Math 524: Linear Algebra

Notes #3.2 — Linear Maps

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Outline

Student Learning Targets, and Objectives

Target Isomorphic Vector Spaces
- Objective: Know the condition for finite-dimensional vector spaces to be isomorphic
- Objective: Be familiar with the definition and special properties of **operators**

Target Product Spaces
- Objective: Be able to form product spaces, and find dimensions, and bases for them

Target Quotient Spaces
- Objective: Be able to form quotient spaces, and define and use the quotient map; as well as find dimensions

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⟨⟨SUPPLEMENTAL⟩⟩

Target Dual Space of a Vector Space & the Dual of a Linear Map
- Objective: Be familiar with the language and notation of duality; and properties of dual maps

Invertible Linear Maps

Definition (Invertible, Inverse)

- A linear map $T \in \mathcal{L}(V, W)$ **invertible** if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity map on V and TS equals the identity map on W
- A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an **inverse** of T (note that the first I is the identity map on V and the second I is the identity map on W).

Inverse is Unique

Theorem (Inverse is Unique)

An invertible linear map has a unique inverse.

Proof (Inverse is Unique)

Suppose $T \in \mathcal{L}(V,W)$ is invertible, and S_1 and S_2 are inverses of T:

$$\boxed{S_1} = S_1 I = S_1(TS_2) = (S_1 T)S_2 = IS_2 = \boxed{S_2}.$$

Notation (The Inverse of T: T^{-1})

If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V,W)$ is invertible, then $\exists! T^{-1} \in \mathcal{L}(W,V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$.

Invertibility ⇔ Injectivity and Surjectivity

Theorem (Invertibility is Equivalent to Injectivity and Surjectivity)

A linear map is invertible if and only if it is injective and surjective.

Proof (Invertibility is Equivalent to Injectivity and Surjectivity)

Suppose $T \in \mathcal{L}(V,W)$.

$\Rightarrow$ First, we assume T is invertible

(i) To show injectivity, let $u, w \in V$, and $T(u) = T(w)$:

$$\boxed{u} = T^{-1}(T(u)) = T^{-1}(T(w)) = \boxed{w}. \quad \surd_{\text{injective}}$$

(ii) To show surjectivity, let $w \in W$, then $w = T(T^{-1}(w))$, which shows $w \in \mathrm{range}(T) \to W = \mathrm{range}(T)$. $\surd_{\text{surjective}}$

Proof :: Invertibility is Equivalent to Injectivity and Surjectivity 1/2

Proof (Invertibility is Equivalent to Injectivity and Surjectivity)

Suppose $T \in \mathcal{L}(V,W)$.

$\Leftarrow$ Now, assume T is injective and surjective; we need to show T is invertible.

$\forall w \in W$, let $S(w) \in V$ (unique) such that $T(S(w)) = w$ (existence + uniqueness of such an element follows from surjectivity + injectivity of T). Clearly $TS = I_W$.

To show that $ST = I_V$, let $v \in V$:

$$T((ST)(v)) = (TS)(T(v)) = I(T(v)) = T(v)$$

so $(ST)(v) = v$, which makes $ST = I_V$.

We must also show that S is linear...

Proof :: Invertibility is Equivalent to Injectivity and Surjectivity 2/2

Proof (Invertibility is Equivalent to Injectivity and Surjectivity)

Suppose $T \in \mathcal{L}(V,W)$.

$\Leftarrow$ The final piece is to show that S is linear. Let $w_1, w_2 \in W$:

$$T(S(w_1) + S(w_2)) = T(S(w_1)) + T(S(w_2)) = w_1 + w_2$$

thus $S(w_1) + S(w_2) \in V$ (unique) that maps to $w_1 + w_2 \in W$; by definition of $S \Rightarrow S(w_1 + w_2) = S(w_1) + S(w_2)$. $\surd_{\text{additive property}}$

Let $w \in W$ and $\lambda \in \mathbb{F}$:

$$T(\lambda S(w)) = \lambda T(S(w)) = \lambda w$$

thus $\lambda S(w) \in V$ is the unique element that T maps to $\lambda w \in W$. By the definition of $S \Rightarrow S(\lambda w) = \lambda S(w)$. $\surd_{\text{homogeneity property}}$

$\surd_{\text{additive property}} + \surd_{\text{homogeneity property}} \Rightarrow S$ is linear.

Examples: Non-Invertible Linear Maps

Example (Linear Maps that are not Invertible)

- The differentiation map: $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is not invertible since $\text{null}(D) = \text{span}\,(1) \neq \{0\}$ [Not Injective]
- The multiplication-by-z^q map: $T \in \mathcal{L}(\mathcal{P}(\mathbb{F}), \mathcal{P}(\mathbb{F}))$ defined by $Tp \equiv T(p) = z^q p(z)$, for $z \in \mathbb{F}$ is not invertible for $q \geq 1$ since $\text{span}\,(1) \not\in \text{range}(T)$. [Not Surjective]
- The shift map: $T \in \mathcal{L}(\mathbb{F}^\infty, \mathbb{F}^\infty)$ defined by $T(z_1, z_2, z_3, \dots) = (z_2, z_3, z_4, \dots)$ is not invertible since $(z, 0, 0, \dots) \in \text{null}(T)$. [Not Injective]

Isomorphic Vector Spaces

Definition (Isomorphism, Isomorphic)

- An **isomorphism** is an invertible linear map.
- Two vector spaces are called **isomorphic** if there is an isomorphism from one vector space onto the other one.

Isomorphic spaces are the "same" in some sense; since the maps $T : V \mapsto W$, and $T^{-1} : W \mapsto V$ "tie" the elements from the spaces together anything we do in one space can be translated to the other space (via the map).

Comment (Linear Maps are Homomorphisms [Text adopted from Wikipedia])

In algebra, a homomorphism is a structure-preserving map between two algebraic structures of the same type (such as two groups, two rings, or two *vector spaces*). A homomorphism may also be an isomorphism, an endomorphism, an automorphism, etc.

Morphisms: auto-, endo-, homo-, iso-

Wikipedia & Dictionary.com

- a **morphism** is a structure-preserving map from one mathematical structure to another one of the same type.
- a **homomorphism** is a structure-preserving map between two algebraic structures of the same type (such as two groups, two rings, or two vector spaces). **"homo" ≈ "same"**
- an **isomorphism** is a structure-preserving mapping between two structures of the same type that can be reversed by an inverse mapping. **"iso" ≈ "equal"**
- an **automorphism** is an *isomorphism* from a mathematical object to itself. It is, in some sense, a symmetry of the object, and a way of mapping the object to itself while preserving all of its structure. The set of all automorphisms of an object forms a group, called the automorphism group. **"auto" ≈ "self"**
- an **endomorphism** is a morphism from a mathematical object to itself. An endomorphism that is also an *isomorphism* is an *automorphism*. **"endo" ≈ "within"**

Dimension Shows Whether Vector Spaces are Isomorphic

Theorem (Dimension Shows Whether Vector Spaces are Isomorphic)

Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

Proof (Dimension Shows Whether Vector Spaces are Isomorphic)

⇒ First suppose V and W are isomorphic finite-dimensional vector spaces. Thus there exists an isomorphism T from V onto W. Because T is invertible, we have $\text{null}(T) = \{0\}$ and $\text{range}(T) = W$. Thus $\dim(\text{null}(T)) = 0$ and $\dim(\text{range}(T)) = \dim(W)$.

Using the [Fundamental Theorem of Linear Maps (Notes#3.1)]

$$\dim(V) = \dim(\text{null}(T)) + \dim(\text{range}(T))$$

gives $\dim(V) = \dim(W)$, completing the proof in one direction.

Proof :: Dimension Shows Whether Vector Spaces are Isomorphic

Proof (Dimension Shows Whether Vector Spaces are Isomorphic)

$\Leftarrow$ Next, suppose V and W are finite-dimensional vector spaces with the same dimension, n. Let $v_1, \ldots, v_n$ be a basis of V and $w_1, \ldots, w_n$ be a basis of W. Let $T \in \mathcal{L}(V, W)$ be defined by

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

Then T is a well-defined linear map because $v_1, \ldots, v_n$ is a basis of V. Also, T is **surjective** because $w_1, \ldots, w_n$ spans W.

Also, $\text{null}(T) = \{0\}$ because $w_1, \ldots, w_n$ is linearly independent; thus T is **injective**.

Because T is injective and surjective, it is an isomorphism. Hence V and W are isomorphic.

Why Not Stay in $\mathbb{F}^n$???

Since every finite-dimensional vector space is isomorphic to some $\mathbb{F}^n$, why not just study $\mathbb{F}^n$ instead of more general vector spaces?

Investigation of $\mathbb{F}^n$ would soon lead to other vector spaces. For example, we would encounter the null-space and range-space of linear maps.

Although each of these vector spaces is isomorphic to some $\mathbb{F}^n$, thinking of them that way often adds complexity but no new insight.

Thinking of vector spaces "in the abstract" makes us focus on the *structures* which drive the properties, instead of getting bogged down in "implementation details."

E.g. [terminus.sdsu.edu/SDSU/Math254/Lectures/2.3/index.php?subdir=2.3_Lecture&page=17]

$\mathcal{L}(V, W)$ and $\mathbb{F}^{m\times n}$ are Isomorphic — $\mathcal{M} : \mathcal{L}(V, W) \mapsto \mathbb{F}^{m\times n}$

Theorem ($\mathcal{L}(V, W)$ and $\mathbb{F}^{m\times n}$ are Isomorphic)

Let $v_1, \ldots, v_n$ be a basis for V, and $w_1, \ldots, w_m$ a basis for W; then $\mathcal{M}$ (the "matrixification") is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m\times n}$.

Proof ($\mathcal{L}(V, W)$ and $\mathbb{F}^{m\times n}$ are Isomorphic)

$\mathcal{M}$ is linear. We need to show that it is injective and surjective.

injective: Let $T \in \mathcal{L}(V, W)$, and $\mathcal{M}(T) = 0$, then $T(v_k) = 0$, $k = 1, \ldots, n$. Thus $T = 0$, and $\text{null}(\mathcal{M}(T)) = \{0\} \Leftrightarrow \mathcal{M}$ is injective.

surjective: Let $A \in \mathbb{F}^{m\times n}$, $T : V \mapsto W$ such that

$$T(v_k) = \sum_{j=1}^{m} a_{j,k} w_j, \quad k = 1, \ldots, n$$

thus $\mathcal{M}(T) = A$, and $\text{range}(\mathcal{M}(T)) = \mathbb{F}^{m\times n}$.

$\dim(\mathcal{L}(V, W)) = (\dim(V))\,(\dim(W))$

Theorem ($\dim(\mathcal{L}(V, W)) = (\dim(V))\,(\dim(W))$)

Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional, and

$$\dim(\mathcal{L}(V, W)) = (\dim(V))\,(\dim(W))$$

Rewind

$\mathcal{L}(V, W)$ and $\mathbb{F}^{m\times n}$ are Isomorphic
Let $v_1, \ldots, v_n$ be a basis for V, and $w_1, \ldots, w_m$ a basis for W; then $\mathcal{M}$ (the "matrixification") is isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m\times n}$.

Dimension Shows Whether Vector Spaces are Isomorphic
Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.

Dimension of $\mathbb{F}^{m\times n}$
With addition and scalar multiplication defined as above, $\mathbb{F}^{m\times n}$ is a vector space with $\dim(\mathbb{F}^{m\times n}) = mn$.

Linear Maps as Matrix Multiplication

To "complete" our language / framework, we formally define:

Definition (Matrix of a Vector, $\mathcal{M}(v)$)

Suppose $v \in V$, and $v_1, \ldots, v_n$ is a basis of V. The matrix of v with respect to the basis is

$$\mathcal{M}(v) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{F}^{n\times 1}$$

where $c_1, \ldots, c_n$ are the (unique) scalars (a.k.a. "coordinates") such that $v = c_1 v_1 + \cdots + c_n v_n$.

Matrix "Slicing" — Columns of $\mathcal{M}(T)$

Notation (The k-th Column of a Matrix)

Let $A \in \mathbb{F}^{m\times n}$, then $a_k \in \mathbb{F}^{m\times 1}$ is the k-th column of A.

Theorem (Columns of $\mathcal{M}$)

Let $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W, $k \in [1, \ldots, n]$, then the k-th column of $\mathcal{M}(T)$ is given by

$$\mathcal{M}(T)_k = \mathcal{M}(v_k)$$

Linear Maps Act Like Matrix Multiplication

Theorem (Linear Maps Act Like Matrix Multiplication)

Let $T \in \mathcal{L}(V, W)$, $v \in V$; $v_1, \ldots, v_n$ a basis of V; $w_1, \ldots, w_m$ a basis of W. Then

$$\mathcal{M}(T(v)) = \mathcal{M}(T)\mathcal{M}(v)$$

Proof (Linear Maps Act Like Matrix Multiplication)

Given $v \in V$, write $v \stackrel{!}{=} c_1 v_1 + \cdots + c_n v_n$

$$\begin{aligned} T(v) &= c_1 T(v_1) + \cdots + c_n T(v_n) \\ \mathcal{M}(T(v)) &= \mathcal{M}(c_1 T(v_1) + \cdots + c_n T(v_n)) \\ &= c_1 \mathcal{M}(T(v_1)) + \cdots + c_n \mathcal{M}(T(v_n)) \\ &= c_1 \mathcal{M}(T)_1 + \cdots + c_n \mathcal{M}(T)_n \\ &= \mathcal{M}(T)\mathcal{M}(v) \end{aligned}$$

Operators, $\mathcal{L}(V)$

Linear maps from a vector space to itself are important, and get their own name and notation:

Definition (Operator, $\mathcal{L}(V)$)

- A linear map from a vector space to itself is called an **operator**.
- The notation $\mathcal{L}(V)$ denotes the set of all operators on V. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

Theorem (For $\mathcal{L}(V)$: Injectivity $\Leftrightarrow$ Surjectivity in Finite Dimensions)

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$; the following are equivalent:

- *T is invertible;*
- *T is injective;*
- *T is surjective.*

Proof :: For $\mathcal{L}(V)$: Injectivity $\Leftrightarrow$ Surjectivity in Finite Dimensions

Proof (For $\mathcal{L}(V)$: Injectivity $\Leftrightarrow$ Surjectivity in Finite Dimensions)

T invertible $\Rightarrow$ injective + surjective. $\surd_{\text{since } \Leftrightarrow \text{ holds}}$

Assume T is injective: $\Rightarrow \text{null}(T) = \{0\}$;
[Fundamental Theorem of Linear Maps (Notes#3.1)]:

$$\dim(\text{range}(T)) = \dim(V) - \dim(\text{null}(T)) = \dim(V)$$

which makes T surjective; injective + surjective $\Rightarrow$ invertible. $\surd$

Assume T is surjective: $\Rightarrow \text{range}(T) = V$;
[Fundamental Theorem of Linear Maps (Notes#3.1)]:

$$\dim(\text{null}(T)) = \dim(V) - \dim(\text{range}(T)) = 0$$

which makes T injective; injective + surjective $\Rightarrow$ invertible. $\surd$

Example $T \in \mathcal{L}(\mathcal{P}_m(\mathbb{F}))$

Example ($T \in \mathcal{L}(\mathcal{P}_m(\mathbb{F}))$)

(Q) Show that $\forall q \in \mathcal{P}(\mathbb{F})\ \exists p \in \mathcal{P}(\mathbb{F}) : ((z^2+5z+7)p(z))'' = q$.

(A) $\dim(\mathcal{P}(\mathbb{F})) = \infty$, so the theorem does not apply.

However, for *any* given $q \in \mathcal{P}(\mathbb{F})$, $\deg(q) = m < \infty$; we restrict ourselves to $\mathcal{P}_m(\mathbb{F})$: Let $T \in \mathcal{L}(\mathcal{P}_m(\mathbb{F}))$ be defined by

$$Tp = ((z^2+5z+7)p(z))''$$

multiplication by (z^2+5z+7) takes us to $\mathcal{P}_{m+2}(\mathbb{F})$; then the derivatives takes us to $\mathcal{P}_{m+1}(\mathbb{F})$, and $\mathcal{P}_m(\mathbb{F})$. (yes, $T \in \mathcal{L}(\mathcal{P}_m(\mathbb{F}))$)

Only sub-quadratic polynomials have zero second derivative, therefore $\text{null}(T) = \{0\}$ ($\Rightarrow T$ is injective).

The theorem guarantees surjectivity; so there must be a polynomial $p \in \mathcal{P}_m(\mathbb{F}) : T(p) = q$. $\surd$

Example, continued — Matrixification

Again, we consider $T \in \mathcal{L}(\mathcal{P}_m(\mathbb{F}), \mathcal{P}_m(\mathbb{F}))$ defined by $T(z) = ((z^2+5z+7)p(z))''$; we will compute $\mathcal{M}(T, \{1, z, \dots, z^m\}, \{1, z, \dots, z^m\})$.

First off, we need to transform the input basis vectors:

$$\begin{cases} T(1) &= & & & & & 2 \\ T(z) &= & & & 6z & + & 10 \\ T(z^2) &= & 12z^2 & + & 30z & + & 14 \\ T(z^3) &= & 20z^3 & + & 60z^2 & + & 42z \\ T(z^4) &= & 30z^4 & + & 100z^3 & + & 84z^2 \\ T(z^5) &= & 42z^5 & + & 150z^4 & + & 140z^3 \\ & \vdots & & & & & \\ T(z^k) &= & \underbrace{(k^2+3k+2)}_{(k+2)(k+1)}z^k & + & \underbrace{(5k^2+5k)}_{5(k+1)k}z^{k-1} & + & \underbrace{(7k^2-7k)}_{7k(k-1)}z^{k-2} \end{cases}$$

Example, continued — Matrixification

Now, we can assemble matrices:

Input Basis (columns); *Output Basis* (rows)

	1	z	z^2	z^3	z^4	z^5	$\cdots$	$\cdots$	$\cdots$	z^k
1	2	10	14							
z		6	30	42						
z^2			12	60	84					
z^3				20	100	140				
z^4					30	150	$\ddots$			
z^5						42	$\ddots$	$\ddots$		
$\vdots$							$\ddots$	$\ddots$	$\ddots$	
z^{k-2}								$\ddots$	$\ddots$	$(7k^2-7k)$
z^{k-1}									$\ddots$	$(5k+5k^2)$
z^k										$(3k+2+k^2)$

Anti-Examples on Infinite-Dimensional Vector Spaces

Example (When $\dim(V) = \infty$, Neither Injectivity nor Surjectivity implies Invertibility)

- The multiplication-by-z^p, $p \in \mathbb{Z}^+$ operator on $\mathcal{P}(\mathbb{F})$ is injective, but not surjective. $\{1, \dots, z^{p-1}\} \notin \text{range}(*z^p)$
- The sequence-[left]shift operator on $\mathbb{F}^\infty$ is surjective but not injective. $(z_1, 0, 0, \dots) \in \text{null}()$
- (we need both Surjectivity and Injectivity for Invertibility)

⟨⟨⟨ Live Math ⟩⟩⟩

e.g. 3D-{**1**, 4, 5}

Live Math :: Covid-19 Version 3D-1

3D-1: Suppose $T \in \mathcal{L}(U, V)$, and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Solution

Using the definitions for products, and associativity:

$$\begin{aligned} (ST)(T^{-1}S^{-1}) &= S(TT^{-1})S^{-1} = SS^{-1} = I \\ (T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}S)T = T^{-1}T = I \end{aligned}$$

which by definition makes (ST) invertible, with inverse $(T^{-1}S^{-1})$.

Products of Vector Spaces

Convention: When dealing with more than one vector space, all the vector spaces in use will be over the same field.

Definition (Products of Vector Spaces)

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$.

- The **product** $V_1 \times \cdots \times V_m$ is defined by
$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}$$
- Addition on $V_1 \times \cdots \times V_m$ is defined by
$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$$
- Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by
$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$$

Products of Vector Spaces

Theorem (Product of Vector Spaces is a Vector Space)

Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

The proof follows directly from how the spaces are joined, and then joining additive-identity-objects and additive-inverse-object in the same way. Closure(s) follow from closure(s) of the joined spaces.

Example: $\mathbb{F}^p \times \mathbb{F}^q$ and $\mathbb{F}^{p+q}$, $p, q \in \mathbb{Z}^+$

- $s \in \mathbb{F}^p$ are lists of length p: $s = (s_1, \dots, s_p)$
- $t \in \mathbb{F}^q$ are lists of length q: $t = (t_1, \dots, t_q)$
- $u \in \mathbb{F}^p \times \mathbb{F}^q$ are lists of length 2, where the first list-element is a list of length p, and the second list-element is a list of length q: $u = ((u_{11}, \dots, u_{1p}), (u_{21}, \dots, u_{2q}))$.
- $v \in \mathbb{F}^{p+q}$ are lists of length $p+q$: $v = (v_1, \dots, v_{p+q})$
- Clearly $\mathbb{F}^p \times \mathbb{F}^q \not\equiv \mathbb{F}^{p+q}$, but they are isomorphic; and yes, the isomorphism is "obvious!"

A Basis for $\mathcal{P}_2(\mathbb{F}) \times \mathbb{F}^3$

- A basis for $\mathcal{P}_2(\mathbb{F})$ is given by $\{1, z, z^2\}$
- A basis for $\mathbb{F}^3$ is given by $\{(1,0,0), (0,1,0), (0,0,1)\}$
- Elements in $\mathcal{P}_2(\mathbb{F}) \times \mathbb{F}^3$ are of the form $(p(z) \in \mathcal{P}_2(\mathbb{F}), v \in \mathbb{F}^3)$
- We can build a basis:

$$\left\{ \begin{array}{lll} (1,\,(0,0,0)) & (z,\,(0,0,0)), & (z^2,\,(0,0,0)) \\ (0,\,(1,0,0)) & (0,\,(0,1,0)), & (0,\,(0,0,1)) \end{array} \right\}$$

- This gives away the "big secret" revealed in the next theorem: is seems like $\dim(\mathcal{P}_2(\mathbb{F}) \times \mathbb{F}^3) = \dim(\mathcal{P}_2(\mathbb{F})) + \dim(\mathbb{F}^3)$

Dimension of a Product is the Sum of Dimensions

Theorem (Dimension of a Product is the Sum of Dimensions)

Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional, and

$$\dim(V_1 \times \cdots \times V_m) = \dim(V_1) + \cdots + \dim(V_m)$$

Proof (Dimension of a Product is the Sum of Dimensions)

Choose a basis for each V_k, for each basis vector, consider the element in $V_1 \times \cdots \times V_m$ which is the appropriately zero-padded version of the basis vector. The list of all such vectors is linearly independent, and spans $V_1 \times \cdots \times V_m$. The length of this basis is $\dim(V_1) + \cdots + \dim(V_m)$

Products and Direct Sums

Theorem (Products and Direct Sums)

Suppose that $U_1, \ldots, U_m$ are subspaces of V. Define a linear map $T : U_1 \times \cdots \times U_M \mapsto U_1 + \cdots + U_m$ by

$$T(u_1, \ldots, u_m) = u_1 + \cdots + u_m.$$

Then $U_1 + \cdots + U_m$ is a direct sum if and only if T is injective.

Rewind ([Condition for a Direct Sum (Notes#1)])

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$, is by taking each $u_j = 0$.

Proof (Products and Direct Sums)

The linear map T is injective if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each $u_j \in U_j$, is by taking $u_j = 0$. [Injectivity ⇔ null space equals {0} (Notes#3.1)] Thus T is injective if and only if $U_1 + \cdots + U_m$ is a direct sum. [Condition for a Direct Sum (Notes#1)]

A Sum is a Direct Sum if and only if Dimensions Add Up

Theorem (A Sum is a Direct Sum if and only if Dimensions Add Up)

Suppose V is finite dimensional and $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if

$$\dim(U_1 + \cdots + U_m) = \dim(U_1) + \cdots + \dim(U_m).$$

Proof (A Sum is a Direct sum ⇔ Dimensions Add Up)

The linear map T (defined in the previous theorem/proof) is surjective. By the [Fundamental Theorem of Linear Maps (Notes#3.1)], T is injective if and only if

$$\dim(U_1 + \cdots + U_m) = \dim(U_1 \times \cdots \times U_m)$$

Combining [Dimension of a product is the sum of dimensions], and the previous theorem shows that $U_1 \times \cdots \times U_m$ is direct sum if and only if

$$\dim(\mathbf{U_1} + \cdots + \mathbf{U_m}) = \dim(U_1 \times \cdots \times U_m) = \dim(\mathbf{U_1}) + \cdots + \dim(\mathbf{U_m}).$$

Quotients of Vector Spaces "Quotient Spaces"

Definition ($v + U$)

Suppose $v \in V$, and U is a subspace of V. Then $v + U$ is the subset of V defined by

$$v + U = \{v + u : u \in U\}$$

Example ($v + U$)

- Let $U = \{\lambda(1, 2, 3) : \lambda \in \mathbb{R}\}$ (a line $\in \mathbb{R}^3$ through the origin.)
- Let $v = (1, 0, 0) \in \mathbb{R}^3$ (not on the line)
- Then $v + U = \{(1 + \lambda, 2\lambda, 3\lambda) : \lambda \in \mathbb{R}\}$ (a line $\in \mathbb{R}^3$ NOT through the origin $\Rightarrow$ Not a Vector Space.)

Affine Subset, Parallel

Definition (Affine Subset, Parallel)

- An **affine subset** of V is a subset of V of the form $v + U$ for some $v \in V$ and some subspace U of V.
- For $v \in V$ and U a subspace of V, the affine subset $v + U$ is said to be **parallel** to U.

Example ($v + U$)

- If $P = \{(x_1, 0, x_3, 0, 0) \in \mathbb{R}^5 : x_1, x_3 \in \mathbb{R}\}$ then the affine subsets parallel to P are the planes in $\mathbb{R}^5$ that are parallel to the x_1–x_3 plane... (in the "obvious" 5-dimensional sense!)

Quotient Space, V/U

Definition (Quotient Space, V/U)

Suppose U is a subspace of V. Then the quotient space V/U is the set of all affine subsets of V parallel to U:

$$V/U = \{v + U : v \in V\}$$

Example (Quotient Spaces)

- If L is a line in $\mathbb{R}^n$ containing the origin, then $\mathbb{R}^n/L$ is the set of all lines in $\mathbb{R}^n$ parallel to L.
- If P is a plane in $\mathbb{R}^n$ $(n \geq 2)$ containing the origin, then $\mathbb{R}^n/P$ is the set of all planes in $\mathbb{R}^n$ parallel to P.

Two Affine Subsets Parallel to U are Equal or Disjoint

Theorem (Two Affine Subsets Parallel to U are Equal or Disjoint)

Suppose U is a subspace of V and $v, w \in V$. Then the following are equivalent:

(a) $v - w \in U$

(b) $v + U = w + U$ *"The Affine Subsets are Equal"*

(c) $(v + U) \cap (w + U) \neq \varnothing$

Stated in the negative:

Theorem (Two Affine Subsets Parallel to U are Equal or Disjoint)

(a$^-$) $v - w \notin U$

(b$^-$) $v + U \neq w + U$

(c$^-$) $(v + U) \cap (w + U) = \varnothing$ *"The Affine Subsets are Disjoint"*

Proof :: Two Affine Subsets Parallel to U are Equal or Disjoint

Proof (Two Affine Subsets Parallel to U are Equal or Disjoint)

(b)⇒(c) This one is for free.

(a)⇒(b) Suppose **(a)** $v - w \in U$. If $u \in U$, then

$$v + u = w + ((v - w) + u) \in w + U$$

so $v + U \subset w + U$. In the same way $w + U \subset v + U$; and therefore **(b)** $v + U = w + U$

(c)⇒(a) Suppose **(c)** $(v + U) \cap (w + U) \neq \varnothing$. $\exists u_1, u_2 \in U$

$$v + u_1 = w + u_2 \quad v - w = u_2 - u_1 \in U$$

which shows **(a)** $v - w \in U$.

(a)⇒(b)⇒(c)⇒(a) $\surd$

Addition and Scalar Multiplication on V/U — Quotient Space is a Vector Space

Definition (Addition and Scalar Multiplication on V/U)

Suppose U is a subspace of V. The **addition** and **scalar multiplication** are defined on V/U by:

$$\begin{aligned} (v + U) + (w + U) &= (v + w) + U \\ \lambda(v + U) &= (\lambda v) + U \end{aligned}$$

for $v, w \in V$, $\lambda \in \mathbb{F}$.

Theorem (Quotient Space is a Vector Space)

Suppose U is a subspace of V. Then V/U, with the operations of addition and scalar multiplication as defined above, is a vector space.

Proof :: Quotient Space is a Vector Space

Problem: Non-Unique Representation

The representation of an affine subset parallel to U is not unique.
Let $v, \widehat{v}, w, \widehat{w} \in V$: $(v+U) = (\widehat{v}+U)$, and $(w+U) = (\widehat{w}+U)$.
We must show $(v+w)+U = (\widehat{v}+\widehat{w})+U$.

Proof (Quotient Space is a Vector Space)

+ By [Two Affine Subsets Parallel to U are Equal or Disjoint] we have $(v-\widehat{v}) \in U$, and $(w-\widehat{w}) \in U$. Since U is a subspace of V (and therefore closed under addition), we have $(v-\widehat{v})+(w-\widehat{w}) \in U$ and therefore $(v+w)-(\widehat{v}+\widehat{w}) \in U$. Invoking [Two Affine Subsets Parallel to U are Equal or Disjoint] again, gives $(v+w)+U = (\widehat{v}+\widehat{w})+U$.

* Let $\lambda \in \mathbb{F}$. Since U is a subspace of V (and therefore closed under scalar multiplication, we have $\lambda(v-\widehat{v}) \in U$. Thus $\lambda v - \lambda\widehat{v} \in U$. Yet another invocation of the theorem gives us $(\lambda v)+U = (\lambda\widehat{v})+U$.

Proof :: Quotient Space is a Vector Space — wrap-up

Proof (Quotient Space is a Vector Space — wrap-up)

Now, addition and scalar multiplication are well-defined on V/U; it remains to show that V/U is a vector space:

0 The **additive identity** is $(0+U)$

− The **additive inverse** of $(v+U)$ is $((-v)+U)$

Now, the rest of the vector space properties follow the fact that V is a vector space, and the definitions of $(+, *, 0, -)$.

We leave the details as an excercise for a dark and stormy night...

Quotient Map, π — Dimension of a Quotient Space

Definition (Quotient Map, π)

Suppose U is a subspace of V. The **quotient map** π is the linear map $\pi : V \mapsto V/U$ defined by

$$\pi(v) = v + U$$

for $v \in V$. (A more complete notation would be $\pi(v, U)$, but usually the subspace U is assumed to be "obvious from context.")

Theorem (Dimension of a Quotient Space)

Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim(V/U) = \dim(V) - \dim(U)$$

Sometimes, $\mathrm{codim}(U) = \dim(V/U)$ — *"the co-dimension of U in V."*

Proof :: Dimension of a Quotient Space

Proof (Dimension of a Quotient Space)

Let $\pi : V \mapsto V/U$. We have $\mathrm{null}(\pi) = U$; and $\mathrm{range}(\pi) = V/U$; therefore

$$\dim(V) = \underbrace{\dim(U)}_{\dim(\mathrm{null}(\pi))} + \underbrace{\dim(V/U)}_{\dim(\mathrm{range}(\pi))}$$

re-arranging gives the result.

We used [Two Affine Subsets Parallel to U are Equal or Disjoint], and [The Fundamental Theorem of Linear Maps (Notes#3.1)]...

The Induced Map $\tilde{T}$

Definition (The Induced Map $\tilde{T}$)

Suppose $T \in \mathcal{L}(V, W)$. Define $\tilde{T} : (V/\text{null(T)}) \mapsto W$ by

$$\tilde{T}(v + \text{null}(T)) = T(v)$$

This is well-defined since for $u, v \in V$: $u + \text{null}(T) = v + \text{null}(T)$. [Two Affine Subsets Parallel to U are Equal or Disjoint] guarantees $(u - v) \in \text{null}(T)$;
i.e. $0 = T(u - v) = T(u) - T(v) \Rightarrow T(u) = T(v)$. $\surd$

Null Space and Range of $\tilde{T}$

Theorem (Null Space and Range of $\tilde{T}$)

Suppose $T \in \mathcal{L}(V, W)$, then

- *1. $\tilde{T}$ is a linear map from $(V/(\text{null}(T))$ to W*
- *2. $\tilde{T}$ is injective*
- *3. $\text{range}(\tilde{T}) = \text{range}(T)$*
- *4. $(V/\text{null}(T))$ is isomorphic to $\text{range}(T)$*

Proof (Null Space and Range of $\tilde{T}$)

- 1. The usual "closure mechanics..." nothing exciting here
- 3. The is true by the construction of $\tilde{T}$

Proof :: Null Space and Range of $\tilde{T}$

Proof (Null Space and Range of $\tilde{T}$)

- 2. Let $v \in V$ and $\tilde{T}(v + \text{null}(T)) = 0$. Then $T(v) = 0 \Rightarrow v \in \text{null}(T)$. [Two Affine Subsets Parallel to U are Equal or Disjoint] implies that $(v + \text{null}(T)) = (0 + \text{null}(T)) \Rightarrow \text{null}(\tilde{T}) = 0 \Rightarrow \tilde{T}$ is injective. $\surd$
- 4. Parts (2.) and (3.) show that if we consider $\tilde{T}$ as a mapping into $\text{range}(T)$, then $\tilde{T}$ is an isomorphism from $V/(\text{null}(T))$ onto $\text{range}(T)$.

More Abstraction?

If you want a more ~~linear~~ abstract [Math-320/520-ish] algebraic view of the Quotient Space / Quotient Map framework...

The elements of the Quotient Space (V/U) are **equivalence classes**

$$[v] = v + U = \{v + U : v \in V\}$$

under the **equivalence relation** $v_1 \sim v_2 \Leftrightarrow v_1 + U = v_2 + U$.

We have scalar multiplication and addition on the equivalence classes defined by

- $\lambda[v] = [\lambda v]$ $\forall \lambda \in \mathbb{F}$, $v \in V$
- $[v_1] + [v_2] = [v_1 + v_2]$, $v_1, v_2 \in V$

The quotient map simply maps v to its equivalence class $[v]$.

⟨⟨⟨ Live Math ⟩⟩⟩

e.g. 3E-{**5**, 13}

Live Math :: Covid-19 Version 3E-5

3E-5: Suppose $W_1, \ldots, W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic spaces.

Solution

We show that Γ defined below is an isomorphism:

$$\begin{aligned} \Gamma : \mathcal{L}(V, W_1 \times \cdots \times W_m) &\mapsto \mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m) \\ (\Gamma(T_1, \ldots, T_m))(v) &= (T_1(v), \ldots, T_m(v)) \end{aligned}$$

Γ, being a "stacking" of linear maps, is "obviously" a linear map (it is linear in each component, etc...)

We show that Γ is both injective (one-to-one) and surjective (onto); thus showing that is an isomorphism...

Live Math :: Covid-19 Version 3E-5

Injectivity

If $(T_1, \ldots, T_m) \in \mathcal{L}(V, W_1 \times \cdots \times W_m)$, and $\Gamma(T_1, \ldots, T_m) = 0$, then $T_k = 0$, $k = 1, \ldots, m$. Thus $\text{null}(\Gamma) = \{0\}$ which makes Γ injective due to [Injectivity ⇔ null space equals $\{0\}$ (Notes#3.1)]

Surjectivity

Let $T \in \mathcal{L}(V, W_1 \times \cdots \times W_m)$. Define $T_k \in \mathcal{L}(V, W_k)$ by

$$T(v) = (T_1(v), \ldots, T_m(v))$$

for $v \in V$. Then $\Gamma(T_1, \ldots, T_m) = T$... and Γ is surjective.

Isomorphic Spaces

[Invertibility is Equivalent to Injectivity and Surjectivity] + Definition of Isomorphism (as an invertible linear map) ⇝ The Spaces are Isomorphic.

Suggested Problems

3.D—1, 2, 3, 4, 5, 6

3.E—2, 4, 5, 13

Assigned Homework — HW#3.2, Due Date in Canvas/Gradescope

3.D—2, 3

3.E—2, 4

$\in$ {Midterm#1 Material}

Note: Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually "scheduled.")

Upload homework to www.Gradescope.com

Duality

"In mathematics, a duality, generally speaking, translates concepts, theorems or mathematical structures into other concepts, theorems or structures, in a one-to-one fashion, often (but not always) by means of an involution operation: if the dual of A is B, then the dual of B is A. Such involutions sometimes have fixed points, so that the dual of A is A itself. For example, Desargues' theorem is self-dual in this sense under the standard duality in projective geometry."

"Many mathematical dualities between objects of two types correspond to pairings, bilinear functions from an object of one type and another object of the second type to some family of scalars. For instance, linear algebra duality corresponds in this way to bilinear maps from pairs of vector spaces to scalars, the duality between distributions and the associated test functions corresponds to the pairing in which one integrates a distribution against a test function, and Poincaré duality corresponds similarly to intersection number, viewed as a pairing between submanifolds of a given manifold"

https://en.wikipedia.org/wiki/Duality_(mathematics)

Duality

For the time being, we will not deep-dive into duality... we will return to the topic later in the class with a slightly different perspective. Still, a quick look at the results in this section provides some useful scaffolding for future concepts.

At the end of the section, we have some formal definitions of concepts we probably recognize, and definitely need...

The Dual Space and the Dual Map

Linear maps into the scalar field $\mathbb{F}$ play a special role in linear algebra, and thus they get a special name:

Definition (Linear Functional)

A **linear functional** on V is a linear map from $V \mapsto \mathbb{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.

Examples of Linear Functionals — Dual Space $V' = \mathcal{L}(V, \mathbb{F})$

Example (Linear Functionals)

- $\varphi : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by $\varphi(x_1, x_2, x_3) = 4x_1 - 5x_2 + 6x_3$
- Fix $(c_1, \dots, c_n) \in \mathbb{F}^n$, then define $\varphi : \mathbb{F}^n \mapsto \mathbb{R}$ by $\varphi(x_1, \dots, x_n) = c_1 x_1 + \cdots + c_n x_n$
  - This is the Math 254-familiar dot product $\varphi(\vec{x}) = \vec{c} \cdot \vec{x}$.
- $\varphi : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}$ defined by $\varphi(p) = p''(1) + 2p'(2) + 3p(p)$
- $\varphi : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}$ defined by $\varphi(p) = \int_{-1}^{1} p(x)\, dx$

Definition (Dual Space $V' = \mathcal{L}(V, \mathbb{F})$)

The dual space of V, denoted V' is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbb{F})$.

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Dimension of the Dual Space

Theorem ($\dim(V') = \dim(V)$)

Suppose V is finite-dimensional. Then V' is also finite-dimensional and $\dim(V') = \dim(V)$.

Proof ($\dim(V') = \dim(V)$)

$$\dim(V') = \dim(\mathcal{L}(V, \mathbb{F})) = \dim(V)\dim(\mathbb{F}) = \dim(V)$$

Definition (Dual Basis)

If $v_1, \dots, v_n$ is a basis of V, then the **dual basis** of $v_1, \dots, v_n$ is the list $\varphi_1, \dots, \varphi_n$ of element of V' where each φ_j is the linear functional on V such that

$$\varphi_j(v_k) = \delta_{jk}$$

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The Dual Basis is a Basis of the Dual Space

Example (The Dual Basis of the Standard Basis of $\mathbb{F}$)

$\varphi_j(x_1, \dots, x_n) = x_j$

Theorem (The Dual Basis is a Basis of the Dual Space)

Suppose V is finite-dimensional. Then the dual basis of a basis of V is a basis of V'.

Proof (The Dual Basis is a Basis of the Dual Space)

Suppose $v_1, \dots, v_n$ is a basis of V. Let $\varphi_1, \dots, \varphi_n$ denote the dual basis.

Linear Independence: Let $a_1, \dots, a_n \in \mathbb{F}$ such that

$$a_1\varphi_1 + \cdots + a_n\varphi_n = 0$$

Now, $(a_1\varphi_1 + \cdots + a_n\varphi_n)(v_j) = a_j$, $(j = 1, \dots, n)$; hence $a_j = 0$ $(j = 1, \dots, n)$. $\surd$ (lin.indep. + n elements $\Rightarrow$ basis)

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Dual Map, T'

Definition (Dual Map, T')

If $T \in \mathcal{L}(V, W)$, then the **dual map** of T is the linear map $T' \in \mathcal{L}(W', V')$ defined by $T'(\varphi) = \varphi \circ T$ for $\varphi \in W'$.

Sorting it out:

With $T \in \mathcal{L}(V, W)$ $(T : V \mapsto W)$, and $\varphi \in W' = \mathcal{L}(W, \mathbb{F})$, then $T'(\varphi)$ is defined the above composition $V \mapsto W \mapsto \mathbb{F}$; *i.e.* $T'(\varphi) \in \mathcal{L}(V, \mathbb{F}) = V'$.

Linearity:

- if $\varphi, \psi \in W'$, then

$$T'(\varphi + \psi) = (\varphi + \psi) \circ T = \varphi \circ T + \psi \circ T = T'(\varphi) + T'(\psi)$$

- if $\varphi \in W'$, and $\lambda \in \mathbb{F}$, then

$$T'(\lambda\varphi) = (\lambda\varphi) \circ T = \lambda(\varphi \circ T) = \lambda T'(\varphi)$$

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Example :: Dual Map

Here we take D' to mean the dual map of the differentiation operator D, and let ∂ denote the derivative (as not to overload the prime(′) on a single slide...)

Example

Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ be defined by $Dp = \partial p$.

- $\varphi : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}$ defined by $\varphi(p) = p(3)$. Then $D'(\varphi) : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}$ defined by
$$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(\partial p) = \partial p(3)$$
- $\varphi : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}$ defined by $\varphi(p) = \int_{-1}^{1} p(t)\,dt$. Then
$$(D'(\varphi))(p) = (\varphi \circ D)(p) = \varphi(Dp) = \varphi(\partial p) = \int_{-1}^{1} \partial p\,dt = p(1) - p(-1)$$

Algebraic Properties of Dual Maps

Theorem (Algebraic Properties of Dual Maps)

- $(S + T)' = S' + T'\ \forall S, T \in \mathcal{L}(V, W)$
- $(\lambda S)' = \lambda S'\ \forall \lambda \in \mathbb{F}, S \in \mathcal{L}(V, W)$
- $(ST)' = T'S'\ \forall S \in \mathcal{L}(V, W), T \in \mathcal{L}(U, V)$

Proof (Algebraic Properties of Dual Maps)

The first two are standard "linearity procedure." For the third, let $\varphi \in W'$:

$$(ST)'(\varphi) \overset{1}{=} \varphi \circ (ST) \overset{2}{=} (\varphi \circ S) \circ T \overset{3}{=} T'(\varphi \circ S) \overset{4}{=} T'(S'(\varphi)) \overset{5}{=} (T'S')(\varphi)$$

① ③ ④ — definition of the dual map
② — associativity
⑤ — definition of composition.

The Null Space and Range of the Dual of a Linear Map

Definition (Annihilator, U^0)

For $U \subset V$, the **annihilator** of U, denoted U^0 is defined by
$$U^0 = \{\varphi \in V' : \varphi(u) = 0\ \forall u \in U\}$$

Example (Annihilator)

- Suppose U is the subspace of $\mathcal{P}(\mathbb{F})$ consisting of all polynomial multiples of z^2. If φ is the linear functional on $\mathcal{P}(\mathbb{F})$ defined by $\varphi(p) = p'(0)$, then $\varphi \in U^0$.
- Let $e_1, \dots, e_{2n}$ be the standard basis for $\mathbb{F}^{2n}$, and let $\varphi_1, \dots, \varphi_{2n}$ denote the dual basis of $((\mathbb{F}^{2n})')$. Suppose $U = \mathrm{span}(e_1, e_3, e_5, \dots, e_{2n-1})$, then $U^0 = \mathrm{span}(\varphi_2, \varphi_4, \varphi_6, \dots, \varphi_{2n})$.

The Annihilator :: Subspace Properties, Dimension, Null Space

Theorem (The Annihilator is a Subspace)

Suppose $U \subset V$, then U^0 is a subspace of V'.

Theorem (Dimension of the Annihilator)

Suppose V is finite-dimensional and U is a subspace of V. Then
$$\dim(U) + \dim\left(U^0\right) = \dim(V)$$

Theorem (The Null Space of T')

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then

- $\mathrm{null}(T') = (\mathrm{range}(T))^0$
- $\dim(\mathrm{null}(T')) = \dim(\mathrm{null}(T)) + \dim(W) - \dim(V)$

The Dual Map, T' :: Injectivity, Surjectivity, Range

Theorem (T Surjective $\Leftrightarrow$ T' Injective)

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Then T is surjective if and only if T' is injective.

Theorem (The Range of T')

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Then

- $\dim(\text{range}(T')) = \dim(\text{range}(T))$
- $\text{range}(T') = (\text{null}(T))^0$

Theorem (T Injective $\Leftrightarrow$ T' Surjective)

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Then T is injective if and only if T' is surjective.

The Matrix of the Dual of a Linear Map

Definition (Transpose, A^T)

The transpose of a matrix A, denoted A^T, is the matrix obtained from A by interchanging the rows and columns. More specifically, if $A \in \mathbb{F}^{m\times n}$, then $A^T \in \mathbb{F}^{n\times m}$ is matrix whose entries are given by the equation

$$(A^T)_{k,j} = A_{j,k}$$

Theorem (The Transpose of the Product of Matrices)

If $A \in \mathbb{F}^{m\times n}$, and $B \in \mathbb{F}^{n\times p}$, then

$$(AB)^T = B^T A^T \in \mathbb{F}^{p\times m}$$

The Matrix of the Dual of a Linear Map — The Rank of a Matrix

Theorem (The Matrix of T' is the Transpose of the Matrix of T)

Suppose $T \in \mathcal{L}(V,W)$. Then $\mathcal{M}(T') = (\mathcal{M}(T))^T$.

Definition (Row Rank, Column Rank)

Suppose $A \in \mathbb{F}^{m\times n}$

- The **row rank** of A is the dimension of the span of the rows of A in $\mathbb{F}^{1\times n}$.
- The **column rank** of A is the dimension of the span of the columns of A in $\mathbb{F}^{m\times 1}$.

The Rank of a Matrix

Theorem (Dimension of $\text{range}(T)$ equals column rank of $\mathcal{M}(T)$)

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Then $\dim(\text{range}(T))$ equals the column rank of $\mathcal{M}(T)$.

Theorem (Row Rank Equals Column Rank)

Suppose $A \in \mathbb{F}^{m\times n}$. The the row rank of A equals the column rank of A.

Definition (Rank)

The rank of a matrix $A \in \mathbb{F}^{m\times n}$ is the column rank of A.

Suggested Problems

3.F—1, 2, 3, 5, 8, 32