F}^m,\ n > m\}$$

Proof (Homogeneous System of Linear Equations)

$T : \mathbb{F}^n \mapsto \mathbb{F}^m$ is a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$, and we have a homogeneous system of m linear equations with n variables $x_1, \dots, x_n$. From [A Map to a Smaller Dimensional Space is not Injective] we see that T is not injective (one-to-one) if $n > m$.

Inhomogeneous System of Linear Equations

Theorem (Inhomogeneous System of Linear Equations)

An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

$$\{T : \mathbb{F}^n \mapsto \mathbb{F}^m,\ m > n\}$$

Proof (Inhomogeneous System of Linear Equations)

$T : \mathbb{F}^n \mapsto \mathbb{F}^m$ is a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$, and we have a homogeneous system of m linear equations with n variables $x_1, \dots, x_n$. From [A Map to a Larger Dimensional Space is not Surjective] we see that T is not surjective (onto) if $n < m$.

Therefore $\exists w \in \mathbb{F}^m : T(v) \neq w, \forall v \in V$.

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e.g. 3B-{**1**, 2, 17^+, 18^+, 31}

Solutions to $^+$-marked problems are longer/more challenging.

Live Math :: Covid-19 Version — 3B-1

3B-1: Give an example of a linear map T such that $\dim(\mathrm{null}(T)) = 3$ and $\dim(\mathrm{range}(T)) = 2$

Solution

By [Fundamental Theorem of Linear Maps] we have $T : V \mapsto W$, and $\dim(V) = 5$; also, we must have $\dim(W) \geq 2$ (since the $\mathrm{range}(T) \subset W$).

To generate an example building on our previous knowledge and intuition, let $V = \mathbb{R}^5$, and $W = \mathbb{R}^2$, and *e.g.*

$$T(x_1, x_2, x_3, x_4, x_5) = (\sqrt{\pi}\, x_2, \hbar\, x_4),$$

with

$$\mathrm{null}(T) = \{(x_1, 0, x_3, 0, x_5) \,:\, x_1, x_3, x_5 \in \mathbb{R}\}, \quad \mathrm{range}(T) = \mathbb{R}^2$$

Representing a Linear Map by a Matrix

Rewind (Linear Maps and Basis of Domain)

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \mapsto W$ such that $T(v_j) = w_j$, $j = 1, \ldots, n$.

We now formally introduce matrices, which is an efficient method of recording the values of the $T(v_j)$ in terms of a basis of W.

Representing a Linear Map by a Matrix

Definition (Matrix; matrix element $a_{k,\ell}$)

Let $m, n \in \mathbb{Z}^+$, an m-by-n matrix A is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns ($A \in \mathbb{F}^{m \times n}$):

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix}$$

where $a_{k,\ell}$ refers to the entry in row $\#k$, column $\#\ell$ of A.

Notation ($\mathbb{F}^{m \times n}$, or $\mathbb{F}^{m,n}$)

For $m, n \in \mathbb{Z}^+$, $\mathbb{F}^{m \times n}$ is the set of all m-by-n matrices with entries in $\mathbb{F}$

Representing a Linear Map by a Matrix

Warning (Notation)

(1) The book uses $A_{k,\ell}$ to denote the elements; I prefer $a_{k,\ell}$. In a large part of the linear algebra literature $A_{k,\ell}$ denotes the submatrix formed from A by deleting row-k and column-ℓ.

(2) In [Math 254], we consistently define matrices $A \in \mathbb{R}^{n \times m}$, here we follow Axler's notation and $A \in \mathbb{F}^{m \times n}$ (in both settings, the first letter is always the number of rows (here m), and the second the number of columns (here n)).

The matrix-vector product Ax will define a linear map $T : \mathbb{F}^n \mapsto \mathbb{F}^m$

Representing a Linear Map by a Matrix

Definition (Matrix of a Linear Map, $\mathcal{M}(T)$)

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The **matrix of T** with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries are defined by

$$T(v_k) = \sum_{j=1}^{m} a_{j,k} w_j$$

When the bases are not "obvious from context", we use the notation $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$.

Rewind ($\mathfrak{W}$-Coordinates) ([Math 254 (Notes#3.4)])

In Math 254 notation we have $(a_{1,k}, \dots, a_{m,k}) = [T(v_k)]_{\mathfrak{W}}$

Polynomial Differentiation

Definition (Standard Basis for $\mathcal{P}_n(\mathbb{R})$)

The standard basis for $\mathcal{P}_n(\mathbb{R})$ is $\{1, x, x^2, \dots, x^n\}$.

Example (Polynomial Differentiation)

(Q) Let $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation defined by $Dp = p'$. Find the matrix of D with respect to the standard bases of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$:

(A) Hopefully, we know $D(x^k) = kx^{k-1}$; that gives us

$$\mathcal{M}(D, \mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R})) = \mathcal{M}(D) = \begin{array}{c|cccc} & 1 & x & x^2 & x^3 \\ \hline 1 & 0 & 1 & 0 & 0 \\ x & 0 & 0 & 2 & 0 \\ x^2 & 0 & 0 & 0 & 3 \end{array}$$

Example :: Polynomial Differentiation

Example (Polynomial Differentiation (continued))

$$\mathcal{M}(D, \mathfrak{B}_{\mathcal{P}_3}, \mathfrak{B}_{\mathcal{P}_2}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad \mathrm{rref}(\mathcal{M}(D)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dim(\mathcal{P}_3(\mathbb{R})) = 4, \quad \dim(\mathcal{P}_2(\mathbb{R})) = 3$$

$$\begin{cases} \dim(\mathrm{null}(\mathcal{M}(D)) &= 1 \\ \mathrm{null}(\mathcal{M}(D)) &= \{(x_1, 0, 0, 0) : x_1 \in \mathbb{R}\} \\ \mathrm{null}(D) &= \mathrm{span}(1) \end{cases}$$

← **Not Injective**

$$\begin{cases} \dim(\mathrm{range}(\mathcal{M}(D)) &= 3 \\ \mathrm{range}(\mathcal{M}(D)) &= \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\} \\ \mathrm{range}(D) &= \mathrm{span}(1, x, x^2) \end{cases}$$

Matrix Addition

Definition (Matrix Addition)

The sum of $A, B \in \mathbb{F}^{m \times n}$ produces $C \in \mathbb{F}^{m \times n}$, with

$$c_{i,j} = a_{i,j} + b_{i,j}, \quad i = 1, \dots, m, \; j = 1, \dots, n$$

Theorem (The Matrix of the Sum of Linear Maps)

Let $S, T \in \mathcal{L}(V, W)$, then

$$\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$$

Scalar Multiplication of a Matrix

Definition (Scalar Multiplication of a Matrix)

The product of a scalar $\lambda \in \mathbb{F}$, and a matrix $A \in \mathbb{F}^{m \times n}$ produces $C \in \mathbb{F}^{m \times n}$, with

$$c_{i,j} = \lambda a_{i,j}, \quad i = 1, \dots, m, \; j = 1, \dots, n$$

Theorem (The Matrix of a Scalar Times a Linear Maps)

Let $\lambda \in \mathbb{F}$, and $T \in \mathcal{L}(V, W)$, then

$$\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$$

The Vector Space $\mathbb{F}^{m \times n}$

Theorem (Dimension of $\mathbb{F}^{m \times n}$)

With addition and scalar multiplication defined as above, $\mathbb{F}^{m \times n}$ is a vector space with $\dim(\mathbb{F}^{m \times n}) = mn$.

The additive identity in $\mathbb{F}^{m \times n}$ is the m-by-n matrix with all zero entries.

Closure under addition and scalar multiplication of $\mathbb{F}^{m \times n}$ follows from the closures of $\mathbb{F}$ and the definitions.

Matrix Multiplication

Rewind (Product (Composition) of Linear Maps)

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the **product** $(ST) \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(T(u))$$

$\forall u \in U$.

Rewind (Matrix of a Linear Map, $\mathcal{M}(T)$)

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The **matrix of T** with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries are defined by

$$T(v_k) = \sum_{j=1}^{m} a_{j,k} w_j$$

When the bases are not "obvious from context", we use the notation $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$.

Matrix Multiplication is defined so that $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$...

Matrix Multiplication

Definition (Matrix Multiplication)

Suppose $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times p}$, then $C = AB \in \mathbb{F}^{m \times p}$ where the (j, k)-element is given by

$$c_{j,k} = \sum_{s=1}^{n} a_{j,s} b_{s,k}, \quad \text{Dot product of } A\text{-row-}j \text{ and } B\text{-col-}k$$

Theorem (The Matrix of the Product of Linear Maps)

If $T \in \mathcal{L}(U, V)$, and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

Rewind (Dot Product of Vectors) ([MATH 254 (NOTES#1.3)])

Consider two vectors $v, w \in \mathbb{F}^n$. The **dot product** is defined as the sum of the element-wise products:

$$v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k$$

Matrix-Matrix and Matrix-Vector Multiplication

This should be familiar territory [http://terminus.sdsu.edu/SDSU/Math254] , we sweep some of Axler's notation under the rug, and remind ourselves:

- We can think of $x \in \mathbb{F}^n$ as a matrix $\in \mathbb{F}^{n \times 1}$.
- We often consider a matrix $A \in \mathbb{F}^{m \times n}$ in terms of its columns $a_j \in \mathbb{R}^m$, $j = 1, \dots, n$

Theorem (Linear Combination of Columns)

Let $A \in \mathbb{F}^{m \times n}$, and $x \in \mathbb{F}^n$, then

$$Ax = \sum_{s=1}^{n} x_j a_j \quad \in \mathbb{F}^m$$

The matrix-vector product is a **linear combination** *of the columns of A, with the scalars multiplying the columns coming from the elements of x.*

$\langle\langle\langle$ Live Math $\rangle\rangle\rangle$

e.g. 3C-{**1**, 4, 5, 12}

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3C-1: Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that wrt. each choice of bases of V — $\mathfrak{B}_V$, and W — $\mathfrak{B}_W$, the matrix of T — $\mathcal{M}(T, \mathfrak{B}_V, \mathfrak{B}_W)$ has at least $\dim(\mathrm{range}(T))$ non-zero entries.

Solution

Let $\mathfrak{B}_V = v_1, \dots, v_n$ be a basis of V — $\dim(V) = n$, and $\mathfrak{B}_W = w_1, \dots, w_m$ be a basis of W.

Suppose the columns $\mathcal{M}(T, \mathfrak{B}_V, \mathfrak{B}_W)_\ell$, $\ell = 1, \dots, k$ are zero-vectors $\Leftrightarrow T(v_\ell) = 0$, for $\ell = 1, \dots, k$; and $T(v_\ell) \neq 0$ when $\ell > k$.

Since $v_1, \dots, v_n$ are linearly independent (being a basis), we have $\dim(\mathrm{null}(T)) \geq k$.

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Now, we use [Fundamental Theorem of Linear Maps]

$\dim(V)$	$=$	$\dim(\mathrm{range}(T)) + \dim(\mathrm{null}(T))$	FTLM
n	$=$	$\dim(\mathrm{range}(T)) + \dim(\mathrm{null}(T))$	$\dim(V) = n$
$n-k$	$=$	$\dim(\mathrm{range}(T)) + \dim(\mathrm{null}(T)) - k$	subtract k
$n-k$	$\geq$	$\dim(\mathrm{range}(T))$	$\dim(\mathrm{null}(T)) - k \geq 0$

Since k is the count of zero-columns in $\mathcal{M}(T, \mathfrak{B}_V, \mathfrak{B}_W)$, the matrix has at least one non-zero entry in $(n-k)$ columns; and by the above relation $(n-k) \geq \dim(\mathrm{range}(T))$...

Hence, $\mathcal{M}(T, \mathfrak{B}_V, \mathfrak{B}_W)$ has at least $\dim(\mathrm{range}(T))$ non-zero entries.

Suggested Problems

3.A—1, 4, 5^+, 6^+, 8^a, 9^a, 11, 14

3.B—1, 2, 5, 6, 9, 17^+, 18^+, 31

3.C—1, 2–3–4–5, 12

^{a}-marked problems have an "analysis flavor" (if that's your thing!)

Solutions to $^+$-marked problems are longer/more challenging.

Assigned Homework HW#3.1, Due Date in Canvas/Gradescope

3.A—4, 14

3.B—5, 6, 9

3.C—2, 3

Note: Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually "scheduled.")

Upload homework to www.Gradescope.com

General Definition of Homogeneity

Definition (Homogeneity of Degree k)

If $f : V \mapsto W$ is a function over a field $\mathbb{F}$, and $k \in \mathbb{Z}$, then f is said to be homogeneous of degree k if

$$f(\alpha v) = \alpha^k f(v)$$

$\forall \alpha \in \mathbb{F} \setminus \{0\}, v \in V$.