

# Math 524: Linear Algebra

## Notes #4 — Polynomials

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## Student Learning Targets, and Objectives

### Target Division Algorithm for Polynomials

**Objective** Know that the quotient-remainder form of division “looks essentially the same” for polynomial and scalars

### Target Factorization of Polynomials over $\mathbb{C}$ and $\mathbb{R}$

**Objective** Be able to identify, at least in the abstract, zeros and factors for polynomials

**Objective** Understand how the structure of the polynomial factorization theorems over  $\mathbb{C}$  and  $\mathbb{R}$  differ, and the impact on the existence (or lack thereof) of real zeros for polynomials.

## Introduction

We need some polynomial properties in order to effectively discuss and understand **operators**.

This is a quick overview of properties and results, many of which we have seen in other contexts.

We consider polynomials in real and/or complex variables; so as usual  $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ .

We start off with some additional notation relating to complex numbers.

## Complex Conjugate and Absolute Value

### Definition (Notation — $\operatorname{Re}(z)$ , $\operatorname{Im}(z)$ )

For  $z \in \mathbb{C}$ ,  $z = a + bj$ ,  $a, b \in \mathbb{R}$

- The **real part** of  $z$ , denoted  $\operatorname{Re}(z)$ , is defined by  $\operatorname{Re}(z) = a$
- The **imaginary part** of  $z$ , denoted  $\operatorname{Im}(z)$ , is defined by  $\operatorname{Im}(z) = b$

### Definition (Complex Conjugate $z^*$ (or $\bar{z}$ ), absolute value (or magnitude) $|z|$ )

For  $z \in \mathbb{C}$ ,

- The **complex conjugate** is defined by  $z^* = \operatorname{Re}(z) - \operatorname{Im}(z)i$
- The **magnitude** of  $z$  is defined by  $|z| = \sqrt{(\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2}$

# Properties of Complex Numbers

## Theorem (Properties of Complex Numbers)

Let  $w, z \in \mathbb{C}$ , then:

- $z + z^* = 2\operatorname{Re}(z)$
- $z - z^* = 2(\operatorname{Im}(z))i$
- $zz^* = |z|^2 = (\operatorname{Re}(z))^2 + (\operatorname{Im}(z))^2$
- $(w + z)^* = w^* + z^*$ , and  $(wz)^* = w^*z^*$
- $(z^*)^* = z$
- $|\operatorname{Re}(z)| \leq |z|$ , and  $|\operatorname{Im}(z)| \leq |z|$
- $|z^*| = |z|$
- $|wz| = |w| |z|$
- $|w + z| \leq |w| + |z|$

## Uniqueness of Coefficients for Polynomials

$p : \mathbb{F} \mapsto \mathbb{F}$  is called a polynomial  $p \in \mathcal{P}(\mathbb{F})$  with coefficients in  $\mathbb{F}$  if there exists  $a_0, a_1, \dots, a_m \in \mathbb{F}$ :

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_mz^m$$

$\forall z \in \mathbb{F}$ . If  $a_m \neq 0$ , then  $\deg(p) = m$ .

**Theorem** (If a Polynomial is the Zero Function, then all Coefficients are 0)

Suppose  $a_0, a_1, \dots, a_m \in \mathbb{F}$ , if

$$a_0 + a_1z + a_2z^2 + \cdots + a_mz^m = 0, \quad \forall z \in \mathbb{F}$$

then  $a_0 = a_1 = \cdots = a_m = 0$ .

The zero-polynomial has  $\deg(p) = -\infty$ .

[By Definition]

## Uniqueness of Coefficients for Polynomials

### Theorem (Unique Coefficients)

*The coefficients of a polynomial are **uniquely determined**.*

### Proof (Unique Coefficients)

If a polynomial had two different sets of coefficients, then subtracting the two representations of the polynomial would give a contradiction to the previous theorem.

Another way of stating the same thing is to say that  $\{1, z, \dots, z^m\}$  is a basis for  $\mathcal{P}_m(\mathbb{F})$ :

### Basis

⇒ linear independence

⇒ only linear combination to equal 0 has all zero coefficients.



## The Division Algorithm for Polynomials

If  $p, s \in \mathbb{Z}^+$ ,  $s \neq 0$ , then there exist  $q, r \in \mathbb{Z}^+$  :  $p = sq + r$ , and  $r < s$ . ([QUOTIENT-REMAINDER THEOREM] w/o all the details)

The analogous result for polynomials:

### Theorem (Division Algorithm for Polynomials)

*Suppose  $p, s \in \mathcal{P}(\mathbb{F})$ , with  $s \neq 0$ . Then  $\exists! q, r \in \mathcal{P}(\mathbb{F})$ :  $p = sq + r$ , and  $\deg(r) < \deg(s)$ .*

We can use some of our linear map “tools” to prove the result... which makes it worth doing.

## Proof :: Division Algorithm for Polynomials

## Proof (Division Algorithm for Polynomials)

Let  $n = \deg(p)$  and  $m = \deg(s)$ . If  $n < m$ , then take  $q = 0$  and  $r = p$  to get the desired result. Thus we can assume that  $n \geq m$ .

**Uniqueness** — Let  $T \in \mathcal{L}(\mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F}), \mathcal{P}_n(\mathbb{F}))$  be defined by:

$$T((q, r)) = sq + r, \text{ if } (q, r) \in \text{null}(T), \text{ then } sq + r = 0$$

$\Rightarrow (q = 0, r = 0)$ . (If not, then  $\deg(sq) \geq m \Rightarrow sq \neq -r$ .)

Thus  $\dim(\text{null}(T)) = 0$ .  $\checkmark$

**Existence** — We know

$$\begin{aligned} \dim(\mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F})) &= \dim(\mathcal{P}_{n-m}(\mathbb{F})) + \dim(\mathcal{P}_{m-1}(\mathbb{F})) \\ &= (n - m + 1) + (m - 1 + 1) = (n + 1) \end{aligned}$$

With help from [FUNDAMENTAL THEOREM OF LINEAR MAPS (NOTES#3.1)] it now follows that  $\dim(\text{range}(T)) = (n + 1)$ , and  $(n + 1) = \dim(\mathcal{P}_n(\mathbb{F}))$ , therefore  $\text{range}(T) = \mathcal{P}_n(\mathbb{F})$ ; and we have existence of  $q \in \mathcal{P}_{n-m}(\mathbb{F})$ , and  $r \in \mathcal{P}_{m-1}(\mathbb{F})$  so that  $p = T((q, r)) = sq + r$ .  $\checkmark$

## Zeros of Polynomials

### Definition (Zero of a Polynomial)

A scalar  $\lambda \in \mathbb{F}$  is called a **zero** (or **root**) of a polynomial  $p \in \mathcal{P}(\mathbb{F})$ , if  $p(\lambda) = 0$ .

### Definition (Factor)

A polynomial  $s \in \mathcal{P}(\mathbb{F})$  is called a **factor** of  $p \in \mathcal{P}(\mathbb{F})$ , if  $\exists q \in \mathcal{P}(\mathbb{F})$  such that  $p = sq$ .

### Theorem (Each Zero of a Polynomial Corresponds to a Degree-1 Factor)

Suppose  $p \in \mathcal{P}(\mathbb{F})$  and  $\lambda \in \mathbb{F}$ . Then  $p(\lambda) = 0$  *if and only if*  $\exists q \in \mathcal{P}(\mathbb{F})$ :

$$p(z) = (z - \lambda)q(z), \quad \forall z \in \mathbb{F}.$$

The proof is a simple application of the Division Algorithm.

## Zeros of Polynomials

### Theorem (A Polynomial Has At Most As Many Zeros As Its Degree)

Suppose  $p \in \mathcal{P}(\mathbb{F})$ , with  $\deg(P) \geq 0$ . Then  $p$  has at most  $m$  distinct zeros in  $\mathbb{F}$ .

The proof gives us an excuse to use mathematical induction!

### Proof (A Polynomial Has At Most As Many Zeros As Its Degree)

**m = 0:**  $p(z) = a_0 \neq 0$ , which has no zeros.  $\checkmark$

**m = 1:**  $p(z) = a_0 + a_1z$ ,  $a_1 \neq 0$ ;  $\lambda = -a_0/a_1$  is the one zero.

**m > 1:** Assume the theorem is true  $\forall q \in \mathcal{P}_{m-1}(\mathbb{F})$ . Let  $p \in \mathcal{P}_m(\mathbb{F})$ , if  $p$  has a zero  $\lambda \in \mathbb{F}$ , then (previous theorem)  $\exists q \in \mathcal{P}_{m-1}(\mathbb{F})$ :

$$p(z) = (z - \lambda)q(z), \quad \forall z \in \mathbb{F}.$$

$\deg(q) = (m - 1)$ . We have:  $\{\text{zeros of } p\} = \{\lambda\} \cup \{\text{zeros of } q\}$ .

$\#\text{zeros}(p) = 1 + \#\text{zeros}(q) \leq 1 + (m - 1) = m$ .  $\checkmark$

# Fundamental Theorem of Algebra

## Theorem (Fundamental Theorem of Algebra)

*Every non-constant polynomial with complex coefficients has a zero.*

For a proof, see a course on Complex Analysis (e.g. [MATH 532]).

Whereas there are formulas for the roots of polynomials of degrees 2, 3, and 4; the Abel-Ruffini theorem states

## Theorem (Abel-Ruffini Theorem)

*There is no algebraic solution — that is, solution in radicals — to the general polynomial equations of degree five or higher with arbitrary coefficients.*

The theorem is named after Paolo Ruffini, who provided an incomplete proof in 1799, and Niels Henrik Abel, who provided a proof in 1824. (Galois later proved more general statements, and provided a construction of a polynomial of degree 5 whose roots cannot be expressed in radicals from its coefficients.)

Factorization of a Polynomials Over  $\mathbb{C}$ Theorem (Factorization of a Polynomial Over  $\mathbb{C}$ )

*If  $p \in \mathcal{P}(\mathbb{F})$  is a non-constant polynomial, then  $p$  has a unique factorization (except for the permutation order to the factors, which does not matter, due to the commutativity of multiplication of complex numbers) of the form:*

$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$$

*where  $c, \lambda_1, \dots, \lambda_m \in \mathbb{C}$ .*

The proof relies on [THE FUNDAMENTAL THEOREM OF ALGEBRA (NOTES#3.1)], and mathematical induction... it does not provide any good excuses to exercise our linear algebra skills, so we skip it.

Factorization of a Polynomials Over  $\mathbb{R}$ 

A polynomial with real coefficients may have no real zeros. For example, the polynomial  $1 + x^2$  has no real zeros.

The failure of the Fundamental Theorem of Algebra over  $\mathbb{R}$  accounts for the differences between operators on real and complex vector spaces.

(To be explored in (painful?) detail...)

# Polynomials with Real Coefficients have Complex Zeros in Pairs

## Theorem (Polynomials with Real Coefficients have Complex Zeros in Pairs)

*Suppose  $p \in \mathcal{P}(\mathbb{F})$  is a polynomial with real coefficients. If  $\lambda \in \mathbb{C}$  is a zero of  $p$ , then so is  $\lambda^*$ .*

## Proof (Polynomials with Real Coefficients have Complex Zeros in Pairs)

Let  $p \in \mathcal{P}(\mathbb{F})$  be a polynomial with real coefficients, and  $\lambda \in \mathbb{C}$  a zero of  $p$ , then:

$$\begin{aligned} \sum_{k=0}^m a_k \lambda^k &= 0 && \text{take the complex conjugate} \\ \sum_{k=0}^m a_k (\lambda^*)^k &= 0 && \text{and there it is! } \checkmark \end{aligned}$$



Factorization of a Polynomial Over  $\mathbb{R}$ 

Polynomial factorization over  $\mathbb{R}$  is not nearly as pretty as factorization over  $\mathbb{C}$ :

Theorem (Factorization of a Polynomial Over  $\mathbb{R}$ )

Let  $p \in \mathcal{P}(\mathbb{R})$  is a non-constant polynomial. Then  $p$  has a unique factorization of the form

$$p(x) = c(x - \lambda_1) \dots (x - \lambda_m)(x^2 + b_1x + c_1) \dots (x^2 + b_Mx + c_M)$$

where  $c, \lambda_1, \dots, \lambda_m, b_1, \dots, b_M, c_1, \dots, c_M \in \mathbb{R}$ , with  $b_k^2 < 4c_k$   
 $k = 1, \dots, M$ .

Theorem (Factorization of a Polynomial Over  $\mathbb{C}$ )

If  $p \in \mathcal{P}(\mathbb{F})$  is a non-constant polynomial, then  $p$  has a unique factorization of the form

$$p(z) = c(z - \lambda_1) \dots (z - \lambda_m)$$

where  $c, \lambda_1, \dots, \lambda_m \in \mathbb{C}$ .

⟨⟨⟨ Live Math ⟩⟩⟩

e.g.  $4-5^{\circ}$ , **8**,  $11^{+}$

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**4-8:** Define  $T : \mathcal{P}(\mathbb{R}) \mapsto \mathbb{R}^{\mathbb{R}}$  by

$$T(p) = \begin{cases} \frac{p - p(3)}{x - 3} & x \neq 3 \\ p'(3) & x = 3 \end{cases}$$

Show (i)  $T(p) \in \mathcal{P}(\mathbb{R}) \forall p \in \mathcal{P}(\mathbb{R})$ , and (ii)  $T$  is a linear map.

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✳

**Solution**

✳

(i) means that  $T : \mathcal{P}(\mathbb{R}) \mapsto \mathcal{P}(\mathbb{R})$ ; we note that  $\mathcal{P}(\mathbb{R}) \subset \mathbb{R}^{\mathbb{R}}$

We also note that for  $x \neq 3$ ,  $T(p)$  produces a function — *i.e.* the  $p$  in the numerator is the polynomial, *not* its value  $p(x)$ .

\*

## Solution

\*

- (i) Let  $p \in \mathcal{P}(\mathbb{R})$ , then  $\hat{p} = p - p(3) \in \mathcal{P}(\mathbb{R})$ , and  $\hat{p}(3) = 0$ . By [EACH ZERO OF A POLYNOMIAL CORRESPONDS TO A DEGREE-1 FACTOR],  $\exists q \in \mathcal{P}(\mathbb{R})$  so that

$$p(x) - p(3) = (x - 3)q(x), \quad \forall x \in \mathbb{R}$$

[DEF. DERIVATIVES AND CONTINUITY (CALCULUS)]  $\Rightarrow p'(3) = q(3)$ . Therefore  $T(p) = q$ , and  $q \in \mathcal{P}(\mathbb{R})$  which shows (i).

- (ii) Linearity follows from the definition of Linear Maps, the usual “mechanics,” and the fact that derivative computations are linear under addition and scalar multiplication ( $\forall p, r \in \mathcal{P}(\mathbb{R}), \forall \alpha \in \mathbb{R}$ ):

$$(p + r)'(3) = p'(3) + r'(3) \quad (\alpha p)'(3) = \alpha p'(3).$$

## Suggested Problems

4—2, 3, 5<sup>ⓐ</sup>, 8, 11<sup>+</sup>

These problems have a strong linear algebra “flavor,” the rest deal more with polynomial / complex arithmetic properties.

ⓐ-marked problem are really good, but may be just a bit too much for homework

<sup>+</sup>-marked problems have longer/more challenging solutions.

## Assigned Homework

## HW#4, Due Date in Canvas/Gradescope

4—2, 3

**Note:** Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually “scheduled.”)

Upload homework to [www.Gradescope.com](http://www.Gradescope.com)

## Supplements

⟨PLACEHOLDER⟩