

# Math 524: Linear Algebra

## Notes #5 — Eigenvalues, Eigenvectors, and Invariant Subspaces

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### Student Learning Targets, and Objectives

#### Target Invariant Subspaces

- Objective Know how the restriction operator and invariant subspaces are connected
- Objective Be familiar with the 1-D “line-type” subspaces and their connection with eigenvalues and eigenvectors.
- Objective Know, and be able to use, the fact that eigenvectors corresponding to distinct eigenvalues are linearly independent.

#### Target Eigenvalues, Eigenvectors, and Eigenspaces

- Objective Know that every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue and an upper-triangular matrix with respect to some basis
- Objective Know the the definitions of Eigenspaces of operators; and understand the discussion of how “collecting” enough eigenvalues can guarantee invertibility of an operator.



### Outline

- 1 Student Learning Targets, and Objectives
  - SLOs: Invariant Subspaces; Eigen-{values, vectors, spaces}
- 2 Invariant Subspaces
  - Invariant Subspaces
- 3 Eigenvectors and Upper-Triangular Matrices
  - Polynomials Applied to Operators
  - Existence of Eigenvalues
  - Upper-Triangular Matrices
- 4 Eigenspaces and Diagonal Matrices
  - Toward Better Understanding of an Operator...
- 5 Problems, Homework, and Supplements
  - Suggested Problems
  - Assigned Homework
  - Supplements



### Introduction

We now turn our attention to one of the cornerstones of Linear Algebra, the study of **Operators on finite-dimensional vector spaces**.

#### Rewind (Operator, $\mathcal{L}(V)$ )

- A linear map from a vector space to itself is called an **operator**.
- The notation  $\mathcal{L}(V)$  denotes the set of all operators on  $V$ . In other words,  $\mathcal{L}(V) = \mathcal{L}(V, V)$ .

We will use our newly acquired abstract understanding of vector spaces and linear maps to the study of eigen-values and eigen-vectors.

Time-Target:  $3 \times 75$ -minute lectures.



## Invariant Subspaces

Let  $T \in \mathcal{L}(V)$ . If we have [DIVIDE-AND-CONQUER “THEOREM”]

$$V = U_1 \oplus \cdots \oplus U_m$$

where each  $U_j$  is a proper subspace of  $V$  (i.e.  $\dim(U_j) < \dim(V)$ ); then it is sufficient to understand the action of  $T$  on each  $U_j$ .

Notation (Restriction,  $T|_{U_j}$ )

$T|_{U_j}$  is the **restriction** of the linear map  $T \in \mathcal{L}(V)$  to the subspace  $U_j$ .

This only makes sense if  $T|_{U_j} : U_j \mapsto U_j$ , or if you want  $T|_{U_j} \in \mathcal{L}(U_j)$ . Such subspaces get their own name...

Definition (Invariant Subspace)

Let  $T \in \mathcal{L}(V)$ . A subspace  $U$  of  $V$  is called **invariant** under  $T$  if  $\forall u \in U \Rightarrow T(u) \in U$ .



## Invariant Subspaces

### Example (Invariant Subspaces)

Suppose  $T \in \mathcal{L}(V)$ , then the following subspace are invariant under  $T$

- $\{0\}$ 
  - If  $u \in \{0\}$ , then  $u = 0$ . [LINEARITY]  $T(u) = T(0) = 0 \in \{0\}$ .
- $V$ 
  - If  $u \in V$ , then — since  $T \in \mathcal{L}(V)$  —  $T(u) \in V$
- $\text{null}(T)$  — (could be  $\{0\}$ )
  - If  $u \in \text{null}(T)$ , then  $T(u) = 0 \in \text{null}(T)$  [LINEARITY]
- $\text{range}(T)$  — (could be  $V$ )
  - If  $u \in \text{range}(T)$ , then  $T(u) \in \text{range}(T)$ , by definition of range.



## Invariant Subspaces

Question

Must an operator  $T \in \mathcal{L}(V)$  have any invariant subspaces other than  $\{0\}$  and  $V$ ?

—  $\text{null}(T)$  and  $\text{range}(T)$  do not necessarily provide useful insight.

We will see that the answer is yes, as long as  $\dim(V) > 1$  (for  $\mathbb{F} = \mathbb{C}$ ), or  $\dim(V) > 2$  (for  $\mathbb{F} = \mathbb{R}$ ).

Example

Let  $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$  is defined by  $Dp = p'$ , then for any fixed  $m$ ,  $\mathcal{P}_m(\mathbb{R})$  is an invariant subspace of  $\mathcal{P}(\mathbb{R})$ .

In this case:

$$\dim(\text{null}(D)) = 1, \text{ and } \dim(\text{range}(D)) = m = (\dim(\mathcal{P}_m(\mathbb{R})) - 1).$$



## Eigenvalues and Eigenvectors :: Eigenvalues

We will look at invariant subspaces in careful detail; first we turn our attention to the case of invariant subspaces with  $\dim = 1$ .

Consider the 1-dimensional (“line”-type) subspaces: let  $v \neq 0 \in V$ , and define  $U = \{\lambda v : \lambda \in \mathbb{F}\} \equiv \text{span}(v)$ .

If  $U$  is invariant under  $T \in \mathcal{L}(V)$ , then  $T(v) \in U$  ( $\forall v \in U$ ), and hence  $\exists \lambda \in \mathbb{F}$  such that

$$T(v) = \lambda v$$

The converse holds: if  $T(v) = \lambda v$  for some  $v \in V$ , and  $\lambda \in \mathbb{F}$ , then  $\text{span}(v)$  is an invariant subspace of  $V$  under the linear map  $T$ .



## Eigenvalues and Eigenvectors :: Eigenvalues

In the past, we have surely seen eigenvalues (and eigenvectors) defined for *matrices*, here we generalize the concept to operators on all finite-dimensional subspaces...

### Definition (Eigenvalue)

Suppose  $T \in \mathcal{L}(V)$ . A scalar  $\lambda \in \mathbb{F}$  is called an **eigenvalue** of  $T$  if there exists  $v \neq 0 \in V$  such that  $T(v) = \lambda v$ .



## Eigenvalues and Eigenvectors :: Eigenvalues

### Theorem (Equivalent Conditions to be an Eigenvalue)

Suppose  $V$  is finite-dimensional,  $T \in \mathcal{L}(V)$ , and  $\lambda \in \mathbb{F}$ . The following are equivalent:

- (a)  $\lambda$  is an eigenvalue of  $T$
- (b)  $T - \lambda I$  is not injective
- (c)  $T - \lambda I$  is not surjective
- (d)  $T - \lambda I$  is not invertible

**Recall:**  $I \in \mathcal{L}(V) : I(v) = v, \forall v \in V$ .

(a)  $\Leftrightarrow$  (b), by rearranging  $T(v) = \lambda v$

(b)  $\Leftrightarrow$  (c)  $\Leftrightarrow$  (d) by [INJECTIVITY  $\Leftrightarrow$  SURJECTIVITY IN FINITE DIMENSIONS (NOTES#3.2)]



## Eigenvalues and Eigenvectors :: Eigenvectors

### Definition (Eigenvector)

Suppose  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbb{F}$ . A vector  $v \in V$  is called an **eigenvector** of  $T$  corresponding to the eigenvalue  $\lambda$  if  $v \neq 0$ , and  $T(v) = \lambda v$ .

**Note:** Eigenvalues can be 0, but

Eigenvectors cannot be the zero-vector.

⚠ Eigenvectors corresponding to  $\lambda = 0$  come from  $\text{null}(T)$ .

Since  $T(v) = \lambda v$  if and only if  $(T - \lambda I)v = 0$ ,  $v \neq 0 \in V$  is an eigenvector of  $T$  corresponding to the eigenvalue  $\lambda$  if and only if  $v \in \text{null}(T - \lambda I)$ .



## Eigenvalues and Eigenvectors ::

### Example (Rotation over $\mathbb{R}$ and $\mathbb{C}$ )

Suppose  $T \in \mathcal{L}(\mathbb{F}^2)$ , is defined by  $T(x, y) = (-y, x)$ .

$\mathbb{F} = \mathbb{R}$   $T$  is a counterclockwise rotation by  $\pi/2$  about the origin in  $\mathbb{R}^2$ . There is no real scaling of a vector such that  $(-y, x) = \lambda(x, y)$ .  $T$  has no eigenvalue(s) and no eigenvector(s).

$\mathbb{F} = \mathbb{C}$  We are looking for  $\lambda \in \mathbb{F}$  such that  $(-y, x) = \lambda(x, y)$ :

$$\begin{cases} \lambda x = -y \\ \lambda y = x \end{cases} \Rightarrow -y = \lambda x = \lambda(\lambda y) = \lambda^2 y$$

EIGENVALUES:  $\lambda^2 = -1 \Rightarrow \lambda = \pm i$ .

EIGENVECTORS:  $\begin{cases} (-y, x) = +i(x, y) \Rightarrow (x, y) = (w, -wi) \\ (-y, x) = -i(x, y) \Rightarrow (x, y) = (w, +wi) \end{cases}$



## Eigenvectors Corresponding to Distinct Eigenvalues are Linearly Independent

### Theorem (Linearly Independent Eigenvectors)

Let  $T \in \mathcal{L}(V)$ . Suppose  $\lambda_1, \dots, \lambda_m$  are distinct eigenvalues of  $T$ , and  $v_1, \dots, v_m$  are the corresponding eigenvectors; then  $v_1, \dots, v_m$  is linearly independent.

### Proof (Linearly Independent Eigenvectors)

[BY CONTRADICTION] Suppose  $v_1, \dots, v_m$  is linearly dependent. Let  $k$  be the smallest positive integer such that  $v_k \in \text{span}(v_1, \dots, v_{k-1})$ . We can find  $a_1, \dots, a_{k-1} \in \mathbb{F}$  such that

$$(1) \quad v_k = a_1 v_1 + \dots + a_{k-1} v_{k-1}$$

$$T(v_k) = T(a_1 v_1 + \dots + a_{k-1} v_{k-1})$$

$$(2) \quad \lambda_k v_k = a_1 \lambda_1 v_1 + \dots + a_{k-1} \lambda_{k-1} v_{k-1}$$

$$\lambda_k(1) - (2) \quad 0 = a_1(\lambda_k - \lambda_1)v_1 + \dots + a_{k-1}(\lambda_k - \lambda_{k-1})v_{k-1}$$

Since  $v_1, \dots, v_{k-1}$  is linearly independent,  $\lambda_k$  is magically equal to all the distinct  $\lambda_1, \dots, \lambda_{k-1}$ . **Contradiction!**



## Number of Eigenvalues $\leq \dim(V)$

### Theorem (Number of Eigenvalues)

Suppose  $V$  is finite-dimensional. Then each operator on  $V$  has at most  $\dim(V)$  distinct eigenvalues.

### Proof (Number of Eigenvalues)

Let  $m = \dim(V)$ . We can find at most  $m$  linearly independent vectors in  $V$ ; eigenvectors corresponding to distinct eigenvalues are linearly independent (by previous theorem); so we can find at most  $m$  eigenvectors; thus at most  $m$  distinct eigenvalues.



## Restriction and Quotient Operators

If  $T \in \mathcal{L}(V)$ , and  $U$  is a subspace of  $V$  invariant under  $T$ , then  $U$  determines two other operators:

Definition (Restriction Operator  $T|_U$ ; and Quotient Operator  $T/U$ )

Suppose  $T \in \mathcal{L}(V)$ , and  $U$  is a subspace of  $V$  invariant under  $T$

- The **restriction operator**  $T|_U \in \mathcal{L}(U)$  is defined by

$$T|_U(u) = T(u), \quad u \in U$$

- The **quotient operator**  $T/U \in \mathcal{L}(V/U)$  is defined by

$$(T/U)(v + U) = T(v) + U, \quad v \in V$$



## Restriction and Quotient Operators :: Example

1/2

### Example

Define  $T \in \mathcal{L}(\mathbb{F}^2)$  by  $T(x, y) = (y, 0)$ . Let  $U = \{(x, 0) : x \in \mathbb{F}\}$

- $U$  is invariant under  $T$  and  $T|_U$  is the 0-operator on  $U$ :
  - $T(x, 0) = (0, 0) \in U$ .
  - So  $U$  is invariant under  $T$  and  $T|_U$  is the 0-operator on  $U$ .
- $\nexists$  a subspace  $W$  of  $\mathbb{F}^2$  that is invariant under  $T$ , and  $U \oplus W = \mathbb{F}^2$ .
  - Since  $\dim(\mathbb{F}^2) = 2$ ,  $\dim(U) = 1$ , we must have  $\dim(W) = 1$ . If  $W$  is invariant under  $T$ , then all  $w \in W$  are eigenvectors. However, the only eigenvalue is  $\lambda = 0$ , and  $U$  contains the corresponding eigenvectors.

Thus  $W$  cannot be invariant under  $T$ .



Restriction and Quotient Operators :: Example

2/2

Example

Define  $T \in \mathcal{L}(\mathbb{F}^2)$  by  $T(x, y) = (y, 0)$ . Let  $U = \{(x, 0) : x \in \mathbb{F}\}$

•  $T/U$  is the 0-operator on  $\mathbb{F}^2/U$ :

•  $(x, y) \in \mathbb{F}^2$

$$\begin{aligned} (T/U)((x, y) + U) &= T(x, y) + U \\ &= (y, 0) + U \\ &= 0 + U \end{aligned}$$

the last equality holds because  $(y, 0) \in U$ .

This example shows that sometimes the restriction and quotient operators do not provide (enough) information about  $T$ . Here, both are the 0-operators on their respective spaces, even though  $T$  is not.



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e.g. 5A-{12}



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5A-12

**5A-12:** Define  $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$  by  $(Tp)(x) = xp'(x) \forall x \in \mathbb{R}$ . Find all eigenvalue and eigenvectors of  $T$ .

**Solution**

We use the eigenvalue/eigenvector characterization  $T(p) = \lambda p$  — here  $xp'(x) = \lambda p(x)$ . We can write any  $p \in \mathcal{P}_4(\mathbb{R})$  in the form  $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ , which gives us

$$xp'(x) = a_1x + 2a_2x^2 + 3a_3x^3 + 4a_4x^4 = \lambda(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) = \lambda p(x)$$

In order for the equality to hold, the coefficients for each power must be equal in the left and right expressions. Collecting those relations give us...



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5A-12

$$\begin{cases} 0a_0 &= \lambda a_0 \\ 1a_1 &= \lambda a_1 \\ 2a_2 &= \lambda a_2 \\ 3a_3 &= \lambda a_3 \\ 4a_4 &= \lambda a_4 \end{cases}$$

For any  $j \in \{0, 1, 2, 3, 4\}$ : a solution is given by

$$\{ a_j \neq 0, \lambda = j, a_{k \neq j} = 0 \},$$

which allows us to identify 5 eigenvalue-eigenvector pairs:

$$\{(0, 1), (1, x), (2, x^2), (3, x^3), (4, x^4)\}$$

Technically, any non-zero scaling of the eigenvectors  $\{1, x, x^2, x^3, x^4\}$  is also an eigenvector.



## Polynomials Applied to Operators

The main reason that a richer theory exists for operators (which map a vector space into itself) than for more general linear maps is that operators can be raised to powers — composed with themselves / applied multiple times:

### Definition ( $T^m$ )

Suppose  $T \in \mathcal{L}(V)$  and  $m$  is a positive integer

- $T^m$  is defined by  $T^m = \underbrace{T \circ \cdots \circ T}_{m \text{ times}}$
- $T^0$  is defined to be the identity operator on  $V$
- If  $T$  is invertible, with inverse  $T^{-1}$ , then  $T^{-m} = (T^{-1})^m$



## The Operator $p(T)$

### Definition (The Operator $p(T)$ )

Suppose  $T \in \mathcal{L}(V)$  and  $p \in \mathcal{P}(\mathbb{F})$  is a polynomial given by

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_mz^m, \quad z \in \mathbb{F}$$

Then  $p(T)$  is the operator defined by

$$p(T) = a_0I + a_1T + a_2T^2 + \cdots + a_mT^m$$

### Example ("The Gateway to Differential Equations.")

Suppose  $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$  is the differentiation operator defined by  $Dq = q'$ , with  $p$  being the polynomial defined by  $p(x) = x^2 + k$ , then  $p(D) = D^2 + k$ , and

$$p(D)q = q'' + kq, \quad \forall q \in \mathcal{P}(\mathbb{R})$$

$p(D)q = 0$  is the Helmholtz Equation (in 1D).

[https://en.wikipedia.org/wiki/Helmholtz\\_equation](https://en.wikipedia.org/wiki/Helmholtz_equation)



## Product of Polynomials

### Definition (Product of Polynomials)

If  $p, q \in \mathcal{P}(\mathbb{F})$ , then  $pq \in \mathcal{P}(\mathbb{F})$  is the polynomial defined by

$$(pq)(z) = p(z)q(z), \quad z \in \mathbb{F}$$

### Theorem (Multiplicative Properties)

Suppose  $p, q \in \mathcal{P}(\mathbb{F})$  and  $T \in \mathcal{L}(V)$ , then

- $(pq)(T) = p(T)q(T)$
- $p(T)q(T) = q(T)p(T)$

The proof is purely "mechanical" (distributive property + bookkeeping)



## Existence of Eigenvalues

### Theorem (Existence of Eigenvalues)



Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.



### Proof (Existence of Eigenvalues)

Suppose  $V$  is a complex vector space with dimension  $n > 0$  and  $T \in \mathcal{L}(V)$ . Let  $v \neq 0 \in V$ , then

$$v, T(v), T^2(v), \dots, T^n(v)$$

is not linearly independent, because  $V$  has dimension  $n$  and we have  $(n+1)$  vectors. Thus there exist complex numbers  $a_0, a_1, \dots, a_n$ , such that

$$0 = a_0v + a_1T(v) + a_2T^2(v) + \cdots + a_nT^n(v).$$

Not all  $a_1, \dots, a_n$  can be zero, since that would force  $a_0 = 0$  (and this would make the  $(n+1)$  vectors linearly independent)...



## Existence of Eigenvalues

### Proof (Existence of Eigenvalues)

Now, let the  $a$ 's be the coefficients of a polynomial; which by the [FUNDAMENTAL THEOREM OF ALGEBRA] has a factorization

$$p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n = c(z - \lambda_1) \cdots (z - \lambda_m)$$

where  $c$  is a nonzero complex number,  $\lambda_j \in \mathbb{C}$ , and the equation holds  $\forall z \in \mathbb{C}$  (here  $m$  is not necessarily equal to  $n$ , because  $a_n$  may equal 0).

We then have

$$\begin{aligned} 0 &= a_0v + a_1T(v) + a_2T^2(v) + \cdots + a_nT^n(v) \\ &= (a_0I + a_1T + a_2T^2 + \cdots + a_nT^n)(v) \\ &= c(T - \lambda_1I) \cdots (T - \lambda_mI)v \end{aligned}$$

Thus  $(T - \lambda_jI)$  is not injective for at least one  $j$ .  $\Leftrightarrow T$  has an eigenvalue.



## Upper-Triangular Matrices

### Definition (Matrix of an Operator, $\mathcal{M}(T)$ )

Suppose  $T \in \mathcal{L}(V)$  and  $v_1, \dots, v_n$  is a basis of  $V$ . The **matrix of  $T$**  with respect to this basis is the  $(n \times n)$  matrix

$$\mathcal{M}(T) = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}$$

whose entries  $a_{j,k}$  are defined by

$$T(v_k) = a_{1,k}v_1 + \cdots + a_{n,k}v_n$$

If the basis is not "obvious from context," then we use the notation  $\mathcal{M}(T, (v_1, \dots, v_n))$ .



## Upper-Triangular Matrices :: Comments

Note that matrices of operators are square, rather than the more general rectangular case which we considered earlier for linear maps.

If  $T$  is an operator on  $\mathbb{F}^n$  and no basis is specified, assume that the basis in question is the standard basis. The  $j$ th column of  $\mathcal{M}(T)$  is then  $T$  applied to the  $j$ th basis vector.

A central ~~goal~~ milestone of linear algebra is to show that given an operator  $T \in \mathcal{L}(V)$ , there exists a basis of  $V$  with respect to which  $T$  has a reasonably simple matrix.

For instance, we might try to choose a basis of  $V$  such that  $\mathcal{M}(T)$  has many 0's.



## Upper-Triangular Matrices

If  $V$  is a finite-dimensional complex vector space, there is a basis of  $V$  with respect to which the matrix of  $T$  looks like

	$v$	$w_1$	$\cdots$	$w_{n-1}$
$v$	$\lambda$	*	*	*
$w_1$	0	*	*	*
$\vdots$	$\vdots$	*	*	*
$w_{n-1}$	0	*	*	*

Let  $\lambda$  be an eigenvalue of  $T$  (existence is guaranteed); and let  $v$  be the corresponding eigenvector. Extend  $v$  to a basis of  $V$ :

$v, w_1, \dots, w_{n-1}$  [LINEARLY INDEPENDENT LIST EXTENDS TO A BASIS (NOTES#2)]. Then the matrix of  $T$  with respect to this basis has the form given.





## Upper-Triangular Matrices

### Definition (Diagonal of a Matrix)

The diagonal of a square matrix consists of the entries along the line from the upper left corner to the bottom right corner.

— The  $a_{i,i}$ -entries.

### Definition (Upper-Triangular Matrix)

A matrix is called upper triangular if all the entries below the diagonal equal 0.

—  $a_{i,j} = 0 \forall i > j$ . “The strictly lower-triangular part is filled with zeros.”



## Conditions for Upper-Triangular Matrix

### Theorem (Conditions for Upper-Triangular Matrix)

Suppose  $T \in \mathcal{L}(V)$  and  $v_1, \dots, v_n$  is a basis of  $V$ . Then the following are equivalent:

- (a) the matrix of  $T$  with respect to  $v_1, \dots, v_n$  is upper triangular
- (b)  $T(v_k) \in \text{span}(v_1, \dots, v_k), \forall k$
- (c)  $U_k = \text{span}(v_1, \dots, v_k)$  is invariant under  $T \forall k$

### Proof (Conditions for Upper-Triangular Matrix)

(a)  $\Leftrightarrow$  (b) follows from the definition, and (c)  $\Rightarrow$  (b). The only part that requires work is (b)  $\Rightarrow$  (c).

Suppose (b) holds. Fix  $k \in \{1, \dots, n\}$ . From (b) we know  $T(v_i) \in \text{span}(v_1, \dots, v_i) \subset \text{span}(v_1, \dots, v_k), i \in \{1, \dots, k\}$ . Thus if  $v = a_1 v_1 + \dots + a_k v_k$ , then  $T(v) \in \text{span}(v_1, \dots, v_k)$ , which shows that  $\text{span}(v_1, \dots, v_k)$  is invariant under  $T$ .



## Over $\mathbb{C}$ , Every Operator has an Upper-Triangular Matrix

### Theorem (Over $\mathbb{C}$ , Every Operator has an Upper-Triangular Matrix)

Suppose  $V$  is a finite-dimensional complex vector space and  $T \in \mathcal{L}(V)$ . Then  $T$  has an upper-triangular matrix with respect to some basis of  $V$ .

### Comment

The result does **not** hold on real vector spaces, because the first vector in a basis with respect to which an operator has an upper-triangular matrix is an eigenvector of the operator. Thus if an operator on a real vector space has no eigenvalues, then there is no basis with respect to which the operator has an upper-triangular matrix.

We skip the proof... but fear not, Axler provides 2 proofs in the book (pp.149–150).



## Determination of Invertibility from Upper-Triangular Matrix

The following two theorems indicate why we have gone through so much trouble to (isomorphically) link operators on abstract vector spaces to operators on  $T \in \mathcal{L}(\mathbb{R}^n)$ ...

### Theorem (Determination of Invertibility from Upper-Triangular Matrix)

Suppose  $T \in \mathcal{L}(V)$  has an upper-triangular matrix with respect to some basis of  $V$ . Then  $T$  is invertible if and only if all the entries on the diagonal of that upper-triangular matrix are nonzero.

### Theorem (Determination of Eigenvalues from Upper-Triangular Matrix)

Suppose  $T \in \mathcal{L}(V)$  has an upper-triangular matrix with respect to some basis of  $V$ . Then the eigenvalues of  $T$  are precisely the entries on the diagonal of that upper-triangular matrix.

Unfortunately, identifying bases which reveal eigenvalues is non-trivial.





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e.g. 5B-**{4, 5, 6, 10}**



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5B-4

**5B-4:** Suppose  $P \in \mathcal{L}(V)$  and  $P^2 = P$ .  
Prove that  $V = \text{null}(P) \oplus \text{range}(P)$ .

⌘ (i)  $\text{null}(P) \cap \text{range}(P) = \{0\}$  ⌘

Let  $u \in \text{null}(P) \cap \text{range}(P)$ . Then  $P(u) = 0$ , and  $\exists w \in W : u = P(w)$ .  
Applying  $P$  to  $u = P(w)$  gives

$$0 = P(u) = P^2(w) = P(w) = u,$$

hence the only vector in  $\text{null}(P) \cap \text{range}(P)$  is  $u = 0$ .



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5B-4

⌘ (ii)  $V = \text{null}(P) + \text{range}(P)$  ⌘

Next, let  $v \in V$ , then

$$v = v + 0 = v + (P(v) - P(v)) = (v - P(v)) + P(v),$$

where

$$P(v - P(v)) = P(v) - P^2(v) = P(v) - P(v) = 0 \Rightarrow (v - P(v)) \in \text{null}(P),$$

and, by definition

$$P(v) \in \text{range}(P).$$

Thus

$$v = u + w, \quad u \in \text{null}(P), \quad w \in \text{range}(P)$$

Since  $v \in V$  was arbitrary,  $V = \text{null}(P) + \text{range}(P)$ .

⌘ (i) + (ii)  $\Rightarrow V = \text{null}(P) \oplus \text{range}(P)$  ⌘

[DIRECT SUM OF TWO SUBSPACES (NOTES#1)]



Eigenspaces and Diagonal Matrices

Definition (Diagonal Matrix)

A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.

**Note:** zeros ARE allowed on the diagonal.

If an operator has a diagonal matrix with respect to some basis, then the entries along the diagonal are precisely the eigenvalues of the operator.

Definition (Eigenspace,  $E(\lambda, T)$ )

Suppose  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbb{F}$ . The **Eigenspace** of  $T$  corresponding to  $\lambda$  denoted  $E(\lambda, T)$  is defined to be

$$E(\lambda, T) = \text{null}(T - \lambda I)$$

$E(\lambda, T)$  is the set of all eigenvectors of  $T$  corresponding to  $\lambda$ , along with the 0 vector.



## The Operator Restricted to an Eigenspace

### The Operator Restricted to an Eigenspace:

If  $\lambda$  is an eigenvalue of  $T \in \mathcal{L}(V)$ , then

$$T|_{E(\lambda, T)}(v) = \lambda v, \quad \forall v \in E(\lambda, T)$$

this indicates that eigenspaces are (non-trivial, and highly useful) **invariant subspaces**; and we get a very simple description (scalar multiplication) of the operator when restricted to such a subspace.

*The is where our notation and language is starting to pay dividends!*



## Sum of Eigenspaces is a Direct Sum

### Theorem (Sum of Eigenspaces is a Direct Sum)

Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Suppose also that  $\lambda_1, \dots, \lambda_m$  are distinct eigenvalues of  $T$ . Then

$$E(\lambda_1, T) + \dots + E(\lambda_m, T)$$

is a direct sum:

$$E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$$

Furthermore,

$$\dim(E(\lambda_1, T)) + \dots + \dim(E(\lambda_m, T)) \leq \dim(V)$$

Eigenvalues and eigenspaces give us excellent understanding of operator behavior, so this is worth showing... However, we need a help-result...



## Filling in a Gap... Dimension of a Direct Sum of Finite Dimensional Subspaces

### Theorem (Dimension of a Direct Sum of Finite Dimensional Subspaces)

Suppose  $U_1, \dots, U_m$  are finite-dimensional subspaces of  $V$  such that  $U_1 + \dots + U_m$  is a direct sum. Then  $U_1 \oplus \dots \oplus U_m$  is finite-dimensional and

$$\dim(U_1 \oplus \dots \oplus U_m) = \dim(U_1) + \dots + \dim(U_m)$$

### Proof (Dimension of a Direct Sum of Finite Dimensional Subspaces)

Let  $W, U_1, \dots, U_m$  be subspaces of  $V$  such that

$$W = U_1 \oplus \dots \oplus U_m$$

Choose a (finite) basis  $\{u_{j,k}\}$  for each  $U_j$ . Concatenate the bases into a single list  $\{w_\ell\}$ . By construction the (finite) list  $\{w_\ell\}$  spans  $W = U_1 + \dots + U_m$ . Thus  $W$  is finite dimensional...  $\rightsquigarrow$



## Filling in a Gap... Dimension of a Direct Sum of Finite Dimensional Subspaces

### Proof (Dimension of a Direct Sum of Finite Dimensional Subspaces)

⚠ We need to show that the vectors in  $\{w_\ell\}$  are linearly independent (and thus a basis for  $W$ ), so that the dimension of  $W$  equals the number of vectors in  $\{w_\ell\}$ , and therefore

$$\dim(W) = \dim(U_1) + \dots + \dim(U_m).$$

**Assume:**  $a_1 w_1 + \dots + a_N w_N = 0$ ; we can group the sum in  $m$  groups (depending on what space  $U_j$  was the original source of the basis vector); and thus write this sum as  $\hat{u}_1 + \dots + \hat{u}_m = 0$ , where each  $\hat{u}_j \in U_j$ .

Since  $W = U_1 \oplus \dots \oplus U_m$ , this forces  $0 \equiv \hat{u}_j \in U_j \forall j$ ; but since each such vector is formed by a linear combination of the basis vectors  $\{u_{j,k}\}$ , all the coefficients in each of those linear combinations must be 0; translated back to  $a_1 w_1 + \dots + a_N w_N = 0$ ,  $a_i \equiv 0$ , which makes  $\{w_\ell\}$  linearly independent.  $\checkmark$



## Sum of Eigenspaces is a Direct Sum

With the help-result in hand, we can show the theorem:

Proof (Sum of Eigenspaces is a Direct Sum)

To show that  $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$  is a direct sum, let

$$u_1 + \cdots + u_m = 0$$

where  $u_j \in E(\lambda_j, T)$ . Since the eigenvectors corresponding to distinct eigenvalues are linearly independent [LINEARLY INDEPENDENT EIGENVECTORS], we get  $u_j \equiv 0$ . Now, using [CONDITION FOR A DIRECT SUM (NOTES#1)] this implies that  $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$  is a direct sum. Using the [HELP-RESULT], we now get

$$\begin{aligned} \dim(E(\lambda_1, T) + \cdots + E(\lambda_m, T)) &= \\ \dim(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)) &= \\ \dim(E(\lambda_1, T)) + \cdots + \dim(E(\lambda_m, T)) &\leq \dim(V) \end{aligned}$$



## Diagonalizable Operators

Definition (Diagonalizable)

An operator  $T \in \mathcal{L}(V)$  is called **diagonalizable** if the operator has a diagonal matrix with respect to some basis of  $V$ .

Theorem (Conditions Equivalent to Diagonalizability)

Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Let  $\lambda_1, \dots, \lambda_m$  denote the distinct eigenvalues of  $T$ . Then the following are equivalent:

- $T$  is diagonalizable.
- $V$  has a basis consisting of eigenvectors of  $T$  "Eigenbasis"
- $\exists$  1-D subspaces  $U_1, \dots, U_n$  of  $V$ , each invariant under  $T$ , such that  $V = U_1 \oplus \cdots \oplus U_n$
- $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$
- $\dim(V) = \dim(E(\lambda_1, T)) + \cdots + \dim(E(\lambda_m, T))$



## Conditions Equivalent to Diagonalizability

Proof (Conditions Equivalent to Diagonalizability)

- $(a) \Leftrightarrow (b)$   $T \in \mathcal{L}(V)$  has a diagonal matrix  $\text{diag}(\lambda_1, \dots, \lambda_n)$  if and only if  $T(v_k) = \lambda_k v_k$  for each  $k$ .  $\checkmark$
- $(b) \Rightarrow (c)$  Suppose (b) holds; i.e.  $V$  has a basis  $v_1, \dots, v_n$  consisting of eigenvectors of  $T$ . For each  $k$ , let  $U_k = \text{span}(v_k)$ . By construction each  $U_k$  is a 1-D subspace of  $V$  that is invariant under  $T$ . Because  $v_1, \dots, v_n$  is a basis of  $V$ , each vector in  $V$  can be written uniquely as a linear combination of  $v_1, \dots, v_n$ . That is, each vector in  $V$  can be written uniquely as a sum  $u_1 + \cdots + u_n$ , where  $u_k \in U_k$ . Thus  $V = U_1 \oplus \cdots \oplus U_n$ .  $\checkmark$



## Conditions Equivalent to Diagonalizability

Proof (Conditions Equivalent to Diagonalizability)

- $(c) \Rightarrow (b)$  Suppose (c) holds; thus there are 1-dimensional subspaces  $U_1, \dots, U_n$  of  $V$ , each invariant under  $T$ , such that  $V = U_1 \oplus \cdots \oplus U_n$ .  $\forall k$ , let  $v_k \neq 0 \in U_k$ . Then each  $v_k$  is an eigenvector of  $T$ . Because each vector in  $V$  can be written uniquely as a sum  $u_1 + \cdots + u_n$ , where  $u_k = \alpha_k v_k \in U_k$ , we see that  $v_1, \dots, v_n$  is a basis of  $V$ .  $\checkmark$
- $(b) \Rightarrow (d)$  Suppose (b) holds; thus  $V$  has a basis consisting of eigenvectors of  $T$ . Hence every vector in  $V$  is a linear combination of eigenvectors of  $T$ , which implies that

$$V = E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

Now [SUM OF EIGENSPACES IS A DIRECT SUM] shows that (d) holds:

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T). \quad \checkmark$$



## Conditions Equivalent to Diagonalizability

Proof (Conditions Equivalent to Diagonalizability)

(d) $\Rightarrow$ (e) [HELP-RESULT]  $\checkmark$

(e) $\Rightarrow$ (b) Suppose (e) holds, *i.e.*  $\dim(V) = \dim(E(\lambda_1, T)) + \dots + \dim(E(\lambda_m, T))$ . Select a basis for each  $E(\lambda_j, T)$ ; concatenate the basis into a list  $v_1, \dots, v_n$  of eigenvectors of  $T$  ( $n = \dim(V)$ , by (e)). For linear independence, suppose  $a_1 v_1 + \dots + a_n v_n = 0$ ; let  $u_j$  denote the sum of the group of vectors from  $E(\lambda_j, T)$ ; and we get  $u_1 + \dots + u_m = 0$ . Now, [LINEARLY INDEPENDENT EIGENVECTORS] forces  $u_j = 0$ , which in turn forces  $a_i = 0$ , which makes  $v_1, \dots, v_n$  linearly independent, and a basis for  $V$  by [LINEARLY INDEPENDENT LIST OF THE RIGHT LENGTH IS A BASIS (NOTES#2)].  $\checkmark$

We now have (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (c)  
(b)  $\Rightarrow$  (d)  $\Rightarrow$  (e)  $\Rightarrow$  (b)

which means (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (c)  $\Leftrightarrow$  (d)  $\Leftrightarrow$  (e)  $\checkmark$



## Diagonalizability is not Guaranteed

Unfortunately not every operator is diagonalizable. This can happen even on complex vector spaces, as was shown in one of our previous examples:

Rewind

The  $T \in \mathcal{L}(\mathbb{F}^2)$  defined by  $T(x, y) = (y, 0)$  has a single eigenvalue  $\lambda = 0$ , and  $E(\lambda = 0, T) = \{(x, 0) : x \in \mathbb{F}\}$ .

Since  $\dim(\mathbb{F}^2) = 2$ , and  $\dim(E(0, T)) = 1$ ; we're out of luck

At some point (soon) we have to “do something” about non-diagonalizable operators.



## Enough Eigenvalues Implies Diagonalizability

Theorem (Enough Eigenvalues Implies Diagonalizability)

If  $T \in \mathcal{L}(V)$  has  $n = \dim(V)$  distinct eigenvalues, then  $T$  is diagonalizable.

Proof (Enough Eigenvalues Implies Diagonalizability)

Let  $T \in \mathcal{L}(V)$  have  $n = \dim(V)$  distinct eigenvalues  $\lambda_1, \dots, \lambda_n$ .  $\forall k$ , let  $v_k \in V$  be an eigenvector corresponding to  $\lambda_k$ . By [LINEARLY INDEPENDENT EIGENVECTORS]  $v_1, \dots, v_n$  is linearly independent, and by [LINEARLY INDEPENDENT LIST OF THE RIGHT LENGTH IS A BASIS (NOTES#2)] therefore a basis. With respect to this basis consisting of eigenvectors,  $T$  has a diagonal matrix.

**Note:** this is a one-way result  $\Rightarrow$ , not an if-and-only-if  $\Leftrightarrow$ .



《《《 Live Math 》》》

e.g. 5C-**{8}**



**5C-8:** Suppose  $T \in \mathcal{L}(\mathbb{F}^5)$  and  $\dim(E(8, T)) = 4$ . Prove that  $(T - 2I)$  or  $(T - 6I)$  is invertible.

**Solution**

We remind ourselves that

- $E(\lambda, T) = \text{null}(T - \lambda I)$  are the eigenspaces, and
- the sum of eigenspaces is a direct sum

Therefore

$$\underbrace{\dim(E(8, T))}_4 + \underbrace{\dim(E(2, T))}_{\in \mathbb{Z}^+} + \underbrace{\dim(E(6, T))}_{\in \mathbb{Z}^+} \leq \dim(\mathbb{F}^5) = 5$$

which means that at least one (possibly both) of  $\dim(E(2, T))$  and  $\dim(E(6, T))$  must be zero.  $\rightsquigarrow \exists \hat{\lambda} \in \{2, 6\}$  so that  $E(\hat{\lambda}, T) = \{0\}$ , making  $\hat{\lambda}$  not an eigenvalue, and  $(T - \hat{\lambda}I)$  invertible.



Suggested Problems

**5.A**—1, 2, 3, 4, 8, 9, 10, 12

**5.B**—1(a), 4<sup>‡</sup>, 5\*, 7, 10, 14, 15

**5.C**—1, 2, 8

<sup>‡</sup> This problem has “something” to do with Orthogonal Projections. (Familiar on  $\mathbb{R}^n$ , but here expressed on general vector spaces); also a matrix for which  $P^2 = P$  holds is called *idempotent*. This problem appears in slightly different form in [MATH 543]

\* This problem relates to the eigenvalue structure, and the reason we (in Math 254) look for diagonalizing similarity transformations



Assigned Homework

HW#5, Due Date in Canvas/Gradescope

**5.A**—4, 8, 10(a)

**5.B**—1(a), 7, 14, 15

**5.C**—1, 2

**Note:** Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually “scheduled.”)

Upload homework to [www.Gradescope.com](http://www.Gradescope.com)



Eigenvectors Corresponding to Distinct Eigenvalues are Linearly Independent

Proof ((Alternative) Linearly Independent Eigenvectors)

Let  $\in \mathcal{L}(V)$ . Suppose  $\lambda_1, \dots, \lambda_m$  are distinct eigenvalues of  $T$ , and  $v_1, \dots, v_m$  are the corresponding eigenvectors. Consider  $c_1 v_1 + \dots + c_m v_m = 0$ , where  $c_j \in \mathbb{F}$ . Let  $k$  be the largest index so that  $v_1, \dots, v_k$  is linearly independent, but  $v_1, \dots, v_{k+1}$  is not and let (\*)  $c_1 v_1 + \dots + c_k v_k = v_{k+1}$ . Now:

$$\begin{aligned} \lambda_{k+1} v_{k+1} = T(v_{k+1}) &= T(c_1 v_1 + \dots + c_k v_k) = c_1 T(v_1) + \dots + c_k T(v_k) \\ &= c_1 \lambda_1 v_1 + \dots + c_k \lambda_k v_k \quad (**) \end{aligned}$$

Multiplying (\*) by  $\lambda_{k+1}$  and subtract from (\*\*):

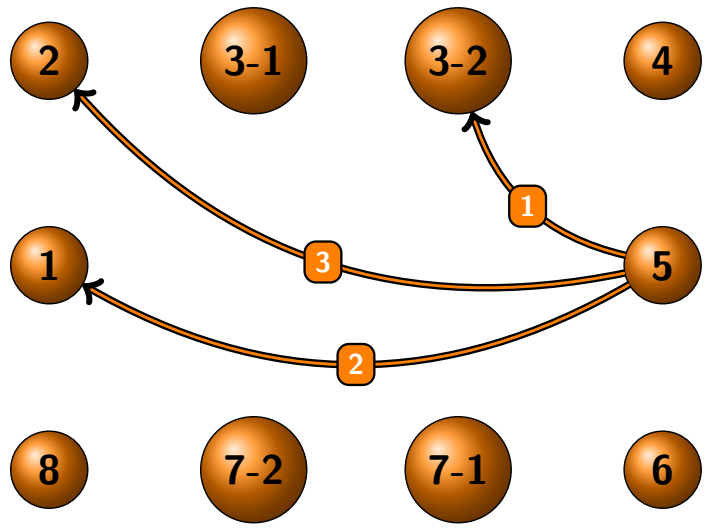
$$c_1 \underbrace{(\lambda_1 - \lambda_{k+1})}_{\neq 0} v_1 + \dots + c_k \underbrace{(\lambda_k - \lambda_{k+1})}_{\neq 0} v_k = 0$$

Since  $v_1, \dots, v_k$  are linearly independent,  $c_1 = \dots = c_k = 0$ , which makes  $v_{k+1} = 0$ , but eigenvectors cannot be the zero vector; hence there is no such value  $k$ , and the vectors are linearly independent.

(It's the same proof, dressed up in a Halloween costume!)



Explicit References to Previous Theorems or Definitions (with count)



Explicit References to Previous Theorems or Definitions

